数理解析研究所講究録 1293 巻 2002 年 1-14

MARTIN BOUNDARY FOR UNION OF CONVEX SETS

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1. Introduction

We study Martin boundary points of a proper subdomain in $\mathbb{R}^n$, where $n \geq 2$, that can be represented as the union of open convex sets. Especially, we give a certain sufficient condition for a boundary point to have exactly one (minimal) Martin boundary point.

In the 1970's, Ancona considered a bounded domain Ω that can be represented as the union of open balls with the same radius. He assumed that

(A) if two balls tangent to each other at a boundary point ξ of Ω, then there is a truncated circular cone, with vertex at ξ and axis in the hyperplane tangent to such balls at ξ, included in Ω.

Under these assumption he showed that each boundary point has exactly one Martin boundary point and it is minimal ([4]).

However, this result is not applicable to domains with wedges. So we consider open convex sets rather than open balls with the same radius. Obviously, we need a different sufficient condition for a boundary point to have exactly one (minimal) Martin boundary point.

We write $\overline{E}$ and ∂E for the closure and the boundary of a set E, respectively. Let $x, y \in \mathbb{R}^n$ $(x \neq y)$ and $r > 0$. We denote by $B(x,r)$ and $S(x,r)$ the open ball and the sphere of center x and radius r, respectively. For $\theta > 0$ let $\Gamma_\theta(x,y)$ stand for the open circular cone of vertex x, axis $\overline{xy}$ and aperture θ, i.e.,

$$\Gamma_\theta(x,y) := \{z \in \mathbb{R}^n : \angle zxy < \theta\}.$$

Let $\rho_0 > 0$ and $A_0 \geq 1$. We consider a proper subdomain D in $\mathbb{R}^n$ such that

(I) D is the union of a family of open convex sets $\{C_\lambda\}_{\lambda \in \Lambda}$ such that $B(z_\lambda, \rho_0) \subset C_\lambda \subset B(z_\lambda . A_0 \rho_0)$.

(II) Let $\xi \in \partial D$. Then there are positive constants $\theta_1 \leq \sin^{-1}(1/A_0)$ and $\rho_1 \leq \rho_0 \cos \theta_1$ such that the union of truncated circular cones $\Gamma_{\theta_1}(\xi, y) \cap B(\xi, 2\rho_1)$ included in D is connected, i.e.,

$$\bigcup_{\substack{y \in D \\ \Gamma_{\theta_1}(\xi,y) \cap B(\xi, 2\rho_1) \subset D}} \Gamma_{\theta_1}(\xi, y) \cap B(\xi, 2\rho_1) \quad \text{is connected.}$$

Remark 1. We note that the union in the condition (II) is non-empty (Lemma 3.2). The condition (II) is the same as Ancona's when $A_0 = 1$ (Ancona's setting).

Throughout this note, we simply write a domain instead of a proper subdomain in $\mathbb{R}^n$. By a Greenian domain we mean a domain with the Green function.

The main result is as follows.

Theorem. *Let D be a Greenian domain satisfying (I). If $\xi \in \partial D$ satisfies (II), then there is exactly one Martin boundary point at ξ and it is minimal.*

Remark 2. We investigated in [3] that the number of minimal Martin boundary points at each boundary point of a John domain is estimated by the John constant. A bounded domain satisfying (I) is a John domain. As seen in Theorem, we obtain a better result under the condition (II).

Corollary. *Suppose that D is a bounded domain satisfying (I) and that each $\xi \in \partial D$ satisfies (II). Then the Martin boundary of D is homeomorphic to its Euclidean boundary. Moreover, each Martin boundary point is minimal.*

The following proposition implies the sharpness of bounds $\theta_1 \le \sin^{-1}(1/A_0)$ and $\rho_1 \le \rho_0 \cos\theta_1$ in the condition (II).

Proposition 1.1. *Let $A_0 > 1$. Suppose either*
(i) $\theta_1 > \sin^{-1}(1/A_0)$, or
(ii) $0 < \theta_1 \le \sin^{-1}(1/A_0)$ and $\rho_1 > \rho_0 \cos\theta_1$.
Then there is a domain D satisfying (I) and $\xi \in \partial D$ satisfies (II), and yet ξ has multiple minimal Martin boundary points.

This note is organized as follows. In Section 2, we shall show a general fact for the support of the measure associated with a kernel function in the Martin representation. In Section 3, we shall show geometrical properties. In Section 4, we shall prove a Carleson type estimate after showing the upper bound of a non-negative subharmonic function on a bounded domain and showing the integrability of the negative power of the distance function. In Section 5, we shall show a (uniform) boundary Harnack principle. In Section 6, we shall prove Theorem and Corollary. In Section 7, we shall give examples for Proposition. In Section 8, we shall give a domain satisfying (I) and (II) at each boundary point but not a uniform domain.

By the symbol A we denote an absolute positive constant whose value is unimportant and may change from line to line. If two positive functions f and g satisfy $A^{-1}f \le g \le Af$ for some constant $A \ge 1$, then we write $f \approx g$ and call A the constant of comparison.

2. General fact

In this section, we show a general fact for the support of the measure of a corresponding to a kernel function in the Martin representation. Let $\xi \in \partial D$ and $x_0 \in D$ be fixed. Let G denote the Green function for D. The Martin kernel (or the Martin boundary point) at ξ, written $K(\cdot,\xi)$, is given as a limit function of the Martin kernels $K(\cdot,y_j) := G(\cdot,y_j)/G(x_0,y_j)$ for some sequence $\{y_j\}$ in D converging to ξ. We say that a property holds quasi-everywhere if it holds except a polar set. A function h on D is called a kernel function at ξ if h is positive and harmonic on D, satisfies $h(x_0) = 1$, vanishes quasi-everywhere on ∂D and is bounded on $D \setminus B(\xi, r)$ for each $r > 0$. We denote by Δ the Martin boundary of D, and by Δ_1 the subset of all minimal elements in Δ. We also write $\Delta(\xi)$ for the set of all Martin boundary points at ξ, and let $\Delta_1(\xi) := \Delta(\xi) \cap \Delta_1$. Let $E \subset D$ and $y \in \Delta_1$. We say that E is minimally thin at y if $\widehat{R}^E_{K(\cdot,y)} \neq K(\cdot,y)$. Here $\widehat{R}^E_u$ denotes the regularized reduced function of a non-negative superharmonic function u relative to E in D.

The following lemma will be used in the proof of Theorem (Section 6).

Lemma 2.1. *Let D be a Greenian domain and $\xi \in \partial D$. If h is a kernel function at ξ, then the support of the measure associated with it in the Martin representation is $\Delta_1(\xi)$. In particular, $\Delta_1(\xi)$ is non-empty.*

Proof. By the Martin representation, there is a unique measure μ on Δ_1 such that

$$h(x) = \int_{\Delta_1} K(x,y)\, d\mu(y) \quad \text{for } x \in D.$$

Let E be a compact subset of $\Delta \setminus \Delta(\xi)$ and let $\{E_j\}$ be a decreasing sequence of compact neighborhoods of E in the Martin topology such that $(E_1 \cap D) \cap B(\xi, r_1) = \emptyset$ for some $r_1 > 0$ and $\bigcap_j E_j = E$. Then we have ([5, Corollary 9.1.4])

$$\widehat{R}^{E_j \cap D}_h(x) = \int_{\Delta_1} \widehat{R}^{E_j \cap D}_{K(\cdot,y)}(x)\, d\mu(y) \quad \text{for } x \in D.$$

Noting that $\lim_{j\to\infty} \widehat{R}^{E_j \cap D}_h$ is bounded and harmonic on D and vanishes quasi-everywhere on ∂D since h is the kernel function at ξ, we have

$$0 = \lim_{j\to\infty} \widehat{R}^{E_j \cap D}_h(x_0) = \int_{\Delta_1} \lim_{j\to\infty} \widehat{R}^{E_j \cap D}_{K(\cdot,y)}(x_0)\, d\mu(y) \tag{2.1}$$

by the monotone convergence. Let $y \in E \cap \Delta_1$. Then $E_j \cap D$ is not minimally thin at y for each j ([5, Lemma 9.1.5]), and so $\lim_{j\to\infty} \widehat{R}^{E_j \cap D}_{K(\cdot,y)}(x_0) = K(x_0, y) = 1$. Hence $\mu(E) = 0$ by (2.1). Thus the lemma follows. □

3. Geometrical properties

Let Ω be a proper subdomain and $x, y \in \Omega$. We write $\delta_\Omega(x)$ for $\operatorname{dist}(x, \partial\Omega)$, the distance from x to $\partial\Omega$, and define the quasi-hyperbolic metric between x and y by

$$k_\Omega(x,y) := \inf_\gamma \int_\gamma \frac{ds}{\delta_\Omega(z)},$$

where the infimum is taken over all rectifiable curves γ in Ω connecting x to y.

Throughout this section we suppose that D is a domain satisfying (I) and that $\xi \in \partial D$ satisfies (II). The main purpose of this section is to show the following lemma.

Lemma 3.1. *Let $\kappa = 6/\sin\theta_1$. There is a positive constant R_ξ with the following property. For each $0 < R < R_\xi$ there is $y_R \in D \cap S(\xi, R)$ such that $\delta_D(y_R) \geq A_\xi^{-1} R$ and*

$$k_{D\cap B(\xi,\kappa R)}(x, y_R) \leq A_\xi \log \frac{R}{\delta_D(x)} + A_\xi \quad \textit{for } x \in D \cap B(\xi, R),$$

where $A_\xi \geq 1$ is independent of x and R.

Remark 3. In general, Lemma 3.1 does not hold for a John domain. We introduced in [3] a geometrical notion, a system of local reference points of order N. That is, for each $0<R<R_\xi$ there are N points, say $y_R^1,\cdots,y_R^N$, in $D\cap S(\xi,R)$ such that $\delta_D(y_R^i)\ge A_\xi^{-1}R$ for $i=1,\cdots,N$ and

$$\min_{i=1,\cdots,N}\{k_{D\cap B(\xi,\kappa R)}(x,y_R^i)\}\le A_\xi\log\frac{R}{\delta_D(x)}+A_\xi\quad\text{for } x\in D\cap B(\xi,R).$$

Lemma 3.1 is the case $N=1$.

In order to prove Lemma 3.1, in view of translation and dilation, we may suppose that $\xi=0$ and $\rho_1=1$ for simplicity. We briefly write $\Gamma(x,y)$ for $\Gamma_{\theta_1}(x,y)$. Let

$$\mathscr{Y}:=\{y\in S(0,1):\Gamma(0,y)\cap B(0,2)\subset D\}.$$

Then the union in the condition (II) is $\bigcup_{y\in\mathscr{Y}}\Gamma(0,y)\cap B(0,2)$, written $\mathscr{C}(0)$. We prove Lemma 3.1 after showing some lemmas.

Lemma 3.2. *There is a positive constant $R_0<\kappa^{-1}$ such that if $C_\lambda\cap B(0,R_0)\ne\emptyset$, then $C_\lambda\cap\mathscr{Y}\ne\emptyset$. In particular, $\mathscr{Y}\ne\emptyset$.*

Proof. We show this by leading a contradiction. Suppose that there is a sequence $\{C_{\lambda_j}\}$ such that $\operatorname{dist}(0,C_{\lambda_j})\to 0$ and $C_{\lambda_j}\cap\mathscr{Y}=\emptyset$. Let $B(z_j,\rho_0)\subset C_{\lambda_j}\subset B(z_j,A_0\rho_0)$. Taking a subsequence if necessary, we may assume that z_j converges, say to z_0. Let $x_j\in\partial C_{\lambda_j}$ be such that $x_j\to 0$. Then, by continuity of the angle $\angle\cdot x_jz_j$ and the distance $|\cdot-x_j|$,

$$\Gamma(0,z_0)\cap B(0,2)\subset\bigcup_j(\Gamma(x_j,z_j)\cap B(x_j,2))\subset\bigcup_j C_{\lambda_j}.$$

Hence $\bigcup_j C_{\lambda_j}\cap\mathscr{Y}\ne\emptyset$, and this contradicts the assumption. Thus the lemma follows. □

Let us take $y_1\in\mathscr{Y}$ and fix. For $0<R<1$ we let $y_R:=Ry_1$. Then $\delta_D(y_R)\ge R\sin\theta_1$.

Lemma 3.3. *There is a positive constant A such that if $0<R<R_0$, then*

$$k_{D\cap B(0,\kappa R)}(Ry,y_R)\le A\quad \textit{for } y\in\mathscr{Y}.$$

Proof. Note that $\mathscr{C}(0)\cap S(0,1)$ is connected since the cone $\mathscr{C}(0)$ is connected. We observe that there is a closed connected subset E of $\mathscr{C}(0)\cap S(0,1)$ and $0<r_0\le\sin\theta_1$ such that $\mathscr{Y}\subset E$ and $\operatorname{dist}(E,\partial\mathscr{C}(0))\ge r_0$. Then $y,y_1\in E$. In view of the compactness of E, we can take a curve γ in $\mathscr{C}(0)\cap S(0,1)$ joining y and y_1 such that $\delta_{\mathscr{C}(0)}(z)\ge r_0/2$ for all $z\in\gamma$ and $\ell(\gamma)\le Ar_0$, where A depends only on a covering constant of E and $\ell(\gamma)$ denotes the length of a curve γ. Let γ_R be the image of γ in $S(0,R)$ under dilation. Then we have

$$k_{D\cap B(0,\kappa R)}(Ry,y_R)\le\int_{\gamma_R}\frac{ds}{\delta_D(z)}\le\frac{Ar_0R}{r_0R/2}=2A.$$

Thus the lemma follows. □

Let $[x,y]$ denote the (open) line segment between x and y. If C is a convex set, then the distance function δ_C is concave on $\overline{C}$, i.e.,

$$\delta_C(z)\ge\frac{|z-y|}{|x-y|}\delta_C(x)+\frac{|x-z|}{|x-y|}\delta_C(y)\quad\text{for } z\in[x,y],\tag{3.1}$$

whenever $x,y\in\overline{C}$ $(x\ne y)$.

Lemma 3.4. *Let $0 < R < R_0$. If $C_\lambda \cap B(0,R) \neq \emptyset$ and $y \in C_\lambda \cap \mathscr{Y}$, then there exists $w \in C_\lambda \cap \Gamma(0,y) \cap B(0,3R/\sin\theta_1)$ such that*

$$\delta_{C_\lambda \cap \Gamma(0,y)}(w) \geq \frac{\sin\theta_1}{3} R.$$

Proof. We can take $w_1 \in C_\lambda \cap \overline{\Gamma(0,y)}$ with $|w_1| \leq R/\sin\theta_1$. In fact, if $x \in C_\lambda \cap B(0,R) \setminus \overline{\Gamma(0,y)}$, then we may take w_1 at which $[x,y]$ intersects $\partial\Gamma(0,y)$, so that

$$|w_1| = \frac{\operatorname{dist}(w_1,[0,y])}{\sin\theta_1} \leq \frac{\operatorname{dist}(x,[0,y])}{\sin\theta_1} \leq \frac{R}{\sin\theta_1}.$$

Note that $|w_1 - y| > 5R/\sin\theta_1$ since $R < \kappa^{-1} = 6^{-1}\sin\theta_1$. Let $w_2 \in [w_1,y] \subset C_\lambda \cap \Gamma(0,y)$ be such that $|w_1 - w_2| = R/\sin\theta_1$. Applying (3.1) to $C := \Gamma(0,y)$, we have

$$\delta_{\Gamma(0,y)}(w_2) \geq \frac{|w_1 - w_2|}{|w_1 - y|}\delta_{\Gamma(0,y)}(y) \geq \frac{R/\sin\theta_1}{1 + R/\sin\theta_1}\sin\theta_1 \geq \frac{2}{3}R.$$

Noting that $|w_2 - z_\lambda| \geq \rho_0 - 2R/\sin\theta_1 \geq 4R$ since $\rho_0 \geq 1 \geq 6R/\sin\theta_1$, we can take $w \in [w_2, z_\lambda] \subset C_\lambda$ with $|w_2 - w| = R/3$. Then (3.1) with $C := C_\lambda$ yields that

$$\delta_{C_\lambda}(w) \geq \frac{|w_2 - w|}{|w_2 - z_\lambda|}\delta_{C_\lambda}(z_\lambda) \geq \frac{R/3}{A_0\rho_0}\rho_0 \geq \frac{\sin\theta_1}{3}R.$$

Hence we have

$$\delta_{\Gamma(0,y)\cap C_\lambda}(w) \geq \min\left\{\frac{2}{3}R - \frac{R}{3}, \frac{\sin\theta_1}{3}R\right\} = \frac{\sin\theta_1}{3}R,$$

and

$$|w| \leq |w - w_2| + |w_2 - w_1| + |w_1| \leq \frac{R}{3} + \frac{R}{\sin\theta_1} + \frac{R}{\sin\theta_1} < \frac{3R}{\sin\theta_1}.$$

Thus the lemma follows. □

Proof of Lemma 3.1. Let $x \in C_\lambda \cap B(0,R)$ and $y \in C_\lambda \cap \mathscr{Y}$. By Lemma 3.4, we can take $w \in C_\lambda \cap \Gamma(0,y) \cap B(0,3R/\sin\theta_1)$ with $\delta_{C_\lambda \cap \Gamma(0,y)}(w) \geq 3^{-1}R\sin\theta_1$. Then we have

$$\delta_D(z) \geq \delta_{C_\lambda}(z) \geq \frac{|x-z|}{|x-w|}\delta_{C_\lambda}(w) \geq \frac{\sin^2\theta_1}{12}|x-z| \quad \text{for } z \in [x,w],$$

by (3.1) with $C := C_\lambda$. Since $[x,w] \subset B(0,\kappa R/2)$, it follows that

$$k_{D\cap B(0,\kappa R)}(x,w) \leq \int_{[x,w]} \frac{ds}{\delta_D(z)} \leq 1 + \int_{\frac{\delta_D(x)}{2}}^{|x-w|} \frac{12}{\sin^2\theta_1}\frac{dt}{t} \leq A\log\frac{R}{\delta_D(x)} + A,$$

where A depends only on θ_1. We also have $k_{D\cap B(0,\kappa R)}(w,Ry) \leq A$. In fact, since $\delta_{\Gamma(0,y)}(Ry) \geq R\sin\theta_1$, it follows from (3.1) with $C := \Gamma(0,y)$ that

$$\delta_D(z) \geq \delta_{\Gamma(0,y)}(z) \geq \frac{|w-z|}{|w-Ry|}\delta_{\Gamma(0,y)}(Ry) \geq \frac{\sin^2\theta_1}{4}|w-z| \quad \text{for } z \in [w,Ry],$$

and so

$$k_{D\cap B(0,\kappa R)}(w,Ry) \leq \int_{[w,Ry]} \frac{ds}{\delta_D(z)} \leq 2 + \int_{\frac{\delta_D(w)}{2}}^{\frac{\delta_D(Ry)}{2}} \frac{4}{\sin^2\theta_1}\frac{dt}{t} \leq A,$$

where A depends only on θ_1. Hence we obtain from Lemma 3.3 that

$$\begin{aligned}k_{D\cap B(0,\kappa R)}(x,y_R) &\le k_{D\cap B(0,\kappa R)}(x,w)+k_{D\cap B(0,\kappa R)}(w,Ry)+k_{D\cap B(0,\kappa R)}(Ry,y_R)\\ &\le A\log\frac{R}{\delta_D(x)}+A.\end{aligned}$$

Thus Lemma 3.1 follows. □

4. **Carleson type estimate**

In this section we show a Carleson type estimate. To this end, we prepare two lemmas. One is a refinement of Domar's theorem ([6, Theorem 2]). Another is the integrability of the negative power of the distance function. This is a local version of [1, Lemma 5].

We note first the following. Let Ω be a domain and $x,y\in\Omega$. We say that x and y are connected by a Harnack chain $\{B(x_j,\delta_\Omega(x_j))\}_{j=1}^N$ if $x\in B(x_1,\frac12\delta_\Omega(x_1))$, $x_{j-1}\in B(x_j,\frac12\delta_\Omega(x_j))$ for $j=2,\cdots,N$ and $x_N=y$. The number N is called the length of the Harnack chain. We observe that if $x\notin B(y,\frac12\delta_\Omega(y))$, then the shortest length of the Harnack chain connecting x and y is comparable to $k_\Omega(x,y)$. Therefore, the Harnack inequality yields that there is a constant $A\ge1$ depending only on the dimension such that if $x,y\in\Omega$ and h is a positive harmonic function on Ω, then

$$\exp(-Ak_\Omega(x,y)-1)\le\frac{h(x)}{h(y)}\le\exp(Ak_\Omega(x,y)+1). \tag{4.1}$$

Lemma 4.1. *Let Ω be a bounded domain. If u is a non-negative subharmonic function on Ω such that*

$$I:=\int_\Omega(\log^+u)^{n-1+\varepsilon}dx<\infty \quad \textit{for some } \varepsilon>0,$$

then there is a positive constant A depending only on ε and the dimension such that

$$u(x)\le\exp\left(2+A\left(\frac{I}{\delta_\Omega(x)^n}\right)^{1/\varepsilon}\right). \tag{4.2}$$

We show first the following lemma. We write $|E|$ for the volume of a set E.

Lemma 4.2. *Let u be a subharmonic function on Ω containing $\overline{B(x,R)}$. Suppose that $u(x)\ge t>0$ and that*

$$R\ge L_n|\{y\in B(x,R):t/e<u(y)\le et\}|^{1/n}, \tag{4.3}$$

where $L_n=(e^2/|B(0,1)|)^{1/n}$. Then there exists $x'\in B(x,R)$ such that $u(x')>et$.

Proof. Suppose to the contrary that $u\le et$ on $B(x,R)$. Noting that (4.3) is equivalent to

$$\frac{|\{y\in B(x,R):t/e<u(y)\le et\}|}{|B(x,R)|}\le\frac{1}{e^2},$$

we have

$$\begin{aligned}t\le u(x)&\le\frac{1}{|B(x,R)|}\int_{B(x,R)}u(y)dy\\ &=\frac{1}{|B(x,R)|}\left(\int_{B(x,R)\cap\{u\le t/e\}}u(y)dy+\int_{B(x,R)\cap\{t/e<u(y)\le et\}}u(y)dy\right)\\ &\le\frac{t}{e}+\frac{et}{e^2}<t.\end{aligned}$$

This is a contradiction, and the lemma follows. □

Proof of Lemma 4.1. Since the right hand side of (4.2) is not less than e^2, it is sufficient to show that

$$\delta_\Omega(x) \le AI^{1/n}(\log u(x))^{-\varepsilon/n} \quad \text{whenever } u(x) > e^2. \tag{4.4}$$

Let $x_1 \in \Omega$ be such that $u(x_1) > e^2$, and let

$$R_j = L_n|\{y \in \Omega : e^{j-2}u(x_1) < u(y) \le e^j u(x_1)\}|^{1/n}.$$

Let us show (4.4) for $x = x_1$. We can choose a finite or infinite sequence $\{x_j\}$ in Ω as follows. By Lemma 4.2, we can iteratively find $x_{j+1} \in B(x_j, R_j)$ with $u(x_{j+1}) > e^j u(x_1)$ whenever $\delta_\Omega(x_j) > R_j$. If $\delta_\Omega(x_j) \le R_j$, then we stop this iteration, otherwise we continue.

We claim that

$$\delta_\Omega(x_1) \le 2\sum_{j=1}^{\infty} R_j. \tag{4.5}$$

Suppose first $\{x_j\}$ is finite. Noting that

$$\delta_\Omega(x_1) \le \sum_{j=1}^{N-1} |x_j - x_{j+1}| + \delta_\Omega(x_N), \tag{4.6}$$

we obtain (4.5) by our choice of $\{x_j\}$. Suppose next $\{x_j\}$ is infinite. Since $u(x_j) \ge e^{j-1}u(x_1) \to \infty$, it follows from the local boundedness of a subharmonic function that x_j goes to the boundary. Hence $\delta_\Omega(x_N) \le \delta_\Omega(x_1)/2$ for some N, and (4.5) follows from (4.6).

To obtain (4.4) for $x = x_1$, it is enough to show that

$$\sum_{j=1}^{\infty} R_j \le AI^{1/n}(\log u(x_1))^{-\varepsilon/n}. \tag{4.7}$$

Let j_1 be the integer such that $e^{j_1} < u(x_1) \le e^{j_1+1}$. Then $j_1 \ge 2$ and

$$R_j \le L_n|\{y \in \Omega : e^{j_1+j-2} < u(y) \le e^{j_1+j+1}\}|^{1/n}.$$

Since the family of intervals $\{(e^{j_1+j-2}, e^{j_1+j+1}]\}_j$ overlaps at most three times, it follows from Hölder's inequality that

$$\begin{aligned}
\sum_{j=1}^{\infty} R_j &\le 3L_n \sum_{j=j_1}^{\infty} |\{y \in \Omega : e^{j-1} < u(y) \le e^j\}|^{1/n} \\
&\le 3L_n \left(\sum_{j=j_1}^{\infty} \frac{1}{j^{(n-1+\varepsilon)/(n-1)}} \right)^{(n-1)/n} \left(\sum_{j=j_1}^{\infty} j^{n-1+\varepsilon} |\{y \in \Omega : e^{j-1} < u(y) \le e^j\}| \right)^{1/n} \\
&\le A j_1^{-\varepsilon/n} \left(\int_\Omega (\log^+ u(y))^{n-1+\varepsilon} dy \right)^{1/n} \\
&\le A(\log u(x_1))^{-\varepsilon/n} I^{1/n},
\end{aligned}$$

where A depends only on ε and n. Thus (4.7) follows and Lemma 4.1 is proved. □

Lemma 4.3. *Let D be a domain satisfying (I) and $\xi \in \partial D$. If $0 < R < \rho_0$, then there are positive constants τ and A depending only on A_0 and the dimension such that*

$$\int_{D\cap B(\xi,R)} \left(\frac{R}{\delta_D(x)}\right)^{\tau} dx \le AR^n.$$

Proof. For each $j \in \mathbb{N}\cup\{0\}$ we put

$$V_j := \left\{x \in D\cap B\left(\xi, R + \frac{A_0+1}{2^{j-1}}R\right) : \frac{R}{2^{j+1}} \le \delta_D(x) < \frac{R}{2^j}\right\}.$$

Let $x \in \bigcup_{j=k+1}^{\infty} V_j$. Then there is C_λ so that $x \in C_\lambda$, and let $B(z_\lambda,\rho_0) \subset C_\lambda \subset B(z_\lambda, A_0\rho_0)$. Let $y, y' \in [x, z_\lambda]$ be such that $\delta_D(y) = R/2^k$ and $\delta_D(y') = (R/2^{k+1} + R/2^k)/2$. Then we see that $x \in \overline{B(y, A_0R/2^k)}$ by (3.1), and that $B(y', R/2^{k+2}) \subset V_k \cap B(y, A_0R/2^k)$. Hence we obtain

$$\left|B\left(y, \frac{5A_0R}{2^k}\right)\right| \le A_1 \left|V_k \cap B\left(y, \frac{A_0R}{2^k}\right)\right|, \tag{4.8}$$

where A_1 depends only on A_0 and the dimension. We also have $\bigcup_{j=k+1}^{\infty} V_j \subset \bigcup_y \overline{B(y, A_0R/2^k)}$, where y is the point associated with x as above. Hence the covering lemma yields that there is $\{y_j\}$ such that $\bigcup_{j=k+1}^{\infty} V_j \subset \bigcup_j \overline{B(y_j, 5A_0R/2^k)}$ and $\{B(y_j, A_0R/2^k)\}$ are mutually disjoint. Then we obtain from (4.8) that

$$\sum_{j=k+1}^{\infty} |V_j| = \left|\bigcup_{j=k+1}^{\infty} V_j\right| \le \sum_j \left|B\left(y_j, \frac{5A_0R}{2^k}\right)\right| \le A_1 \sum_j \left|V_k \cap B\left(y_j, \frac{A_0R}{2^k}\right)\right| \le A_1 |V_k|.$$

Let $t = 1 + 1/2A_0$. Then

$$\begin{aligned} A_1 \sum_{k=0}^{N} t^{k+1}|V_k| &\ge \sum_{k=0}^{N}\sum_{j=k+1}^{N+1} t^{k+1}|V_j| = \sum_{j=1}^{N+1}\sum_{k=0}^{j-1} t^{k+1}|V_j| \ge \sum_{j=1}^{N}\sum_{k=0}^{j-1} t^{k+1}|V_j| \\ &= \sum_{j=1}^{N} \frac{t^{j+1}-t}{t-1}|V_j| = \frac{1}{t-1}\sum_{j=0}^{N} t^{j+1}|V_j| - \frac{t}{t-1}\sum_{j=0}^{N} |V_j|, \end{aligned}$$

and so

$$\sum_{j=0}^{N} t^{j+1}|V_j| \le \frac{t}{1-(t-1)A_1}\sum_{j=0}^{N}|V_j|.$$

Letting $N \to \infty$, we have

$$\sum_{j=0}^{\infty} t^{j+1}|V_j| \le \frac{t}{1-(t-1)A_1}\sum_{j=0}^{\infty}|V_j| \le A|B(\xi, R + 2(A_0+1)R)| \le AR^n.$$

Since $t^j < (R/\delta_D(x))^\tau \le t^{j+1}$ for $x \in V_j$ with $\tau = \log t/\log 2 > 0$, we obtain

$$\int_{D\cap B(\xi,R)} \left(\frac{R}{\delta_D(x)}\right)^{\tau} dx \le \sum_{j=0}^{\infty} t^{j+1}|V_j| \le AR^n.$$

Thus the lemma follows. □

Lemma 4.4 (Carleson type estimate). *Suppose that D is a domain satisfying (I) and that $\xi \in \partial D$ satisfies (II). Let $0 < R < R_\xi$. If h is a positive bounded harmonic function on $D \cap B(\xi, \kappa R)$ vanishing quasi-everywhere on $\partial D \cap B(\xi, \kappa R)$, then*

$$h(x) \le A h(y_R) \quad \text{for } x \in D \cap \overline{B(\xi, \kappa^{-1} R)},$$

where A is independent of x, R and h.

Proof. By (4.1) and Lemma 3.1 we have

$$\frac{h(x)}{h(y_R)} \le A_2 \left(\frac{R}{\delta_D(x)} \right)^\alpha \quad \text{for } x \in D \cap B(\xi, R), \tag{4.9}$$

where A_2 and α are positive constants depending only on A_ξ and the dimension. We note that h has a non-negative subharmonic extension h^* to $B(\xi, R)$ with zero values on $B(\xi, R) \setminus \overline{D}$ ([5, Theorem 5.2.1]). Let $u = h^*/A_2 h(y_R)$. Using the inequality

$$\left[\log \left(\frac{R}{\delta_D(x)} \right) \right]^n \le \left(\frac{n}{\tau} \right)^n \left(\frac{R}{\delta_D(x)} \right)^\tau \quad \text{for } x \in D \cap B(\xi, R),$$

where $\tau > 0$ is as in Lemma 4.3, we obtain from (4.9) and Lemma 4.3 that

$$I = \int_{B(\xi, R)} (\log^+ u)^n dx \le A \int_{D \cap B(\xi, R)} \left(\frac{R}{\delta_D(x)} \right)^\tau dx \le A R^n.$$

Hence it follows from Lemma 4.1 that $u \le A$ on $S(\xi, \kappa^{-1} R)$, and the maximum principle yields that

$$h(x) \le A h(y_R) \quad \text{for } x \in D \cap \overline{B(\xi, \kappa^{-1} R)}.$$

Thus the lemma follows. □

5. Boundary Harnack principle

The purpose of this section is to show a (uniform) boundary Harnack principle, which is useful to obtain properties of Martin kernels. The proofs in this section are based on [2] for a uniform domain.

For $r > 0$ we let

$$U(r) := \{ x \in D : \delta_D(x) < r \}.$$

We denote by $\omega(x, E, U)$ the harmonic measure of a set E for an open set U evaluated at x. We write $|E|$ for the volume of a set E. Let us start with an estimate of a harmonic measure.

Lemma 5.1. *Let D be a domain satisfying (I). Then there are constants $0 < \varepsilon_0 < 1$ and $A_3 \ge 1$ such that if $0 < r < \rho_0/2$, then*

$$\omega(x, U(r) \cap S(x, A_3 r), U(r) \cap B(x, A_3 r)) \le \varepsilon_0 \quad \text{for } x \in U(r).$$

Proof. Let $x \in U(r)$. Then there is C_λ so that $x \in C_\lambda$, and let $B(z_\lambda, \rho_0) \subset C_\lambda \subset B(z_\lambda, A_0 \rho_0)$. Take $w \in [x, z_\lambda]$ with $\delta_D(w) = 2r$. Then we have $|x - w| \le 2A_0 r$ by (3.1), and so $B(w, r) \subset B(x, 3A_0 r) \setminus U(r)$. Hence there is $0 < \varepsilon_0 < 1$ depending only on A_0 and the dimension such that

$$\frac{|U(r) \cap B(x, 3A_0 r)|}{|B(x, 3A_0 r)|} \le \varepsilon_0.$$

Let $A_3 := 3A_0 + 1$. We note that $\omega(\cdot, U(r) \cap S(x, A_3 r), U(r) \cap B(x, A_3 r))$ has a subharmonic extension ω to $B(x, A_3 r)$ with zero values on $B(x, A_3 r) \setminus \overline{U(r)}$ ([5, Theorem 5.2.1]). Hence

$$\omega(x) \leq \frac{1}{|B(x, 3A_0 r)|} \int_{B(x, 3A_0 r)} \omega(y)\, dy \leq \varepsilon_0.$$

Thus the lemma follows. □

Lemma 5.2. *Let D be a domain satisfying (I) and A_3 be as in Lemma 5.1. Then there is a positive constant $A_4 \leq 1$ such that if $r > 0$ and $R > 0$, then*

$$\omega(x, U(r) \cap S(x, R), U(r) \cap B(x, R)) \leq \exp\left(A_3 - A_4 \frac{R}{r}\right) \quad \textit{for } x \in U(r). \tag{5.1}$$

Proof. Note that if $R \leq A_3 r$, then (5.1) clearly holds since the right hand side of (5.1) is not less than 1. Let $k \in \mathbb{N}$ be such that $kA_3 r < R \leq (k+1) A_3 r$. We claim that

$$\sup_{U(r) \cap B(x, R - jA_3 r)} \omega(\cdot, U(r) \cap S(x, R), U(r) \cap B(x, R)) \leq \varepsilon_0^j \tag{5.2}$$

for $j = 0, \cdots, k$, where ε_0 is as in Lemma 5.1. We show this by induction. If $j = 0$, then (5.2) clearly holds. We assume that (5.2) holds for $j-1$, and show (5.2) for j. Let $y \in U(r) \cap S(x, R - jA_3 r)$. Since $S(y, A_3 r) \subset \overline{B(x, R - (j-1) A_3 r)}$, it follows from the assumption, the maximum principle and Lemma 5.1 that

$$\begin{aligned} \omega(y, U(r) \cap S(x, R), U(r) \cap B(x, R)) &\leq \varepsilon_0^{j-1} \omega(y, U(r) \cap S(y, A_3 r), U(r) \cap B(y, A_3 r)) \\ &\leq \varepsilon_0^j. \end{aligned}$$

Since y is an arbitrary point in $U(r) \cap S(x, R - jA_3 r)$, the maximum principle yields (5.2) for j. Finally, noting that $R/A_3 r \leq 2k$, we obtain from (5.2) with $j := k$ that

$$\omega(x, U(r) \cap S(x, R), U(r) \cap B(x, R)) \leq \exp((\varepsilon_0 - 1)k) \leq \exp\left(\frac{\varepsilon_0 - 1}{2A_3} \frac{R}{r}\right).$$

Thus the lemma follows. □

Lemma 5.3. *Suppose that D is a domain satisfying (I) and that $\xi \in \partial D$ satisfies (II). Let $0 < R < R_\xi$. If h is a positive bounded harmonic function on $D \cap B(\xi, \kappa R)$ vanishing quasi-everywhere on $\partial D \cap B(\xi, \kappa R)$, then*

$$\omega(x, D \cap S(\xi, \kappa^{-1} R), D \cap B(\xi, \kappa^{-1} R)) \leq A \frac{h(x)}{h(y_R)} \quad \textit{for } x \in D \cap B(\xi, \kappa^{-2} R),$$

where A is independent of x, R and h.

Proof. By Lemma 4.4, we have $h \leq Ah(y_R)$ on $D \cap B(\xi, \kappa^{-1} R)$. Let A_5 be such that $A_5 h / h(y_R) \leq e^{-1}$ on $D \cap B(\xi, \kappa^{-1} R)$, and put $u := A_5 h / h(y_R)$. Then it follows from (4.1) and Lemma 3.1 that

$$u(x) \geq A \left(\frac{\delta_D(x)}{R}\right)^\alpha \quad \text{for } x \in D \cap B(\xi, \kappa^{-1} R). \tag{5.3}$$

Let $D_j := \{x \in D : \exp(-2^{j+1}) \leq u(x) < \exp(-2^j)\}$ and $U_j := \{x \in D : u(x) < \exp(-2^j)\}$. Then, by (5.3), we have

$$U_j \cap B(\xi, \kappa^{-1} R) \subset V_j := \left\{x \in D : \delta_D(x) \leq A_6 R \exp\left(-\frac{2^j}{\alpha}\right)\right\}.$$

Let $\{R_j\}$ be a sequence defined by $R_0 := \kappa^{-1}R$ and

$$R_j := \left(\kappa^{-1} - \frac{6(\kappa^{-1}-\kappa^{-2})}{\pi^2}\sum_{k=1}^{j}\frac{1}{k^2}\right)R.$$

Then $R_j \downarrow \kappa^{-2}R$. We briefly write $\omega_0 := \omega(\cdot, D\cap S(\xi,\kappa^{-1}R), D\cap B(\xi,\kappa^{-1}R))$, and put

$$d_j := \begin{cases} \sup_{D_j\cap B(\xi,R_j)} \dfrac{\omega_0}{u} & \text{if } D_j\cap B(\xi,R_j)\neq\emptyset, \\ 0 & \text{if } D_j\cap B(\xi,R_j)=\emptyset. \end{cases}$$

It suffices to show that $\sup_{j\geq 0} d_j$ is bounded by a constant independent of R and u. Let $j>0$ and $x\in U_j\cap B(\xi,R_j)$. Then the maximum principle yields that

$$\omega_0(x) \leq \omega(x, U_j\cap S(\xi,R_{j-1}), U_j\cap B(\xi,R_{j-1})) + d_{j-1}u(x). \tag{5.4}$$

Since $B(x,R_{j-1}-R_j)\subset B(\xi,R_{j-1})$, the first term of the right hand side of (5.4) is not greater than

$$\omega(x, V_j\cap S(x,R_{j-1}-R_j), V_j\cap B(x,R_{j-1}-R_j)) \leq \exp\left(A_3 - A_4\frac{R_{j-1}-R_j}{A_6R\exp(-2^j/\alpha)}\right)$$

by Lemma 5.2. Let us divide the both sides of (5.4) by $u(x)$ and take the supremum over $D_j\cap B(\xi,R_j)$. Then we have

$$d_j \leq \exp\left(2^{j+1} + A_3 - A_4\frac{6(\kappa^{-1}-\kappa^{-2})}{\pi^2}\frac{\exp(2^j/\alpha)}{A_6 j^2}\right) + d_{j-1}.$$

Since $d_0 \leq e^2$, we obtain

$$d_j \leq \sum_{j=1}^{\infty}\exp\left(2^{j+1} + A_3 - A_4\frac{6(\kappa^{-1}-\kappa^{-2})}{\pi^2}\frac{\exp(2^j/\alpha)}{A_6 j^2}\right) + d_0 < \infty.$$

Thus the lemma follows. □

Lemma 5.4 (Boundary Harnack principle). *Suppose that D is a domain satisfying (I) and that $\xi\in\partial D$ satisfies (II). Let $0<R<R_\xi$. If u and v are positive bounded harmonic functions on $D\cap B(\xi,\kappa R)$ vanishing quasi-everywhere on $\partial D\cap B(\xi,\kappa R)$, then*

$$\frac{u(y)}{v(y)} \approx \frac{u(y')}{v(y')} \quad \text{for } y,y'\in D\cap B(\xi,\kappa^{-2}R),$$

where the constant of comparison is independent of y, y', R, u and v.

Proof. By Lemma 4.4, the maximum principle and Lemma 5.3, we have

$$u(y) \leq Au(y_R)\omega(y, D\cap S(\xi,\kappa^{-1}R), D\cap B(\xi,\kappa^{-1}R)) \leq Au(y_R)\frac{v(y)}{v(y_R)}$$

for $y\in D\cap B(\xi,\kappa^{-2}R)$. Changing the roles of u and v, we have

$$v(y') \leq Av(y_R)\frac{u(y')}{u(y_R)} \quad \text{for } y'\in D\cap B(\xi,\kappa^{-2}R).$$

Hence two inequalities above yield the lemma. □

Remark 4. We note that the constant of comparison and R_ξ in Lemma 5.4 depends on ξ. For a bounded uniform domain, these constants could be taken uniformly for ξ ([2, Theorem 1]). Using this fact, the first author showed the uniqueness of a kernel function at ξ ([2, Lemma 4 and Proof of Theorem 3]). However, in view of Lemma 2.1, we need not take those constants uniformly in order to prove Theorem. We also note that there is a bounded domain satistying (I) and each boundary point satisfies (II) but not a uniform domain (see example in Section 8).

6. Proof of Theorem and Corollary

Suppose that D is a domain satisfying (I) and that $\xi \in \partial D$ satisfies (II). We note first that every Martin kernel at ξ is a kernel function at ξ. In fact, let $R > 0$ be small enough and $x \in D \setminus B(\xi, \kappa R)$. Applying Lemma 5.4 to $u := G(x, \cdot)$ and $v := G(x_0, \cdot)$, we see that each Martin kernel at ξ is bounded on $D \setminus B(\xi, \kappa R)$, and is kernel function at ξ.

Proof of Theorem. Let $u, v \in \Delta_1(\xi)$ and $R > 0$ be small enough. Then, by definition of the Martin kernel at ξ, there are sequences $\{y_j\}$ and $\{y'_j\}$ in D converging to ξ such that $K(\cdot, y_j) \to u$ and $K(\cdot, y'_j) \to v$, respectively. Since $K(x, y_j) \approx K(x, y'_j)$ for $x \in D \setminus B(\xi, \kappa R)$ by Lemma 5.4 if j is sufficiently large, we have $u(x) \approx v(x)$ for $x \in D \setminus B(\xi, \kappa R)$. Since the constant of comparison is independent of R, it follows from the minimality of u and v and $u(x_0) = 1 = v(x_0)$ that $u \equiv v$. Hence $\Delta_1(\xi)$ is a singleton. Furthermore, it follows from Lemma 2.1 that $\Delta(\xi) = \Delta_1(\xi)$. Theorem is proved. □

Proof of Corollary. Let $x \in D$. By Theorem, we see that $K(x, \cdot)$ extends continuously to $\overline{D} \setminus \{x_0\}$. Moreover, it follows from the first paragraph of this section that $K(\cdot, \xi_1) \neq K(\cdot, \xi_2)$ if $\xi_1, \xi_2 \in \partial D$ are distinct. Thus Corollary follows. □

7. Remark for bounds in condition (II)

Let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and let

$$\mathbb{H}_+ := \{x \in \mathbb{R}^n : x_n > 0\} \quad \text{and} \quad \mathbb{H}_- := \{x \in \mathbb{R}^n : x_n < 0\}.$$

In view of dilation, we give examples for $\rho_0 = 1$.

Example of (i) ($\theta_1 > \sin^{-1}(1/A_0)$). Let $w_0 = (0, \cdots, 0, A_0)$ and let V_1 be the convex hull of $B(w_0, 1) \cup \{0\}$. We consider the domain

$$D := \Big(B(0, A_0 + 1) \setminus \big(\overline{B(0, A_0 - 1) \cap \mathbb{H}_+}\big)\Big) \cup V_1.$$

Then D satisfies (I) and the union $\mathscr{C}(0)$ in the condition (II) at 0 is $B(0, 2\rho_1) \cap \mathbb{H}_-$, that is, the origin satisfies (II). But there are two minimal Martin boundary points at the origin.

Example of (ii) for $1 < A_0 \le 2$ ($0 < \theta_1 \le \sin^{-1}(1/A_0)$ and $\rho_1 > \rho_0 \cos\theta_1$). Let

$$w_1 = (0, 0, \cdots, 0, 1), \quad w_2 = (\sqrt{1 - (2 - A_0)^2}, 0, \cdots, 0, -1) \quad \text{and}$$
$$w_3 = (\sqrt{1 - (2 - A_0)^2}, 0, \cdots, 0, A_0 - 1).$$

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Let V_2 be the convex hull of $B(w_2,1)\cup\{w_3\}$. We consider the domain

$$D:=\left(B(0,5)\setminus(\overline{B(0,3)\cap\mathbb{H}_+})\right)\cup B(w_1,1)\cup V_2.$$

Then D satisfies (I) and $\mathscr{C}(0)=B(0,2\rho_1)\cap\mathbb{H}_-$. But there are two minimal Martin boundary points at the origin.

Example of (ii) for $A_0>2$ $(0<\theta_1\le\sin^{-1}(1/A_0)$ and $\rho_1>\rho_0\cos\theta_1)$. Let $w_1'=(0,\cdots,0,1)$, $w_2'=(1,0,\cdots,0,1-A_0)$ and $w_3'=(1,0,\cdots,0,1)$. Let V_3 be the convex hull of $B(w_2',1)\cup\{w_3'\}$. We consider the domain

$$D:=\left(B(0,5)\setminus(\overline{B(0,3)\cap\mathbb{H}_+})\right)\cup B(w_1,1)\cup V_3.$$

Then D satisfies (I) and $\mathscr{C}(0)=B(0,2\rho_1)\cap\mathbb{H}_-$. But there are two minimal Martin boundary points at the origin.

It is easy to check, in each case, that D is represented as the union of balls $B(z_\lambda,1)$ and V_i, and that V_i includes a ball of radius 1 and is included in a ball of radius A_0 with the same center. We also observe that any truncated circular cone $\Gamma_{\theta_1}(0,y)\cap B(0,2\rho_1)$ is not included in $D\cap\mathbb{H}_+$, so that $\mathscr{C}(0)=B(0,2\rho_1)\cap\mathbb{H}_-$. Moreover, we observe that one limit function obtained by approaching from $D\cap\mathbb{H}_+$ is bounded on $D\cap\mathbb{H}_-$ and another limit function obtained by approaching from $D\cap\mathbb{H}_-$ is bounded on $D\cap\mathbb{H}+$, so that the origin has two minimal Martin boundary points.

8. Example of a domain satisfying (I) and (II) but not a uniform domain

A domain Ω is called a uniform if there exists a positive constant A with the following property. For each pair of points $x_1,x_2\in\Omega$ there is a rectifiable curve γ in Ω joining x_1 and x_2 such that

(i) $\ell(\gamma)\le A|x_1-x_2|$,

(ii) $\min\{\ell(\gamma(x_1,z)),\ell(\gamma(z,x_2))\}\le A\delta_\Omega(z)\quad$ for all $z\in\gamma$,

where $\ell(\gamma)$ and $\gamma(z,w)$ are the length of γ and the subarc of γ between z and w, respectively.

For simplicity, we give an example when $n=2$.

Example. Let $a=(0,2)$, $b=(0,-2)$ and $c=(-2,0)$. Suppose

$$\Omega:=B(a,2)\cup B(b,2)\cup B(c,2).$$

Then Ω satisfies (I) and each boundary point satisfies (II) but not a uniform domain.

In fact, let $p=(0,1)$ and $w=(x,y)$ be a point in $S(p,1)$ such that $x>0$ and $0<y<1$, and let $\overline{w}=(x,-y)$. Then $y=1-(1-x^2)^{1/2}$. Let γ_w be an arbitrary rectifiable curve in Ω joining w and $\overline{w}$. Then γ_w must hit y-axis $\{x=0\}$, and we have

$$\ell(\gamma_w)\ge\operatorname{dist}(w,\{x=0\})=x=\frac{x}{1-(1-x^2)^{1/2}}y=\frac{1}{2}\frac{x}{1-(1-x^2)^{1/2}}|w-\overline{w}|.$$

This inequality shows that a constant A satisfying (i) does not exist since

$$\lim_{x\to0+}\frac{x}{1-(1-x^2)^{1/2}}=+\infty.$$

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