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HAMILTONIAN STABILITY OF CERTAIN H-MINIMAL LAGRANGIAN SUBMANIFOLDS AND RELATED PROBLEMS

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ABSTRACT. In this article we shall provide a survey on the Hamiltonian stability problem and our recent results for certain compact minimal or Hamiltonian minimal Lagrangian submanifolds in complex projective spaces, compact Hermitian symmetric spaces and complex Euclidean spaces.

INTRODUCTION

Let M be a $2n$-dimensional symplectic manifold with a symplectic form ω. An n-dimensional submanifold L in M is called a *Lagrangian submanifold* if the restriction of ω to L vanishes identically.

We say that a compact Lagrangian submanifold in a Kähler manifold M is a *Hamiltonian minimal* or *H-minimal* Lagrangian submanifold if it has extremal volume under all Hamilitonian deformations of the Lagrangian immersion. If a Lagrangian submanifold is minimal in the usual sense that it has extremal volume under every smooth variation of the submanifold, then it is called a *minimal Lagrangian submanifold* in M. A compact H-minimal Lagrangian submanifold in a Kähler manifold M is called *Hamiltonian stable* if the second variation for the volume is nonnegative for all Hamiltonian deformations of the Lagrangian immersion.

Oh [13], [14], [15], [16] developed the fundamental theory for Hamiltonian stability minimal Lagrangian submaniifolds and Hamiltonian minimal Lagrangian subamnifolds in Kähler manifolds. After his works, several interesting results were given also by other differential geometers

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and many problems to be studied are still open. It is one of most fundamental and interesting problems to find or determine compact Hamiltonian stable H-minimal Lagrangian submanifolds in specific Kähler manifolds such as complex Euclidean spaces, complex projective spaces, complex hyperbolic spaces, Hermitian symmetric spaces, homogeneous Einstein-Kähler manifolds and so on.

Recently in [1], [2], [3], we studied the Hamiltonian stability problem for a nice class of compact minimal or Hamiltonian minimal Lagrangian submanifolds in complex projective spaces, compact Hermitian symmetric spaces and complex Euclidean spaces constructed by the Lie theoretic method. In this article we shall provide an exposition on our recent results and their environs.

1. Hamiltonian minimality and Hamiltonian stability of Lagrangian submanifolds in Kähler manifolds

Let M be a $2n$-dimensional symplectic manifold with a symplectic form ω and $\varphi : L \longrightarrow M$ be a Lagrangian immersion of an n-dimensional smooth manifold L. We set $NL := \varphi^{-1}(TM)/\varphi_*TL$, the quotient vector bundle of $\varphi^{-1}(TM)$ by the subbundle φ_*TL. Let $x \in L$ be a point of L and for each vector $v \in (\varphi^{-1}TM)_x$ along L we define a 1-form $\alpha_v \in T_x^*L$ by $\alpha_v(X) := \omega_{\varphi(x)}(V, X)$ for each $X \in T_xL$. Then it induces linear isomorphisms

$$\varpi : NL \longrightarrow T^*L \text{ and } \varpi : C^\infty(NL) \longrightarrow \Omega^1(L).$$

In this way infinitesimal deformations $V \in C^\infty(NL)$ of a Lagrangian submanifold can be described as 1-forms on L.

A smooth family $\{\varphi_t \mid |t| < \varepsilon\}$ of Lagrangian immersions of L into M with $\varphi_0 = \varphi$ is called a *Lagrangian deformation* of φ or L. We set

$$V_t = \frac{\partial \varphi_t}{\partial t} \in C^\infty(\varphi_t^{-1}TM). \tag{1.1}$$

We call $V \in C^\infty(\varphi^{-1}TM)$ an *infinitesimal Lagrangian deformation* if $\alpha_V \in \Omega^1(L)$ is closed. The following fact is elementary but fundamental.

Proposition 1.1. *If $\varphi_t : L \longrightarrow M$ is a Lagrangian deformation, then V_t is an infinitesimal Lagrangian deformation for each t. Conversly, assume that φ_t is a a smooth family of immersions of L into M such that V_t is an infinitesimal Lagrangian deformation for each t. If φ_{t_0} is a*

Lagrangian immersion for some t_0, then φ_t is a Lagrangian immersion for each t.

Next we define a notion of Hamiltonian deformations of a Lagrangian submanifold, which is a smaller class of Lagrangian deformations. Let $\varphi : L \longrightarrow M$ be a Lagrangian immersion. An infinitesimal deformation $V \in C^\infty(\varphi^{-1}TM)$ is called an *infinitesimal Hamiltonian deformation* if $\alpha_V \in \Omega^1(L)$ is exact. A smooth family $\{\varphi_t\}_{|t|<\varepsilon}$ of Lagrangian immersions of L into M with $\varphi = \varphi_0$ is called a *Hamiltonian deformation* of φ if its derivative $V_t = \partial\varphi_t/\partial t$ for each t is an infinitesimal Hamiltonian deformation. Note that if $H^1(L, \mathbf{R}) = \{0\}$, then Lagrangian deformations coincide with Hamiltonian deformations.

Assume that M is a complex n-dimensional Kähler manifold with complex structure tensor field J and Kähler metric g. The Kähler form ω of M is defined by $\omega(X, Y) := g(JX, Y)$. It defines in particular a symplectic structure of M. An immersion $\varphi : L \longrightarrow M$ is a Lagrangian immersion if and only if it satisfies $J_x(\varphi_* T_x L) \subset T_x^\perp L$ for each $x \in L$, and in this case it is also called an n-dimensional *totally real submanifold* of M in the theory of Riemannian submanifolds (cf.[6]). $T_{\varphi(x)} M = \varphi_* T_x L \oplus T_x^\perp L$ for each $x \in L$ along the immersion $\varphi : L \longrightarrow M$ with respect to the metric g. We can identify the normal bundle NL with the bundle $T^\perp L$. Then the complex structure tensor field J induces a bundle isomorphism $NL \longrightarrow \varphi_* TL$ preserving metrics and connections. Since we have $\alpha_V(X) = \omega_{\varphi(x)}(V, \varphi_* X) = g_{\varphi(x)}(JV, \varphi_* X)$ for each $X \in T_x L$, the 1-form α_V corresponds to the vector field JV on L through the linear isomorphism $T_x^* L \cong T_x L \cong \varphi_* T_x L$ with respect to the metric g. Thus we have linear isomorphisms preserving metrics and connections

$$(1.2) \qquad \varpi : T^\perp L \longrightarrow T^* L \text{ and } \varpi : C^\infty(T^\perp L) \ni V \longmapsto \alpha_V \in \Omega^1(L).$$

Definition 1.1. A Lagrangian immersion φ of an n-dimensional compact smooth manifold L into a Kähler manifold M is called *Hamiltonian minimal*, or simply *H-minimal*, if

$$\frac{d}{dt}\mathrm{Vol}(L, {\varphi_t}^* g)\bigg|_{t=0} = 0$$

for all Hamiltonian deformations $\{\varphi_t\}$ of $\varphi = \varphi_0$. In this case we say that (M, L) is an *H-minimal Lagrangian submanifold* immersed in M.

We give a curvature characterization of Hamiltonian minimal for Lagrangian submanifolds. The mean curvature vector field H of a Lagrangian immersion $\varphi : L \longrightarrow M$ into a Kähler manifold is defined by

$$H = \sum_{i=1}^{n} B(e_i, e_i),$$

where B denotes the second fundamental form of the submanifold L in M.

Then φ satisfies the identity

$$d\alpha_H = \varphi^* \rho,$$

where ρ denotes the Ricci form of M. Thus in the case when M is an Einstein-Kähler manifold, we have $d\alpha_H = 0$, that is, α_H is a closed 1-form on L. See [7] and [15].

In [15] it was shown that φ is H-minimal if and only if

$$\delta\alpha_H = 0,$$

where δ denotes the codifferential operator of d with respect to the induced metric on L. Hence a Lagrangian immersion φ into an Einstein-Kähler manifold is H-minimal if and only if α_H is a harmonic 1-form on L.

It is a useful result that if a Lagrangian immersion $\varphi : L \longrightarrow M$ has the parallel mean curvature vector field H with respect to the normal connection, then it is H-minimal.

Definition 1.2. A compact H-minimal Lagrangian submanifold L immersed in a Kähler manifold M is called *Hamiltonian stable* or *H-stable* if

$$\frac{d^2}{dt^2}\mathrm{Vol}(L, {\varphi_t}^* g)\Big|_{t=0} \geq 0$$

for all Hamiltonian deformations $\{\varphi_t\}$ of $\varphi = \varphi_0$.

If a compact Lagrangian submanifold L immersed in a Kähler manifold M is a minimal submanifold in the usual sense, then we call L a *minimal Lagrangian submanifold* of M.

By Hodge's theorem we immediately see the following.

Proposition 1.2. *Let L is a compact H-minimal Lagrangian submanifold in an Einstein-Kähler manifold M. If $H^1(L, \mathbf{R}) = \{0\}$ or more* *generally L has positive Ricci curvature, then L must be a minimal Lagrangian submanifold of M.*

Next we recall the second variational formula for the volume of H-minimal Lagrangian immersion of L into M under Hamiltonian deformations.

Let $\bar{K}$ be the curvature tensor field of M. We denote by $\bar{R}$ the corresponding Ricci operator of $\bar{K}$, that is,

$$\bar{R}(X) = \sum_{i=1}^{2n} \bar{K}(X, e_i)e_i$$

for each vector $X \in TL$. Here $\{e_1, \ldots, e_{2n}\}$ is an orthonormal frame on M.

Define a symmetric covariant tensor field S of degree 3 on L by

$$S(X,Y,Z) := \langle B(X,Y), JZ\rangle \tag{1.3}$$

for $X, Y, Z \in TL$, where B denotes the second fumdamental form of L in M. Oh showed Hamiltonian stability of the Clifford torus.

Theorem 1.1 ([15]). *Let M be a Kähler manifold and $\varphi : L \longrightarrow M$ be an H-minimal Lagrangian immersion of a compact smooth manifold L. If $\{\phi_t\}_{0\leq t\leq 1}$ is a Hamiltonian deformation of $\varphi = \varphi_0$ such that*

$$\left.\frac{\partial}{\partial t}\varphi_t\right|_{t=0} = V$$

is normal to L, then we have

$$\begin{aligned} \left.\frac{d^2}{dt^2}\mathrm{Vol}(L, \varphi_t^* g)\right|_{t=0} = \int_L \big(&\langle \Delta\alpha_V, \alpha_V\rangle - \langle \bar{R}_{\alpha_V}, \alpha_V\rangle \\ &-2\langle \alpha_V \otimes \alpha_V \otimes \alpha_H, S\rangle + \langle \alpha_V, \alpha_H\rangle^2\big)\, dv. \end{aligned} \tag{1.4}$$

Here $\Delta^1 = d\delta + \delta d$ is the Laplacian of L acting on $\Omega^1(L)$ and $\bar{R}_{\alpha_V}$ denotes a tensor field on L defined through ϖ from the restriction $\bar{R}|_{NL}$ of the Ricci operator $\bar{R}$ to NL.

If we denote by $Z^1(L)$ and $B^1(L)$ the vector space of smooth closed 1-forms on L and the vector space of smooth exact 1-forms on L respectively, then we have

$$B^1(L) = d(\Omega^0(L)) \subset Z^1(L) \subset \Omega^1(L).$$

The above second variational formula can be considered as a symmetric bilinear form Π on $B^1(L) = d(\Omega^0(L))$ as follows :

$$
\begin{aligned}
\Pi(\alpha,\beta) := \int_L \big(& \langle \Delta^1 \alpha, \beta\rangle - \langle \bar{R}_\alpha, \beta\rangle \\
& -2\langle \alpha \otimes \beta \otimes \alpha_H, S\rangle + \langle \alpha, \alpha_H\rangle\langle \beta, \alpha_H\rangle \big)\, dv.
\end{aligned} \tag{1.5}
$$

for each $\alpha, \beta \in B^1(L) = d(\Omega^0(L))$. The *null space* for an H-minimal Lagrangian submanifold L is defined as

$$\mathrm{Null}(L) := \left\{ \alpha \in B^1(L) = d(\Omega^0(L)) \mid \Pi(\alpha,\beta) = 0 \text{ for each } \beta \in B^1(L) \right\}.$$

Set $n(L) := \dim \mathrm{Null}(L)$ and we call it the *nullity* of L.

2. Hamilitonian stability of minimal Lagrangian submanifolds in Einstein-Kähler manifolds

Suppose that L is a compact minimal Lagrangian submanifold immersed in an Einstein-Kähler manifold M with Einstein constant κ. Under the correspondence between $C^\infty(NL)$ and $\Omega^1(L) = d(\Omega^0(L)) \oplus \mathrm{Ker}(d^*|\Omega^1(L))$, the Jacobi operator $\mathcal{J}$ as a minimal submanifold corresponds to the linear operator $\tilde{\mathcal{J}} = \Delta^1 - \kappa \mathrm{Id}$, where Id is the identity operator. The second variation of the volume for a compact minimal Lagrangian submanifold under Hamiltonian deformations is described by the restriction of $\tilde{\mathcal{J}}$ to $d(\Omega^0(L))$. The null space of $\mathcal{J}$ on Hamiltonian deformations corresponds to the null space of $\tilde{\mathcal{J}}$ on $d(\Omega^0(L))$, and it is linearly isomorphic to the eigenspace of the Laplacian on $C^\infty(L)$ with eigenvalue κ.

Hence the Hamiltonian stability problem of compact minimal Lagrangian submanifolds in an Einstein-Kähler manifold is reduced to the first positive eigenvalue problem of the Laplacian acting on functions.

Theorem 2.1 ([13]). *Let M be an Einstein-Kähler manifold with Einstein constant κ. A compact minimal Lagrangian submanifold L in M is Hamiltonian stable if and only if $\lambda_1 \geq \kappa$, where λ_1 is the first positive eigenvalue of the Laplacian acting on* $C^\infty(L)$.

Let $\mathcal{K}$ denote the vector space of all Killing vector fields on a compact Einstein-Kähler manifold M with positive Einstein constant κ. Assume that the first eigenvalue of the Laplacian acting on $C^\infty(M)$ is equal to 2κ. We denote by $\mathrm{V}_1(M)$ its eigenspace. By the theorem of Y.Matsushima, we have

$$\mathcal{K} = \{J\mathrm{grad}f \in \mathrm{C}^\infty(TM) \mid f \in \mathrm{V}_1(M)\}.$$

For each $W \in \mathcal{K}$, we have an orthogonal decomposition $W = W^T + W^\perp$, where W^T and $W^\perp$ denote the tangential and the normal components of the restriction of W to the submanifold L in M. Set

$$\mathcal{K}^\perp = \{W^\perp \in \mathrm{C}^\infty(NL) \mid W \in \mathcal{K}\}.$$

Then we have a linear isomorphism

$$\mathcal{K}^\perp \cong \mathcal{K}/\{W \in \mathcal{K} \mid W^\perp = 0\}.$$

If $W = -J\mathrm{grad}f \in \mathcal{K}$ for the first eigenfunction f of the Laplacian acting on $\mathrm{C}^\infty(M)$, then it is easy to check the formula

$$d(f|_L) = \alpha_{W^\perp}$$

on L, which means that each $W^\perp \in \mathcal{K}^\perp$ is an infinitesimal Hamiltonian deformation. Hence, for a suitable constant α, $f|_L + \alpha$ is an eigenfunction of the Laplacian acting on $\mathrm{C}^\infty(L)$ with eigenvalue κ. Set $n_{\mathcal{K}}(L) = \dim\mathcal{K}^\perp$. Since each $W \in \mathcal{K}$ with $W^\perp = 0$ induces a Killing vector field on L, we obtain inequalities

$$n(L) \geq n_{\mathcal{K}}(L) \geq \dim\mathcal{K} - \dim I_0(L),$$

where $I_0(L)$ denotes the identity component of the isometry group of L. Especially when $M = \mathbf{C}P^n$, we have

$$n(L) \geq n_{\mathcal{K}}(L) \geq \dim\mathcal{K} - \dim I_0(L) \geq \frac{n(n+3)}{2}, \tag{2.1}$$

It is important to study the case when M is a Hermitian symmetric space, especially a complex projective space, and more generally a generalized flag manifolds with homogeneous Kähler metrics.

It is an important property for compact minimal Lagrangian submanifolds in a complex projective space $\mathbf{C}P^n$ that if f is the first eigenfunction of the Laplacian on $\mathbf{C}P^n$, then the restriction $f|_L$ of f to L is the eigenfunction of the Laplacian on L with eigenvalue $c(n+1)/2$ (Urbano[29], Ono [18], [19] for generalized flag manifolds including Hermitian symmetric spaces).

Proposition 2.1. *Assume that M is a compact homogenenous Einstein-Kähler manifold with positive Einstein constant κ. Then a compact minimal Lagrangian submanifold L of M is Hamiltonian stable if and* *only if* $\lambda_1 = \kappa$. *Here* λ_1 *is the first eigenvalue of the Laplacian acting on* $C^\infty(L)$. *In particular in the case when* M *is a complex projective space* $\mathbf{C}P^n(c)$ *with constant holomorphic sectional curvature* c, *a compact minimal Lagrangian submanifold* L *of* $\mathbf{C}P^n(c)$ *is Hamiltonian stable if and only if* $\lambda_1 = c(n+1)/2$.

Let $\mathbf{C}P^n(c)$ denote the n-dimenisonal complex projective space with constant holomoprphic sectional curvature c and $\pi : S^{2n+1}(c/4) \longrightarrow \mathbf{C}P^n$ be the Hopf fibration.

Example 2.1. The real projective space $\mathbf{R}P^n$ is a totally real totally geodesic submanifold of the complex projective space $\mathbf{C}P^n$. Thus $\mathbf{R}P^n$ is the simplest example of a compact minimal Lagrangian submanifold embedded in $\mathbf{C}P^n$. Then the inverse image of $\mathbf{R}P^n$ by π is a real quadric $\pi^{-1}(\mathbf{R}P^n) = S^1 \cdot S^n = Q_{2,n+1}(\mathbf{R})$, which is an H-minimal Lagrangian submanifold embedded in $\mathbf{C}^{n+1}$.

Example 2.2. For each $r_1, \cdots, r_{n+1} > 0$ with $r_1^2 + r_2^2 + \cdots + r_{n+1}^2 = 4/c$, let

$$T^{n+1}_{r_1,\ldots,r_{n+1}} = S^1(r_1) \times \cdots \times S^1(r_{n+1}) \subset \mathbf{C}^{n+1}$$

be the $(n+1)$-dimensional standard torus. Then they are H-minimal Lagrangian submanifolds in $\mathbf{C}^{n+1}$. A torus $T^{n+1} = S^1(\frac{1}{\sqrt{c(n+1)}}) \times \cdots \times S^1(\frac{1}{\sqrt{c(n+1)}})$ is a minimal submanifold in $S^{2n+1}(c/4) \subset \mathbf{C}^n$. We set $L := \pi(T^{n+1})$. Then L is a minimal Lagrangian submanifold embedded in $\mathbf{C}P^n$. We call L the *Clifford torus.*

Oh showed Hamiltonian stability of the real projective spaces and the Clifford tori.

Theorem 2.2 ([13]). *The real projective subspace* $\mathbf{R}P^n$ *and the Clifford torus* $T^n = \pi(T^{n+1})$ *are Hamiltonian stable as minimal Lagrangian submanifolds in* $\mathbf{C}P^n$.

Problem 2.1. Classify compact Hamiltonian stable minimal Lagrangian submanifolds in complex projective spaces $\mathbf{C}P^n$.

Urbano and independently Chang have determined Hamitonian stable minimal Lagrangian immersions of compact orientble surfaces of genus 1 into $\mathbf{C}P^2$.

Theorem 2.3 ([29],[4]). *Compact Hamiltonian stable minimal Lagrangia tori in* $\mathbf{C}P^2$ *are only the Clifford tori* T^2.

There is a topological restriction on compact Hamiltonian stable minimal Lagrangian submanifolds in $\mathbf{C}P^n$.

Theorem 2.4 ([1]). *Let L be a compact minimal Lagrangian submanifold immersed in $\mathbf{C}P^n$. If L is Hamiltonian stable, then we have $H_1(L;\mathbf{Z}) \neq \{0\}$ and thus L cannot be simply connected.*

Remark 1. This result does not hold in the case of compact Hermitian symmeric spaces of rank greater than 1 (See Section 5).

Some minimal Lagrangian submanifolds in Hermitian symmetric spaces are related with other interesting submanifolds in differential geometry.

Example 2.3. Palmer showed that the Gauss maps of compact oriented minimal surfaces L in the sphere $S^3(1)$ and isoparemetric hypersurfaces L in the sphere $S^{n+1}(1)$ provide compact minimal Lagrangian submanifolds immersed in the hyperquadrics $Q_n(\mathbf{C}) = \tilde{G}_2(\mathbf{R}^{n+2}) = SO(n+2)/SO(2)\times SO(n)$ and they are not Hamiltonian stable if L is not a sphere ([22], [23]) .

3. Lagrangian submanifolds in $\mathbf{C}^{n+1}$ and $\mathbf{C}P^n$ with parallel second fundamental form

In this section we provide the nice models of compact H-minimal Lagrangian submanifolds in complex Euclidean spaces and complex projective spaces.

The complete classification of totally real submanifolds with parallel fundamental form in complex Euclidean spaces and complex projective spaces was accomplished by H. Naitoh and M. Takeuchi ([8],[9],[10], [11]). The property that the second fundamental form is parallel implies that the mean curvature vector field is parallel. Thus such submanifolds are H-minimal Lagrangian submanifolds in complex Euclidean spaces and complex projective spaces.

Let (U,G) be an Hermitian symmetric pair of compact type with the canonical decomposition $\mathfrak{u} = \mathfrak{g}+\mathfrak{p}$. Set $\dim(U/G) = 2(n+1)$. Let $\langle\ ,\ \rangle$ denote the $\mathrm{Ad}(U)$-invariant inner product of $\mathfrak{u}$ defined by (-1)-times the Killing-Cartan form of the Lie algebra $\mathfrak{u}$. Relative to the complex structure the subspace $\mathfrak{p}$ can be identified with a complex Euclidean space $\mathbf{C}^{n+1}$. We take the decomposition of (U,G) into irreducible Hermitian symmetric pairs of compact type :

$$(U,G) = (U_1,G_1)\oplus\cdots\oplus(U_s,G_s). \tag{3.1}$$

Set $\dim(U_i/G_i) = 2(n_i+1)$ for $i = 1, \cdots, s$. Let $\mathfrak{u}_i = \mathfrak{g}_i + \mathfrak{p}_i$ be the canonical decomposition of (U_i, G_i) for each $i = 1, 2, \cdots, s$. Assume that there is an element $\eta_i \in \mathfrak{p}_i$ satisfying the condition $(\mathrm{ad}\eta_i)^3 + 4(\mathrm{ad}\eta_i) = 0$. Choose positive numbers $c_1 > 0, \cdots, c_s > 0$ with $\sum_{i=1}^{s} 1/c_i = 1/c$. Note that $\langle \eta_i, \eta_i \rangle_{\mathfrak{u}} = 8(n_i + 1)$. Put $a_i = 1/\sqrt{2c_i(n_i+1)}$ for each $i = 1, \cdots, s$. Set $\hat{L}_i = \mathrm{Ad}(G_i)(a_i\eta_i) \subset S^{2n_i+1}(c_i/4) \subset \mathfrak{p}_i$, which is an irreducible symmetric R-space embedded in the complex Euclidean space $\mathfrak{p}_i$.

Set $\eta = a_1\eta_1 + \cdots + a_s\eta_s \in \mathfrak{p}$. Set $\hat{L} = \mathrm{Ad}(G)(\eta) \subset S^{2n+1}(c/4) \subset \mathfrak{p}$, which is a symmetric R-space standardly embedded in the complex Euclidean space $\mathfrak{p}$. Then we have the inclusions

(3.2)
$$\hat{L} = \hat{L}_1 \times \cdots \times \hat{L}_s \subset S^{2n_1+1}(c_1/4) \times \cdots \times S^{2n_s+1}(c_s/4) \subset S^{2n+1}(c/4).$$

and $\hat{L}$ is an $(n+1)$-dimensional totally real submanifold with parallel second fundamental form in the complex Euclidean space $\mathfrak{p} \cong \mathbf{C}^{n+1}$. Thus we see

Proposition 3.1. *The submanifold $\hat{L}$ is a compact H-minimal Lagrangian submanifold embedded in the complex Euclidean space $\mathfrak{p} \cong \mathbf{C}^{n+1}$, which is never minimal.*

In the case of $s = 1$, the space $\hat{L}$ is an *irreducible symmetric R-spaces of $U(r)$ type* (see [26]). The following is a complete list of all irreducible symmetric R-spaces of type $U(r)$:

$$Q_{2,n+1}(\mathbf{R}),\ U(p),\ U(p)/O(p),\ U(2p)/Sp(p),\ T \cdot E_6/F_4.$$

Here $Q_{2,n+1}(\mathbf{R})$ denotes the real quadric defined by

$$Q_{2,n+1}(\mathbf{R}) = \{[x] \in \mathbf{R}P^{n+2} \mid x_1^2 + x_2^2 - x_3^2 - \cdots - x_{n+3}^2 = 0\}.$$

The space $Q_{2,n+1}(\mathbf{R})$ is isomorphic to $(SO(2) \times SO(n+1))/S'(O(1) \times O(n))$, where $S'(O(1) \times O(n))$ is a compact subgroup consisting of matrices of the form

$$\begin{pmatrix} \begin{pmatrix} \varepsilon & \\ & A \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} B & \\ & \varepsilon \end{pmatrix} \end{pmatrix} \in SO(n+3),$$

with $\varepsilon = \pm 1$, $A \in O(1)$ and $B \in O(n)$.

Let $\pi : S^{2n+1}(c/4) \longrightarrow \mathbf{C}P^n(c)$ be the Hopf fibration and put $L = \pi(\hat{L})$. Note that $\pi^{-1}(L) = \hat{L}$. Then the following properties of L are known :

Proposition 3.2 ([11]). (1) *L is an n-dimensional compact totally real submanifold embedded in $\mathbf{C}P^n(c)$ with parallel second fundamental form, and thus L is a symmetric space.*

(2) *L is a minimal submanifold in $\mathbf{C}P^n(c)$ if and only if $c_i(n_i+1) = c(n+1)$ for each $i = 1, \cdots, s$.*

(3) *The dimension of the Euclidean factor of L is equal to $s-1$.*

(4) *L is flat if and only if $s = n+1$. In this case, L is the Clifford torus in $\mathbf{C}P^n$.*

(5) *L has no Euclidean factor if and only if $s = 1$. In this case L is an irreducible symmetric space and a minimal submanifold in $\mathbf{C}P^n$.*

In particular, we see

Proposition 3.3. *Such an L is a compact H-minimal Lagrangian submanifold embedded in $\mathbf{C}P^n(c)$.*

The following conditions are equivalent:

(a) L has no Euclidean factor.
(b) (U, G) is irreducible, i.e. $s = 1$.
(c) L has positive Ricci curvature.
(d) L is an Einstein manifold with positive Einstein constant.

In the case when L has no Euclidean factor, L is isometric to one of the following symmetric spaces:

$$\mathbf{R}P^n(c/4),\ SU(p)/\mathbf{Z}_p,\ SU(p)/SO(p)\mathbf{Z}_p,\ SU(2p)/Sp(p)\mathbf{Z}_{2p},\ E_6/F_4\mathbf{Z}_3.$$

4. Hamiltonian stability of minimal Lagrangian submanifolds with parallel second fundamental form in complex projective spaces

Now we saw the construction of compact H-minimal Lagrangian submanifolds embedded in $\mathbf{C}P^n(c)$ with parallel second fundamental form. We can determine Hamiltonian stability in the case $s = 1$.

Theorem 4.1 ([1]). *Let L be an n-dimensional compact totally real minimal submanifold with parallel second fundamental form embedded in $\mathbf{C}P^n$ in the following list :*

(1) $SU(p)/\mathbf{Z}_p$, $n = p^2 - 1$.
(2) $SU(p)/SO(p)\mathbf{Z}_p$, $n = \frac{(p-1)(p+2)}{2}$.
(3) $SU(2p)/Sp(p)\mathbf{Z}_{2p}$, $n = (p-1)(2p+1)$.
(4) $E_6/F_4\mathbf{Z}_3$, $n = 26$.

Here $p \geq 2$ is an integer. Then L is Hamiltonian stable as a compact minimal Lagrangian submanifold in $\mathbf{C}P^n$ and moreover the nulll space of L is eactly the span of normal projections of Killing vector fields on $\mathbf{C}P^n$.

Combining the results of Theorems 3.2, 2.2 and 4.1, we conclude the following.

Theorem 4.2 ([1]). *All compact n-dimensional totally real minimal submanifolds embedded in $\mathbf{C}P^n$ with parallel second fundamental form and positive Ricci curvature are Hamiltonian stable as compact minimal Lagrangian submanifolds.*

In order ro prove Theorem 4.1, we need to determine the eigenvalues of the Laplacian on functions for such compact symmetric spaces. Here we describe the method to calculate the eigenvalues of the Laplacians on functions by virtue of the theory of spherical functions on compact symmmetric spaces ([27]).

Let G/K be a compact symmetric space with the symmetric pair (G, K). Here G is a compact connected Lie group. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be its canonical decomposition and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{m}$. We fix an AdG-invariant inner product $(\ , \ )$ of $\mathfrak{g}$. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$ and then we have $\mathfrak{t} = \mathfrak{b} + \mathfrak{a}$, where $\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}$. We fix a σ-linear order $<$ on $\mathfrak{t}$. The maximal torus T of G is generated by $\mathfrak{t}$. For each $\alpha \in \mathfrak{t}$, we put

$$(4.1) \quad \tilde{\mathfrak{g}}_\alpha = \{X \in \mathfrak{g}^{\mathbf{C}} \mid (\mathrm{ad}H)X = 2\pi\sqrt{-1}(\alpha, H)X \text{ for each } H \in \mathfrak{t}\}.$$

An element $\alpha \in \mathfrak{t}$ is called a *root* of $\mathfrak{g}$ with respect to $\mathfrak{t}$ if $\tilde{\mathfrak{g}}_\alpha$ is non zero. We denote by $\Sigma(G)$ and $\Sigma^+(G)$ the set of all roots and all positive roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$, respectively. We have the root decomposition

$$\mathfrak{g}^{\mathbf{C}} = \mathfrak{t}^{\mathbf{C}} + \sum_{\alpha \in \Sigma(G)} \tilde{\mathfrak{g}}_\alpha.$$

$$\begin{aligned}
&\Gamma(G) := \{H \in \mathfrak{t} \mid \exp H = e\},\\
(4.2)\qquad &Z(G) := \{\lambda \in \mathfrak{t} \mid (\lambda, H) \in \mathbf{Z} \text{ for each } H \in \Gamma(G)\},\\
&D(G) := \{\lambda \in Z(G) \mid (\lambda, \alpha) \geq 0 \text{ for each } \alpha \in \Sigma^+(G)\}.
\end{aligned}$$

Let $\mathcal{D}(G)$ be the complete set of inequivalent irreducible unitary representation of G. Then for each $(V, \rho) \in \mathcal{D}(G)$ the highest weight λ_ρ of (V, ρ) belongs to $D(G)$ and the mapping $\mathcal{D}(G) \longrightarrow D(G)$ is bijective.

Let A be the torus of G generated by $\mathfrak{a}$ and $\hat{A} = Ao$ be a maximal torus of G/K, where o denotes the origin eK of G/K. For each $\gamma \in \mathfrak{a}$, we put

$$\mathfrak{g}_\gamma = \{X \in \mathfrak{g}^{\mathbf{C}} \mid (\mathrm{ad}H)X = 2\pi\sqrt{-1}(\alpha, H)X \text{ for each } H \in \mathfrak{a}\}.$$

An element $\gamma \in \mathfrak{a}$ is called a *root* of $\mathfrak{g}$ with respect to $\mathfrak{a}$ if $\mathfrak{g}^{\mathbf{C}}_\gamma$ is nonzero. We denote by $\Sigma(G, K)$ and $\Sigma^+(G, K)$ the set of all roots and all positive roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$, respectively. We have the decomposition

$$\mathfrak{g}^{\mathbf{C}}_\gamma = \mathfrak{g}^{\mathbf{C}}_0 + \sum_{\gamma \in \Sigma(G,K)} \mathfrak{g}^{\mathbf{C}}_\gamma.$$

Put

$$\begin{aligned}
&\Gamma(G, K) := \{H \in \mathfrak{a} \mid (\exp H)o = o\},\\
&Z(G, K) := \{\lambda \in \mathfrak{a} \mid (\lambda, H) \in \mathbf{Z} \text{ for each } H \in \Gamma(G, K)\},\\
&D(G, K) := \{\lambda \in Z(G, K) \mid (\lambda, \gamma) \geq 0 \text{ for each } \gamma \in \Sigma^+(G, K)\}.
\end{aligned}$$

Then we have $Z(G, K) \subset Z(G)$ and $D(G, K) \subset D(G)$. Let $\mathcal{D}(G, K)$ be the complete set of inequivalent unitary class one representation of pair (G, K). Then for each $(V_\rho, \rho) \in \mathcal{D}(G, K)$ the subspace $(V_\rho)_K = \{v \in V_\rho \mid \rho(k)v = v \text{ for each } k \in K\}$ is of complex dimension 1 and the bijection induces the bijection $\mathcal{D}(G, K) \longrightarrow D(G, K)$.

Let g be the G-invariant Riemannian metric on G/K induced by $(\ ,\ )$ and Δ be the Laplace-Beltrami operator of $(G/K, g)$ acting on functions. Then the complete set of eigenvlues of Δ is given by

$$(4.3)\qquad \{-a_\rho = 4\pi^2(\lambda_\rho + 2\delta(G), \lambda_\rho) \mid \rho \in \mathcal{D}(G, K)\}.$$

Here we set

$$\delta(G) = \frac{1}{2} \sum_{\alpha \in \Sigma^+(G)} \alpha.$$

The multiplicity of the k-th eigenvalue λ_k of Δ is given by $\sum_\rho d_\rho$, where the summation runs over all $\rho \in \mathcal{D}(G, K)$ such that $\lambda_k = -a_k$, and

$$d_\rho = \dim(V_\rho, \rho) = \prod_{\alpha \in \sum^+(G)} \frac{(\alpha, \lambda_\rho + \delta(G))}{(\alpha, \delta(G))}.$$

Here we give a table on the scalar curvature s of L, the first eigenvalue λ_1 of L and the first eigenvalue $\tilde{\lambda}_1$ of the universal covering space $\tilde{L}$.

$\tilde{L}$	L	n	s	$\tilde{\lambda}_1$	λ_1
S^n	$\mathbf{R}P^n$	n	$\frac{n(n-1)c}{4}$	$\frac{nc}{4}$	$\frac{(n+1)c}{2}$
$SU(p)$	$\frac{SU(p)}{\mathbf{Z}_p}$	p^2-1	$\frac{p^2}{4}\frac{p^2-1}{2}c$	$\frac{p^2-1}{2}c$	$\frac{p^2}{2}c$
$\frac{SU(p)}{SO(p)}$	$\frac{SU(p)}{SO(p)\cdot\mathbf{Z}_p}$	$\frac{(p-1)(p+2)}{2}$	$\frac{p^2(p-1)(p+2)c}{32}$	$\frac{(p-1)(p+2)c}{4}$	$\frac{p(p+1)}{4}c$
$\frac{SU(2p)}{Sp(p)}$	$\frac{SU(2p)}{Sp(p)\cdot\mathbf{Z}_{2p}}$	$(p-1)(2p+1)$	$\frac{p^2(p-1)(2p+1)c}{4}$	$\frac{(p-1)(2p+1)c}{4}$	$\frac{p(2p-1)}{2}c$
$\frac{E_6}{F_4}$	$\frac{E_6}{F_4\cdot\mathbf{Z}_3}$	26	$9\cdot 13c$	$\frac{13}{2}c$	$\frac{27}{2}c$

Now we shall remark on some related open problems.

Problem 4.1. Is it true that all compact n-dimensional totally real submanifolds embedded in $\mathbf{C}P^n$ with parallel second fundamental form are Hamiltonian stable as H-minimal Lagrangian submanifolds ?

Problem 4.2. Is it true that compact H-minimal Lagrangian submanifolds in $\mathbf{C}P^n$ which are Hamiltonian stable have parallel second fundamental form ?

Problem 4.3. Is such a compact Hamiltonian stabe H-minimal Lagrangian submanifold L in $\mathbf{C}P^n$ *globally Hamiltonian stable* or not, that is, volume minimizing with respect to every Hamiltonian deformation of L ?

Remark 2. It is known that the real projective subspace $\mathbf{R}P^n$ of $\mathbf{C}P^n$ is globally Hamiltonian stable ([13],[2]).

5. Hamiltonian stability of symmetric R-spaces canonically embedded in compact Hermitian symmetric spaces

We should remark that there exist compact minimal Lagrangian submanifolds embedded in compact Hermitian symmetric spaces of rank greater than 1 which is NOT Hamiltonian stable.

Let M is a Kähler manifold and let τ be an involutive anti-holomorphic isometry of M. Let $L = \mathrm{Fix}(\tau)$ be the subset of all fixed points of τ. This subset is called a *real form* of M. Then it is known that it is a totally real and totally geodesic submanifold in M with dimension equal to a half of $\dim(M)$, and hence a totally geodesic Lagrangian submanifold in M.

Assume that M is a compact Hermitian symmetric space. Let τ be an involutive anti-holomorphic isometry. It is also a symmetric R-space canonically embedded in a compact Hermitian symmetric space ([26]). Moreover in [26] he showed that a symmetric R-space L canonically embedded in a compact Hermitian symmetric space is stable as a minimal submanifold if and only if L is simply connected.

The theory of symmetric R-spaces is well investigated and we refer to [26] for a complete list of irreducible symmetric R-spaces. By using the results of M. Takeuchi in [26], Y. G. Oh [13] showed that an Einstein, symmetric R-space canonically embedded in a compact Hermitian symmetric space is always Hamiltonian stable. Moreover M. Takeuchi classified all irreducible symmetric R-spaces into five classes : Hermitian and four types corresponding to each of the groups $Sp(r)$, $U(r)$, $SO(2r)$ and $SO(2r+1)$. He also showed that symmetric R-spaces of Hermitian type are always Einstein and hence Hamiltonian stable. Here we give a complete list of Hamiltonian stability of all irreducible symmetric R-spaces of non-Hermitian type which are canonically embedded in Hermitian symmetric spaces.

M	L	Einstein	λ_1	H-stable	stable
$G_{p,q}(\mathbf{C}), p \leq q$	$G_{p,q}(\mathbf{R})$	Yes	$\frac{1}{2}$	Yes	No
$G_{2p,2q}(\mathbf{C}), p \leq q$	$G_{p,q}(\mathbf{H})$	Yes	$\frac{1}{2}$	Yes	Yes
$G_{m,m}(\mathbf{C})$	$U(m)$	No	$\frac{1}{2}$	Yes	No
$SO(2m)/U(m)$	$SO(m), m \geq 5$	Yes	$\frac{1}{2}$	Yes	No
$SO(4m)/U(2m), m \geq 3$	$U(2m)/Sp(m)$	No	$\frac{m}{4m-2}$	No	No
$Sp(2m)/U(2m)$	$Sp(m), m \geq 2$	Yes	$\frac{1}{2}$	Yes	Yes
$Sp(m)/U(m)$	$U(m)/O(m)$	No	$\frac{1}{2}$	Yes	No
$Q_{p+q-2}(\mathbf{C}), 3 \leq q-p$	$Q_{p,q}(\mathbf{R})$	No	$\frac{p}{p+q-2}$	No	No
$Q_{2p-2}(\mathbf{C})$	$Q_{p,p}(\mathbf{R})$	Yes	$\frac{1}{2}$	Yes	No
$Q_q(\mathbf{C})$	$Q_{2,q}(\mathbf{R})$	No	$\frac{1}{2}$	Yes	No
$Q_{q-2}(\mathbf{C})$	$Q_{1,q}(\mathbf{R})$	Yes	$\frac{1}{2}$	Yes	No
$E_6/TSpin(10)$	$P_2(\mathbf{K})$	Yes	$\frac{1}{2}$	Yes	Yes
$E_6/TSpin(10)$	$G_{2,2}(\mathbf{H})/\mathbf{Z}_2$	Yes	$\frac{1}{2}$	Yes	No
E_7/TE_6	$SU(8)/Sp(4)\mathbf{Z}_2$	Yes	$\frac{1}{2}$	Yes	No
$E_7/T \cdot E_6$	$T \cdot E_6/F_4$	No	$\frac{1}{6}$	No	No

Here $G_{p,q}(\mathbf{F})$: Grassmanian manifold of all p-dimensional subspaces of $\mathbf{F}^{p+q}$, for each $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$,
$\mathbf{P}_2(K)$: Cayley projective plane,
$Q_n(\mathbf{C})$: complex quadric of dimension n.
Note that the heading of the third column indicates whether L is Einstein or not.

Problem 5.1. In the above list,

$$\begin{aligned}(M, L) =&(SO(4m)/U(2m), U(2m)/Sp(m)) \ (m \geq 3),\\ &(Q_{p+q-2}(\mathbf{C}), Q_{p,q}(\mathbf{R})) \ (3 \leq q-p, p \geq 2),\\ &(E_7/T \cdot E_6, T \cdot E_6/F_4)\end{aligned}$$

are compact minimal Lagrangian submanifolds embedded in compact Hermitian symmetric spaces which are NOT Hamiltonian stable. Can we find their geometric reasons ?

Problem 5.2. Which Hamitonian stable symmetric R-spaces in the above classification are globally Hamiltonian stable ?

Problem 5.3. More generally let M be a Kähler C-space, that is, a generalized flag manifold equipped with a homogeneous Kähler metric. It is well-known that M is obtained as an adjoint orbit of a compact

Lie group and Kähler C-spaces exhaust simply connected compact homogeneous Kähler manifold. The R-spaces canonically embedded in Kähler C-spaces are real forms of M ([25]), and thus totally geodesic Lagrangian submanifolds of M. Study Hamiltonian stability of R-spaces canonically embedded in Kähler C-spaces M.

6. Hamiltonian minimality and Hamilitonian stability of Lagrangian submanifolds in complex Euclidean spaces

Finally we shall discuss Hamiltonian stability of H-minimal Lagrangian submanifolds in complex Euclidean spaces.

Let L be a Lagrangian submanifold immersed in $\mathbf{C}^{n+1}$ and $\varphi : L \longrightarrow \mathbf{C}^{n+1}$ denote its Lagrangian immersion. In this section, we discuss Lagrangian submanifolds L in the complex Euclidean space $\mathbf{C}^{n+1}$ which are minimally immersed in the hypersphere $S^{2n+1}(c/4)$ of constant positive sectional curvature $c/4$. Then L is an H-minimal Lagrangian submanifold in $\mathbf{C}^{n+1}$. In fact, since the mean curvature vector field of L in $\mathbf{C}^{n+1}$ is given by

$$H_x = -\frac{c(n+1)}{4}\varphi(x)$$

for each point $x \in L$, by the Weingarten formula we see that L has parallel mean curvature vector field in $\mathbf{C}^{n+1}$ with respect to the normal connection.

Lemma 6.1. *Let B denote the second fundamental form of the submanifold L in $\mathbf{C}^{n+1}$. Then L satisfies*

$$\langle B(X,Y), H\rangle = \frac{c(n+1)}{4}\langle X, Y\rangle, \tag{6.1}$$

for all tangent vectors X, Y of L.

Proposition 6.1. *Let L be a Lagrangian submanifold in $\mathbf{C}^{n+1}$ which is minimally immersed in $S^{2n+1}(c/4)$. Then*

$$\langle \alpha_V \otimes \alpha_V \otimes \alpha_H, S\rangle = \frac{c(n+1)}{4}\langle \alpha_V, \alpha_V\rangle \tag{6.2}$$

for each normal vector field V on L.

Proposition 6.2. *Let L be a Lagrangian submanifold in $\mathbf{C}^{n+1}$ which is minimally immersed in $S^{2n+1}(\frac{c}{4})$. Then*

$$\langle \alpha_V, \alpha_H\rangle^2 = \frac{c}{4}(n+1)^2\alpha_V^2(E_1)$$

for every normal vector field V on L, where E_1 denotes a parallel vector field on L with unit length defined by $E_1 = J(H/|H|)$.

Since $\mathbf{C}^{n+1}$ is flat, we have $\bar{R}_{\alpha_V} \equiv 0$. Using this fact and Propositions 6.1 and 6.2, we rewrite the second variation formula as follows :

Proposition 6.3. *Let L be a Lagrangian submanifold in $\mathbf{C}^{n+1}$ which is minimally immersed in $S^{2n+1}(\frac{c}{4})$. Then the second variational formula for volume becomes*

(6.3)
$$\Pi(\alpha_V, \alpha_V) = \int_L \left(\langle \Delta\alpha_V - \frac{c}{2}(n+1)\alpha_V, \alpha_V \rangle + \frac{c}{4}(n+1)^2\alpha_V^2(e_1) \right) dv.$$

Let L be one of the following irreducible symmetric R-spaces of type $U(r)$ standardly embedded in $\mathbf{C}^{n+1}$:

(6.4) $Q_{2,n+1}(\mathbf{R})$, $U(p)$, $U(p)/O(p)$, $U(2p)/Sp(p)$ and $(T^1 \cdot E_6)/F_4$.

Then L is a minimal submanifold in $S^{2n+1}(\frac{c}{4})$ ([28]). Note that their images under the Hopf maps $\pi : S^{2n+1}(c/4) \longrightarrow \mathbf{C}P^n(c)$ are

$$\mathbf{R}P^n,\ SU(p)/\mathbf{Z}_p,\ SU(p)/SO(p)\mathbf{Z}_p,\ SU(2p)/Sp(p)\mathbf{Z}_{2p} \text{ and } E_6/F_4\mathbf{Z}_3,$$

respectively. The irreducible symmetric R-space L of type $U(r)$ is a compact H-minimal Lagrangian submanifold in $\mathbf{C}^{n+1}$ with parallel second fundamental form, which is a minimal submanifold in the hypersphere S^{2n+1} ([28]).

The $(n+1)$-dimensional standard torus

$$T^{n+1}_{r_1,\dots,r_{n+1}} = S^1(r_1) \times \cdots \times S^1(r_{n+1}) \subset \mathbf{C}^{n+1}$$

is the simplest example of compact H-minimal Lagrangian submanifold embedded in $\mathbf{C}^{n+1}$. Oh showed the Hamiltonian stability of the standard torus.

Theorem 6.1 ([15]). *For each $r_1, \cdots, r_{n+1} > 0$, the $(n+1)$-dimensional standard torus*

$$T^{n+1}_{r_1,\dots,r_{n+1}} = S^1(r_1) \times \cdots \times S^1(r_{n+1}) \subset \mathbf{C}^{n+1}$$

is Hamiltonian stable as an H-minimal Lagrangian submanifold embedded in $\mathbf{C}^{n+1}$.

The $(n+1)$-dimensional standard torus $T^{n+1}_{r_1,\dots,r_{n+1}}$ in $\mathbf{C}^{n+1}$ is also the simplest model of totally real submanifolds in complex Euclidean spaces with parallel second fundamental form. In this section we shall duscuss Hamiltonian stability of irreducible symmetric R-space of type $U(r)$ canonically embedded in the complex Euclidean spaces as H-minimal Lagrangian submanifolds.

Let $L = G/K$ be an irreducible symmetric R-space of type $U(r)$ and assume that L is embedded in the complex Euclidean space as a H-minimal Lagrangian submanifold by the standard embedding φ as in Section 3.

By the spherical function theory on compact symmetric spaces, we have

$$B^1(L)^{\mathbf{C}} = d(C^\infty(L)^{\mathbf{C}}) \cong C^\infty(L)^{\mathbf{C}}/\mathbf{C} = \bigoplus_{\Lambda \in D(G,K)\setminus\{0\}} V_\Lambda,$$

where $(V_\Lambda, \rho_\Lambda)$ denotes an irreducible unitary represention space with highest weight Λ. The vector space V_Λ can be regarded as a subspace of $C^\infty(L)$ as follows. Set

$$(V_\Lambda)_K := \{v \in V_\Lambda \mid \rho_\Lambda(k)v = v \text{ for all } k \in K\}.$$

It is known that $(V_\Lambda)_K \neq \{0\}$ if and only if $\Lambda \in D(G,K)$, and $\dim(V_\Lambda)_K = 1$ ([27]). Choose a nonzero element $v_\Lambda \in (V_\Lambda)_K$. For each $v \in V_\Lambda$, we define a function f_v on G/K as

$$f_v(aK) := \langle \rho_\Lambda(a)v_\Lambda, v\rangle_{V_\Lambda}$$

for each $aK \in G/K$. Here $\langle\ ,\ \rangle_{V_\Lambda}$ denotes an ρ_Λ-invariant Hermitian inner product of V_Λ.

We extend the symmetric bilinear form Π on $B^1(L) = d(\Omega^0(L))$ to an Hermitian form on $B^1(L)^{\mathbf{C}} = d(\Omega^0(L))^{\mathbf{C}}$ in a natural way :

$$\begin{aligned}\Pi(\alpha,\overline{\beta}) := \int_L (&\langle \Delta\alpha, \overline{\beta}\rangle - \langle \bar{R}_\alpha, \overline{\beta}\rangle \\ &- 2\langle \alpha\otimes\overline{\beta}\otimes\alpha_H, S\rangle + \langle\alpha,\alpha_H\rangle\langle\overline{\beta},\alpha_H\rangle)dv.\end{aligned} \tag{6.5}$$

for each $\alpha, \beta \in B^1(L)^{\mathbf{C}} = d(\Omega^0(L))^{\mathbf{C}}$. Note that if $\Lambda, \Lambda' \in D(G,K)$ with $\Lambda \neq \Lambda'$, then we have $\Pi(df_v, \overline{df_{v'}}) = 0$ for each $v \in V_\Lambda$ and each $v' \in V_{\Lambda'}$.

Let $\mathfrak{c}(\mathfrak{g})$ be the center of G and choose $E_1 \in \mathfrak{c}(\mathfrak{g})$ with $|E_1| = 1$. We denote also by E_1 the vector field on G/K generated by the element

Theorem 6.2 ([3]). *Let $L = G/K$ be an irreducible symmetric R-space of type $U(r)$ with $\dim L = n+1$. Then the Hemitian form Π on $B^1(L)^{\mathbf{C}} = d(\Omega^0(L))^{\mathbf{C}}$ is given as follows: For each $\Lambda \in D(G,K)$ and each $v \in V_\Lambda$,*

$$(6.6)\quad \begin{aligned}&\Pi(df_v, \overline{df_v})\\ &= \left(a_\Lambda^2 + \frac{c}{2}(n+1)a_\Lambda + \frac{c}{4}(n+1)^2|\langle E_1, \Lambda\rangle|^2\right) \frac{|v_\Lambda|^2_{V_\Lambda}\mathrm{Vol}(L)}{\dim_{\mathbf{C}} V_\Lambda}|v|^2_{V_\Lambda},\end{aligned}$$

where a_Λ is an eigenvalue of the Casimir operator of ρ_Λ with respect to the metric induced from $\mathbf{C}^{n+1}$.

Corollary 6.1. *The Lagrangian submanifold $L = G/K$ is Hamiltonian stable if and only if*

$$II(\Lambda) := a_\Lambda^2 + \frac{c}{2}(n+1)a_\Lambda + \frac{c}{4}(n+1)^2|\langle E_1, \Lambda\rangle|^2 \geq 0$$

for all $\Lambda \in D(G,K)$.

By using the above formula we can show case by case that $II(\Lambda) \geq 0$ for each $\Lambda \in D(G,K)$ and each irreducible symmetric R-space G/K of $U(r)$ type ([3]). Thus we obtain

Theorem 6.3 ([3]). *Every irreducible symmetric R-space of $U(r)$ type :*

$$Q_{2,n+1}(\mathbf{R}), U(p), U(p)/O(p), U(2p)/Sp(p), T\cdot E_6/F_4$$

is Hamiltonian stable as an H-minimal Lagrangian submanifold in the complex Euclidean space.

Problem 6.1. Are these compact Hamiltonian stabe H-minimal Lagrangian submanifolds L in complex Euclidean spaces *globally Hamiltonian stable* or not ?

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