数理解析研究所講究録 1293 巻 2002 年 31-41

Regularity of function-kernels and the vanishing property of potentials in the neighborhood of the point at infinity

Isao HIGUCHI (樋口 功)
Science Division, Aichi Institute of Technology

1. Introduction

Let X be a locally compact but non-compact Hausdorff space satisfying the second countability axiom. A function $G = G(x,y)$ on $X \times X$ is called **a continuous function-kernel on** X if $G(x,y)$ is continuous in the extended sense and satisfies

$$0 < G(x,x) \leq +\infty \quad \text{for } \forall x \in X ,$$

$$0 \leq G(x,y) < +\infty \quad \text{for } \forall (x,y) \in X \times X \text{ with } x \neq y .$$

In general, a positive linear mapping from C_K to C is called **a continuous kernel on** X .

When a continuous function-kernel G satisfies **the continuity principle**, we can verify, under the additional condition that every non-empty open set in X is of positive G-inner capacity, that there exists a positive mesure ξ everywhere dence in X satisfying

(1) $G(x,y)$ is locally $\xi \otimes \xi$ -summable,

(2) $V_G^{\xi}(f)(x) = \int G(x,y)f(y)\,d\xi(y)$ is continuous on X for $\forall f \in C_K$.

Then we can consider V_G^{ξ} as **a continuous kernel on** X.

Let V be a continuous kernel on X . The family $(V_p)_{p>0}$ of continuous kernels on X is colled **a resolvent family associated with** V, if it satisfies the following equalities:

(3) $V_p - V_q = (q-p)V_p \cdot V_q = (q-p)V_q \cdot V_p, \qquad \forall p,\ \forall q > 0$,

(4) $\lim_{p \to 0} V_p = V$.

G.A.Hunt[9] verified that, when a continuous kernel V satisfies the complete maximum principle, we can associate a resolvent family $(V_p)_{p>0}$ with V, under the assumption that $V(C_K) \subset C_0$ and $V(C_K)$ is dense in C_0.

The existence of a resolvent family may be developped to the theory of a semi-group. So we can consider a continuous kernel as the elementary solution of the infinitesimal generator of the semi-group and hence we can enter analytically into the arguement of the generalized Poisson and Dirichlet problems.

Subsequently, G.Lion[18] obtained the same result without the condition that $V(C_K)$ is dense in C_0.

On the other hand, P.A.Meyer[19], J.C.Taylor[20] and F.Hirsch[8] constructed the resolvent family replacing the condition that $V(C_K) \subset C_0$ with the weaker conditions on the vanishing properties of potentials at infinity.

Now, let us recall here the arranged results in the theory of convolution kernels.

A convolution kernel N on a locally compact abelian group X is said to be **a Hunt kernel** when there exists a vaguely continuous semi-group $(\alpha_t)_{t>0}$ of positive measures on X satisfying

$$N = \int_0^\infty \alpha_t \, dt \quad \Big(\text{i.e.,} \int f \, dN = \int_0^\infty \Big\{ \int f \, d\alpha_t \Big\} \, dt \ \text{ for } \forall f \in C_K \Big).$$

Concerning the characterization of a Hunt kernel, the following rerults are well known.

A non-periodic convolution kernel N becomes a Hunt kernel if and only if N satisfies one of the following conditions:

(A) N is **balayable**, that is, there exists a balayaged mesure on every open set not necessarily relatively compact (cf. G.Choquet-J.Deny[1]).

(B) There exists **a resolvent family associated with** N (cf. M.Itô[10]).

(C) N satisfies the domination principle and N is **regular** (cf. M.Itô[10]).

(D) N satisfies the domination principle and has **the dominated convergent property** (cf, M.Itô[10] and M.Kishi[17]).

Remark 1. The author has investigated the relations $(A) \sim (D)$ with respect to the continuous function-kernels and verified already the equivalence of (C) and (D) and obtained the relations $(C) \longrightarrow (A)$ and $(C) \longrightarrow (B)$ (cf. I.Higuchi[4], [5], [6]). Bu t the inverse relations $(A) \longrightarrow (C)$ and $(B) \longrightarrow (C)$ fail to hold in general (cf. I.Higuchi[6] and M.Itô[12]). These facts suggest that the treatments of the function-kernels are more complicated than that of of the convolution kernels.

The regularity of function kernel is concerned deeply with the vanishing property of potentials in the neighborhood of the point at infinity.

The purpose of this paper is to characterize the regularity of **a non-symmetric** continuous function-kernel $G = G(x, y)$ satisfying the complete maximum principle and to prove that at least one of G and $\check{G}$ converges to 0 quasi-everywhere at infinity.

2. preliminaries

The G-potential $G\mu(x)$ of a Radon mesure μ on X is defined by

$$G\mu(x) = \int G(x,y)\ d\mu(x).$$

Put

$$M = \{\ \mu : \text{positive Radon mesure on } X\ \},$$

$$E = E(G) = \{\ \mu \in M\ ;\ \int G\mu(x)\ d\mu(x) < +\infty\ \},$$

$$F = F(G) = \{\ \mu \in M\ :\ G\mu(x) \text{ is finite continuous on } X\ \},$$

$$D = D(G) = \{\ \mu \in M\ :\ G\mu(x) < +\infty \text{ G-n.e. on } X\ \}.$$

And we write their sub-families consisting of the measures with compact support by M_0, E_0, F_0 respectively.

We denote by $P_{M_0}(G)$ the totality of G-potentials of the measures in M_0. The notations of the families of various class of potensiala are also denoted similarly.

A Borel measurable set B is said to be G-negligible if $\mu(B) = 0$ for $\forall \mu \in E_0(G)$. We say that a property P holds G-nearly everywhere on a subset A of X and write simply that P holds G-n.e. on A, when it holds on A except for a G-negligible set.

A lower semi-continuous function u on X is said to be **G-superharmonic** when $0 \le u(x) < +\infty$ G-n.e. on X and for any $\mu \in E_0(G)$, the inequality $G\mu(x) \le u(x)$ G-n.e. on $S\mu$ implies the same inequality on the whole space X .

We denote by $S(G)$ the totality of G-superharmonic functions on X.

For a function $u \in S(G)$ and a closed set $F \subset X$, a positive measure μ' supported by F satisfying the following conditions is called a **balayaged mesure** of u on F, if it exists,

$$G\mu'(x) = u(x) \quad \text{G-n.e. on } F\ ,$$

$$G\mu'(x) \le u(x) \quad \text{on } X\ .$$

We denote by $S_{bal}(F,G)$ the totality of G-superharmonic functions for which the balayaged mesure on F exists and write simply $S_{bal}(G)$ instead of $S_{bal}(X,G)$.

Potential theoretic principles are stated as follows:

(i) We say that G satisfies **the domination principle** and write simply $G \prec G$ when $P_{M_0}(G) \subset S(G)$.

(ii) We say that G satisfies **the complete maximum principle** and write simply $G \prec G+1$ when we have $P_{M_0}(G) \cup \{c\} \subset S(G)$ for $\forall c \ge 0$.

(iii) We say that G satisfies **the balayage principle** if we have

$$P_{M_0}(G) \subset \cap_{k;compact \subset X} S_{bal}(K, G).$$

(iv) We say that G is **balayable** when we have $P_{M_0}(G) \subset S_{bal}(G)$.

(v) We say that G satisfies **the continuity principle** if, for any $\mu \in M_0$, the finite continuity of the restriction of $G\mu(x)$ to $S\mu$ implies the finite continuity of $G\mu(x)$ on the whole space X.

For a non-negative Borel function u and a closed set F, **the G-reduced function of u on F** and **the G-reduced function of u on F at infinity δ**, are defined respectively by

$$R_G^F(u)(x) = \inf \{ \ v(x) \ ; \ v \in S(G), \ v(x) \geq u(x) \ G - n.e. \ on \ F \ \},$$

$$R_G^{F,\delta}(u)(x) = \inf_{\omega \in \Omega_0} R_G^{F \cap C\omega}(u)(x),$$

where Ω_0 denotes the totality of all relatively compact open sets in X.

And we write simply $R_G^\delta(u)(x)$ instead of $R_G^{X,\delta}(u)(x)$.

Put, for a closed set F,

$$S_0(F, G) = \{ \ u \in S(G) \ ; \ R_G^{F,\delta}(u)(x) = 0 \text{ G-n.e. on } X \ \}.$$

And write simply $S_0(G)$ instead of $S_0(X, G)$.

Remark 2. When G satisfies the domination principle, the following (1) and (2) hold:

(1) We have $R_G^F(u) \in S(G)$ for any closed set F and for any $u \in S(G)$.

(2) We have $\hat{R}_G^\delta(u) \in S(G)$ for any $u \in S(G)$, where $\hat{R}_G^\delta(u)(x)$ denotes the lower regularization of $R_G^\delta(u)(x)$.

Further we put, for a closed set F,

$$S_0(F, G) = \{ \ u \in S(G) \ ; \ R_G^{F,\delta}(u)(x) = 0 \text{ G-n.e. on } X \ \},$$

and write simply $S_0(G)$ instead of $S_0(X, G)$.

The kernel G is said to be **regular** when we have $P_{M_0}(G) \subset S_0(G)$.

Remark 3. Suppose that G satisfies the complete maximum principle and that, for $\forall \mu \in M_0$, $G\mu(x)$ converges uniformly to 0 at infinity δ, that is, for $\forall \epsilon > 0$ and for $\forall \mu \in M_0$, there exists an $\omega \in \Omega_0$ satisfying $G\mu(x) < \epsilon$ on $C\omega$. Then G becomes regular. Therefore, regularity means a kind of vanishing property of potentials at infinity δ.

Remark 4 (cf. I.Higuchi[6] and M.Itô[13]). Whwn G satisfies the domination principle, the following statements are equivalent:

(1) G is regular.

(2) $P_{E_0}(G) \subset S_0(G)$.

(3) $P_{F_0}(G) \subset S_0(G)$.

(4) $P_{D_0}(G) \subset S_0(G)$.

(5) $\check{G}$ is regular.

Therefore, it suffices to obtain the weakest condition (3) when we show the regulariry of G and we may use the strongest condition (4) when we apply the regularity of G. And the duality of regularity follows from the equivalence of (1) and (5).

Remark 5 (cf. Theorem 1 and Corollary 1).

(1) When G satisfies the domination principle, we have $S_0(G) \subset S_{bal}(G)$.

(2) If G satisfies the domination principle and is **regular**, then G is **balayable**.

Remark 6. The inverse of (2) in Remark 5 does not necessarily hold in general. In fact, there exists an example of continuous function-kernel G such that G is balayabble but not regular (cf. I.Higuchi[6]).

Three notions of the thinness of a closed set F are defined as follows:

(i) F is said to be **G-thin at infinity** δ , if $P_{M_0}(G) \subset S_0(F,G)$ holds.

(ii) F is said to be **G-1-thin at** δ , if $1 \in S_0(F,G)$ holds.

(iii) F is said to be **G-cap-thin at** δ , if $inf_{\omega\in\Omega_0} cap^i_G(F \cap C\omega) = 0$ holds.

Remark 7. When both G and adjoint $\check{G}$ satisfy the complete maximum principle, the implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) can be shown (cf. Theorem 3 and Theorem 4).

In the rest of this paper, we always assume that **every non-empty open set in X is of positive G-inner capacity**.

3. Thinness and balayability

By using the equivalence of the relative domination principle and the relative balayage principle obtained by M.Kishi[15], we can prove the following

Theorem 1. If G satisfies the domination principle, we have

$$S_0(F,G) \subset S_{bal}(F,G).$$

Remark 8. The inverse inclusion relation of Theorem 1 does not necessarily hold in general. But if we suppose that G satisfies the domination principle and that G is regular, then we have $S_0(G) = S_{bal}(G)$. Further, the following (1) $\sim$ (4) are equivalent:

(1) $u \in S_0(G)$.

(2) $u \in S_{bal}(G)$.

(3) $\hat{R}_G^\delta(u) \in S_0(G)$.

(4) $\hat{R}_G^\delta(u) \in S_{bal}(G)$.

Corollary 1 (cf. I.Higuchi[6]). Suppose that $G \prec G$. Then G is balayable if G is regular.

Remark 9. The inverse of Corollary 1 does not correct in general.
In fact, We denote by $N = N(x,y)$ the Newton kernel on $R^n (n \geq 3)$ and by ξ a positive measure such that $N\xi(x)$ is finite continuous on X and that $\int d\xi < +\infty$. The the continuous function-kernel defined by

$$G(x,y) = N(x,y) + N\xi(x)$$

satisfies the domination principle. And we can prove G is balayable but not regular (cf. I.Higuchi[6]). Therefore, the regularity is a stronger property than the balayability.

Remark 10. Similarly we can prove the following propositions concerning the thinness of a closed set at infinity δ :

(1) Let G satisfy the domination principle and F be G-thin at δ . Then, for a G-potential $G\mu(x)$ of any $\mu \in M_0(G)$, there exists a balayaged mesure of $G\mu$ on every closed set contained in F .

(2) Let G satisfy the complete maximum principle and F be G-1-thin at δ . Then there exists the equilibrium measure of every closed set contained in F .

(3) Let G satisfy the maximum principle and F be G-cap-thin at δ . Then we have $cap_G^i(F) < +\infty$.

4. Thinness and strong balayability

Let G be a continuous function kernel on X and F be a closed set in X. We say that G is **strongly balayable on** F, if, for every G-superharmonic function dominated by some G-potential in $P_{M0}(G)$, there exists a balayaged measure on every closed set contained in F .

A continuous function-kernel $G = G(x, y)$ is said to be **non-degenerate** when for any $y_1, y_2 \in X$, $G(x, y_1)$ and $G(x, y_2)$ are not proportional each other.

The following theorem is a characterization of G-thinness of a closed set at infinity δ and is an answer to the inverse problem of (1) in Remark 10.

Theorem 2. Suppose that G satisfies the domination principle and that G is **non-degenerate**. Then, for any compact set F, the following statements are equivalent:

(1) F is **G-thin at infinity δ.**

(2) The following (a) and (b) hold:

(a) G is **strongly balayable on** F .

(b) $\check{G}$ is **balayable on** F .

(3) Both G and $\check{G}$ are **strongly balayable on** F .

The next result is a characterization of G-1-thinness of F at δ and is an answer to the inverse problem of (2) in Remark 10. This theorem asserts that the G-1-thinness at δ is a stronger property than the G-thinness at δ .

Theorem 3. Suppose that G and $\check{G}$ satisfy the complete maximum principle. Then, for any closed set F, the following statements are equivalent each other:

(1) F is **G-1-thin at infinity δ** .

(2) Next (a) and (b) hold:

(a) F is **G-thin at δ** .

(b) **The equilibrium measure γ_F of F** exists.

And further, if both G and $\check{G}$ are **non-degenerate**, then (1) or (2) is equivalent to the following (3) .

(3) Next (c) and (d) hold:

(c) For any bounded G-superharmonic function u, there exists a G-balayaged measure of u on every closed set contained in F .

(d) For any bounded $\check{G}$-superharmonic function u, there exists a $\check{G}$-balayaged measure of u on every closed set contained in F .

The following theorem is a characterization of G-cap-thinness at infinity δ of a closed set and is an answer to the inverse problem of (3) in Remark 10 . And, by this result, we can assert that the G-cap-thinness is a stronger property than the G-1-thinness.

Theorem 4. Suppose that both G and $\check{G}$ satisfy the complete maximum principle. Then the flowing (1) and (2) are equivalent:

(1) F is **G-cap-thin at infinity δ.**

(2) The following 3 statement hold:

(a) $cap_G^i(F) < +\infty$.

(b) F is **G-1-thin at infinity δ** .

(c) F is **$\check{G}$-1-thin at infinity δ** .

And further, if we suppose that both G and $\check{G}$ are **non-degenerate**, the above (1) or (2) is equivalent the following (3):

(3) The above (a) and both of the following (c) and (d) hold:

(c) For any bounded G-superharmonic function u, there exists a G-balayaged measure of u on every closed set contained in F .

(d) For any bounded $\check{G}$-superharmonic function u, there exists a $\check{G}$-balayaged measure of u on every closed set contained in F .

5. Thinness and the vanishing property of potentials

Remark 11 (cf. I.Higuchi[7]). In the case that G is **symmetric** and that G satisfies the complete maximum principle, we have already verified that the following (1) and (2) are equivalent:

(1) G is **regular** .

(2) For $\forall c > 0$ and for $\forall \mu \in F_0(G)$, we have

$$\inf_{\omega \in \Omega_0} cap_G^i \{x \in C\omega \ ; \ G\mu(x) \geq c\} = 0 .$$

Therefore, the regularity is an extension of the property that G-potential converges to 0 in the neighborhood of the point at infinity.

When G is non-symmetric, that is, $G \neq \check{G}$, the circumstance is more complicated. In the rest of this section, We consider the relations between the regularity and the vanishing property of potentials at infinity δ, camparing the three notions of thinness at δ, when the kernel G is non-symmetric.

Theorem 5. Suppose that both G and $\check{G}$ satisfy the complete maximum principle. Then, for any closed set F, the following three statements are equivalent:

(1) F is **G-thin at infinity** δ .

(2) For $\forall c > 0, and\ \ \forall \mu \in M_0$, the set

$$F \cap \{x \in X\ ;\ G\mu(x) \geq c\}$$

is **G-1-thin at** δ .

(3) For $\forall c > 0$, $\forall d > 0$ and $\forall \mu, \forall \nu \in M_0$, the set

$$F \cap \{x \in X\ ;\ G\mu(x) \geq c\} \cap \{x \in X\ ;\ \check{G}\nu(x) \geq d\}$$

is **G-cap-thin at** δ .

Putting $F = X$, we have following main theorem.

Theorem 6. Suppose that both G and $\check{G}$ satisfy the complete maximum principle. Then the following (1), (2) and (3) are equivqalent one anathor:

(1) G is **regular** .

(2) For $\forall c > 0$, and for $\forall \mu \in M_0$, the set

$$\{x \in X\ ;\ G\mu(x) \geq c\}$$

is **G-1-thin at** δ .

(3) For $\forall c > 0$, $\forall d > 0$ and $\forall \mu$, $\forall \nu \in M_0$, the set

$$\{x \in X\ ;\ G\mu(x) \geq c\} \cap \{x \in X\ ;\ \check{G}\nu(x) \geq d\}$$

is **G-cap-thin at** δ.

And further, if we suppose that both G and $\check{G}$ are **non-degenerate** , then the above (1) $\sim$ (3) are equivalent to the following (4):

(4) Both (a) and (b) hold:

(a) G is **strongly balayable**.

(b) $\check{G}$ is **strongly balayable**.

Remark 12. Roughly speaking, Theorem 6 asserts that, when G and $\check{G}$ satisfy the complete maximum principle, G is regular if at most one of the G-potential and $\check{G}$-potential converges to 0 at infinity δ.

Example. let $Y = Y(x, y)$ be a function-kernel on R^2 defined by

$$Y(x,y) = \begin{cases} 1, & x < y, \\ e^{-(x-y)}, & x \geq y \,. \end{cases}$$

This kernel is called the **Yukawa kernel**. It is not so easyt to prove that yhe Yukawa kernel is regular. We can prove both Y and $\check{Y}$ satisfy the complete maximum principle and that the Y-potentials converge to 0 at infinity δ but the $\check{Y}$-potentials do not converge to 0 at δ.

By our Theorem 6, we can assert that both Y and $\check{Y}$ are regular.

参考文献

[1] G.Choquet-J.Deny, Noyaux de convolution et balayage sur tout ouvert, Lecture Notes in Math.,Springer-Verlag, No. 404(1973), 60-112.

[2] J.Deny, Noyaux de convolution de Hunt et noyaux associées à une famille fondamontale, Ann.Inst Fourier, 12(1962), 643-667.

[3] I.Higuchi, Duality of domination principle for non-symmetric lower semi-continuous function-kernels, Hiroshima Math.J., 5(1975), 551-559.

[4] ——, Régularité et propriété de convergence dominée des potentiels d'un noyau-fonction non-symétrique, Séminaire de Théorie du Potentiel, Paris, No. 6, Lecture Notes in Math.,906(1982), 158-202, Springer-Verlag.

[5] ——, Existence de résolvantes associées à un noyau vérifiant le principe de domination, ibid.,No. 6(1982), 203-224.

[6] ——, La balayabilité au sens fort des noyau-fonctions continues du potentiel, Nagoya Math. J.,108(1987), 145-165.

[7] ——, On the thinness of a closed set in the neighborhood of the point at infinity, 京大数理研, 講究録 1016(1997), 136-145.

[8] F.Hirsch, Conditions nécesaires et suffisantes d'existence de résolvantes, Z. Wahrscheinlichkeitstheorie verw. Gebiete, 29(1974), 73-85.

[9] G.A.Hunt, Markoff processes and potentials, Illinois J., Math. t.1(1957),44-94 and 316-369, t.2(1958), 151-215.

[10] M.Itô, Sur le principe relatif de domination pour les noyaux de convolution, Hiroshima Math. J., 5(1975), 293-350.

[11] ——, Positive eigen elements for an infinitesimal generator of a diffusion semi-group and their integral representations, Potential Theory Copenhagen 1979, Lecture Notes in Math., 787(1979), 164-184.

[12] ——, On weakly regular Hunt diffusion kernels, Hokkaido Math. J., 23(1981),Sp., 303-335.

[13] ——, Remarks on the regulariry for continuous function-kernels satisfying the domination principle, Nagoya Univ. preprint series No. 4(1987).

[14] M.Kishi, Inferior limit of a sequence of potentials, Proc. Japan Acad., 33(1957), Sp, 303-335.

[15] ——, Unicity principles in the potential theory, Osaka Math. J., 13(1961), 41-74.

[16] ——, Maximum principles in the potential theory, Nagoya Math. J., 23(1963), 165-187.

[17] ——, Some remarks on the existence of a resolvent, Ann. Inst. Fourier, 25, fasc, 3-4(1975), 345-352.

[18] G.Lion, Familles d'operateur et frontieres en théorie du potentiel, Ann. Inst. Fourier, 16, Fasc. 2(1966), 389-453.

[19] P.A.Meyer, Caractérisation des noyaux potentiels des semi-groupes discret, Ann. Inst. Fourier, 16, Fasc. 2(1966), 225-240.

[20] J.C.Taylor, On the existence of sub-markovian resolvent, Invent. Math. 17(1972), 85-93.

[21] H.Watanabe, 閉集合の平衡分布の存在について, 日本数学会昭和５９年秋季関数論分科会アブストラクト.