

Congruences for the Burnside module

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Notation 1.

- G : finite group
- $S(G)$: the set of all subgroups of G and G -set by conjugation
- $\Phi(G)$: the conjugacy class set of G
- Π : partially ordered set and G acts on it preserving the partially order
- $\rho : \Pi \rightarrow S(G)$: an order preserving G -map

Notation 2.

For any $\alpha \in \Pi$,

- $\Pi_\alpha := \{\beta \in \Pi \mid \beta \geq \alpha\}$
- $G_\alpha := \{g \in G \mid g\alpha = \alpha\}$

Definition 3.

A pair (Π, ρ) is called a G -poset if it is satisfying the following condition: for any $\alpha \in \Pi$,

$$\rho(\alpha) \triangleleft G_\alpha \text{ and } \rho : \Pi_\alpha \rightarrow S(G)_{\rho(\alpha)} \text{ is injective.}$$

Note that $S(G)_{\rho(\alpha)} = S(\rho(\alpha)) \subset S(G_\alpha)$ and $G_\alpha \subset G_{\rho(\alpha)} = N_G(\rho(\alpha))$, the normalizer of $\rho(\alpha)$ in G .

Definition 4.

A G -poset (Π, ρ) is called complete if

$$\rho : \Pi_\alpha \rightarrow S(G)_{\rho(\alpha)} \text{ is bijective for all } \alpha \in \Pi.$$

Definition 5.

A finite G -CW-complex X with the base point q is called a Π -complex if it is equipped with a specified set $\{X_\alpha \mid \alpha \in \Pi\}$ of subcomplexes X_α of X , satisfying the following four conditions:

- (i) $X_\alpha \ni q$
- (ii) $gX_\alpha = X_{g\alpha}$ for $g \in G$, $\alpha \in \Pi$,

- (iii) $X_\alpha \subseteq X_\beta$ if $\alpha \leq \beta$ in Π , and
 (iv) for any $H \in S(G)$,

$$X^H = \bigvee_{\rho(\alpha)=H} X_\alpha \quad (\text{the wedge sum of } X_\alpha \text{'s}).$$

Example 6.

Let $\alpha \in \Pi$. The G -CW-complex $(G/\rho(\alpha))^+ (= G/\rho(\alpha) \coprod \{*\})$ is a Π -complex ;

$$(G/\rho(\alpha))_\beta^+ = \{g\rho(\alpha) \mid g\alpha \leq \beta\} \coprod \{*\} \text{ for any } \beta \in \Pi.$$

$\implies (G/\rho(\alpha))^+$ is a Π -complex.

Definition 7. ([7])

Let Z and W be Π -complexes.

$$Z \sim W \iff \chi(Z_\alpha) = \chi(W_\alpha) \text{ for all } \alpha \in \Pi$$

The set

$$\Omega(G, \Pi) = \{[Z] \mid Z \text{ is a } \Pi\text{-complex}\}$$

is called **Burnside module**.

$$[Z] + [W] := [Z \vee W]$$

Remark that

$\implies \Omega(G, \mathcal{F})$ is a finitely generated free abelian group (\leftarrow Proposition 2)

Notation 8.

- $S((G), \alpha) := \{K \in S(G) \mid (K/\rho(\alpha)) \in \Phi(G_\alpha/\rho(\alpha))$
and $K/\rho(\alpha)$ is cyclic}
- $\bar{\chi}(X) = \chi(X) - 1$ for any space X

Theorem A.

Let α be an element of Π .

Then we have $\sum_{K \in S((G), \alpha)} \frac{|G_\alpha/\rho(\alpha)|}{|N_{G_\alpha/\rho(\alpha)}(K/\rho(\alpha))|} \cdot \phi(|K/\rho(\alpha)|) \cdot \bar{\chi}(X_\alpha^K) \equiv 0 \pmod{|G_\alpha/\rho(\alpha)|}$, where $\phi(|K/\rho(\alpha)|)$ is the number of generators of the cyclic group $K/\rho(\alpha)$.

«Proof of Theorem A»

Let (Π, ρ) be a G -poset and G_α the isotropy subgroup at α . Given a Π -complex X , we see

the $G_\alpha/\rho(\alpha)$ -CW-complex $X^{\rho(\alpha)}$ is equipped with a Π -complex structure as following: $(X^{\rho(\alpha)})_\alpha = X_\alpha^{\rho(\alpha)}$ for all $\alpha \in \Pi$. By our definition of the Π -complex, it can be shown that $X_\alpha^{\rho(\alpha)} = X_\alpha$ for all $\alpha \in \Pi$. Let $\chi(X)$ be the Euler characteristic of X , and $\bar{\chi}(X) = \chi(X) - 1$. Note that a map $f : \mathcal{F}_c(G_\alpha/\rho(\alpha)) \rightarrow \mathbb{Z}; K/\rho(\alpha) \mapsto \bar{\chi}(X_\alpha^K)$ satisfies a Burnside relation. By Burnside's lemma [6, Lemma 4.1], we have the desired result.

Lem

Suppose that a G -poset (Π, ρ) is complete. Let α be an element of Π and K a subgroup with $K \supset \rho(\alpha)$. For a Π -complex X , it holds that

$$\bar{\chi}(X_\alpha^K) = \sum_{\beta \in \Pi \text{ with } \rho(\beta) = K, \beta < \alpha} \bar{\chi}(X_\beta).$$

Theorem B.

If a G -poset (Π, ρ) is complete, one has

$$\text{Im}(\bar{\chi} : \Omega(G, \Pi) \rightarrow \bigoplus_{\alpha \in \mathcal{A}} \mathbb{Z})$$

$$= \{(x_\alpha) \in \bigoplus_{\alpha \in \mathcal{A}} \mathbb{Z} \mid \sum_{K \in S((G), \alpha)} \frac{|G_\alpha/\rho(\alpha)|}{|N_{G_\alpha/\rho(\alpha)}(K/\rho(\alpha))|} \cdot \phi(|K/\rho(\alpha)|) \cdot x_{\alpha, (K)} \equiv 0 \pmod{|G_\alpha/\rho(\alpha)|}\},$$

where $x_{\alpha, (K)}$ is some integer such that

$$x_{\alpha, (K)} = \begin{cases} x_\alpha & (K = \rho(\alpha)) \\ \sum_{\beta} x_\beta & (K \neq \rho(\alpha), \beta \text{ is some element of } \Pi \text{ with} \\ & \rho(\beta) = K, \beta < \alpha). \end{cases}$$

《Outline of Proof》

First we use S for the right side, and Im for the left side in the equation of Theorem B. Let $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\}$. By [4, Lemma 1.80], we can arrange elements of \mathcal{A} such that

$$\alpha_i \leq \alpha_j \implies i \leq j.$$

Define a map $P_{\leq k} : \bigoplus_{i=1}^m \mathbb{Z}_{\alpha_i} \rightarrow \bigoplus_{i=1}^k \mathbb{Z}_{\alpha_i}$ by k coordinate maps $p_i : \bigoplus_{i=1}^m \mathbb{Z}_{\alpha_i} \rightarrow \mathbb{Z}_{\alpha_i}$ such that

$$P_{\leq k}(x) = (p_1(x), \dots, p_k(x)),$$

where each \mathbb{Z}_{α_i} is a copy of \mathbb{Z} . Note that $S \subset \bigoplus_{i=1}^m \mathbb{Z}_{\alpha_i}$. It will now suffice to prove that

$$P_{\leq m}(S) = P_{\leq m}(\text{Im}).$$

We proceed by induction on k .

• the case of $k = 1$

By Theorem A and the previous Lemma, we have that $P_{\leq 1}(\mathbf{Im}) = P_{\leq 1}(\mathbf{S})$.

• the case of $k = m$

Remark that $P_{\leq m}(\mathbf{Im}) \subset P_{\leq m}(\mathbf{S})$. (\leftarrow the previous Lemma) Suppose that $P_{\leq k-1}(\mathbf{S}) = P_{\leq k-1}(\mathbf{Im})$

We prove $P_{\leq k}(\mathbf{S}) \subset P_{\leq k}(\mathbf{Im})$

Suppose that $P_{\leq k-1}(\mathbf{S}) = P_{\leq k-1}(\mathbf{Im})$. Let $y = (y_{\alpha_1}, y_{\alpha_2}, \dots, y_{\alpha_{k-1}}, y_{\alpha_k}, y_{\alpha_{k+1}}, \dots, y_{\alpha_m})$ be an element of \mathbf{S} . By assumption, there exists an element

$$x = (x_{\alpha_1}, x_{\alpha_2}, \dots, x_{\alpha_{k-1}}, x_{\alpha_k}, x_{\alpha_{k+1}}, \dots, x_{\alpha_m}) \in \mathbf{Im}$$

such that $x_{\alpha_1} = y_{\alpha_1}, x_{\alpha_2} = y_{\alpha_2}, \dots, x_{\alpha_{k-1}} = y_{\alpha_{k-1}}$. Then we have

$$z = y - x = (0, 0, \dots, 0, y_{\alpha_k} - x_{\alpha_k}, y_{\alpha_{k+1}} - x_{\alpha_{k+1}}, \dots, y_{\alpha_m} - x_{\alpha_m}) \in \mathbf{S}.$$

Here we let $z_{\alpha_i} = y_{\alpha_i} - x_{\alpha_i}$, and $n_{\alpha, K} = \frac{|G_{\alpha}/\rho(\alpha)|}{|N_{G_{\alpha}/\rho(\alpha)}(K/\rho(\alpha))|} \cdot \phi(|K/\rho(\alpha)|)$. Consider the case of $\alpha = \alpha_k$. Then we have

$$\sum_{K \in \mathcal{S}(G, \alpha_k)} n_{\alpha_k, K} \cdot z_{\alpha_k, (K)} \equiv 0 \pmod{|G_{\alpha_k}/\rho(\alpha_k)|}.$$

Observe that the coefficient $z_{\alpha_k, (K)}$ ($K \neq \rho(\alpha_k)$) is equal to $\sum_{\beta} z_{\beta}$, where β is some element of Π with $\rho(\beta) = K, \beta < \alpha_k$. Thus the above equation implies

$$n_{\alpha_k, \rho(\alpha_k)} \cdot z_{\alpha_k, (\rho(\alpha_k))} \equiv 0 \pmod{|G_{\alpha_k}/\rho(\alpha_k)|}.$$

Note that $n_{\alpha_k, \rho(\alpha_k)} = \frac{|G_{\alpha_k}/\rho(\alpha_k)|}{|N_{G_{\alpha_k}/\rho(\alpha_k)}(\rho(\alpha_k)/\rho(\alpha_k))|} \cdot \phi(|\rho(\alpha_k)/\rho(\alpha_k)|) = 1$. That is,

$$z_{\alpha_k} \equiv 0 \pmod{|G_{\alpha_k}/\rho(\alpha_k)|}.$$

On the other hand, we have

$$\bigoplus_{\alpha \in \mathcal{A}} \bar{\chi}([(G/\rho(\alpha_k))^+]) = (\bar{\chi}_{\alpha}([(G/\rho(\alpha_k))^+])_{\alpha \in \mathcal{A}} = \overbrace{(0, 0, \dots, 0, |G_{\alpha_k}/\rho(\alpha_k)|, \dots)}^{k-1}).$$

Hence there exists an integer $a \in \mathbb{Z}$ such that

$$y - x - a(\bar{\chi}_{\alpha}([(G/\rho(\alpha_k))^+])) = \overbrace{(0, 0, \dots, 0, 0, \dots)}^k.$$

$$y = x + a(\bar{\chi}_\alpha((G/\rho(\alpha_k))^+)) + \overbrace{(0, 0, \dots, 0, 0, \dots)}^k.$$

By induction, we see immediately that

$$P_{\leq k}(y) = P_{\leq k}(x + a(\bar{\chi}_\alpha((G/\rho(\alpha_k))^+)) \in P_{\leq k}(\mathbf{Im}).$$

Example 9

Let p be a prime number. We set $G = C_p$ (a cyclic group of order p). Since $S(G) = \{\{e\}, G\}$ (e is the unit element of G), and the G -action on $S(G)$ is trivial, a Burnside module $\Omega(G, S(G))$ is a free abelian group generated by $[(G/\{e\})^+]$, $[(G/G)^+]$. Clearly $\Phi(G) = \{\{e\}, G\}$.

First, consider the case of $\alpha = \{e\}$. Since $S((G), \alpha) = \{\{e\}, G\}$, we get

$$\frac{|G|}{|G|} \cdot 1 \cdot x_{\{e\}, (\{e\})} + \frac{|G|}{|G|} \cdot (p-1) \cdot x_{\{e\}, (G)} \equiv 0 \pmod{p}.$$

That is,

$$x_{\{e\}, (\{e\})} \equiv x_{\{e\}, (G)} \pmod{p}.$$

By Theorem 1.7, there exists a Π -complex X such that $\bar{\chi}(X_{\{e\}}) = x_{\{e\}, (\{e\})}$ and $\bar{\chi}(X_G) = x_{\{e\}, (G)}$. Thus we have

$$\bar{\chi}(X_{\{e\}}) \equiv \bar{\chi}(X_G) \pmod{p}.$$

In particular, if X has a Π -complex structure as follows:

$$X_\alpha = \begin{cases} X & (\alpha = \{e\}) \\ X^G & (\alpha = G), \end{cases}$$

the previous expression implies

$$\chi(X) \equiv \chi(X^G) \pmod{p}.$$

Next for $\alpha = G$, since $S((G), \alpha) = \{G\}$, we obtain

$$\frac{1}{1} \cdot 1 \cdot x_{G, (G)} \equiv 0 \pmod{1}.$$

Immediately,

$$x_{G, (G)} \equiv 0 \pmod{1}.$$

This equation is true for any integer, and so there is no relation for Π -complexes.

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