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Function spaces and stochastic processes on fractals

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1 Introduction

Since the late 80's of the last century, there has been a lot of development in the mathematical study of stochastic processes and the corresponding operators on fractals (see, for instance, [2], [12], [13], [17]). On the other hand, there has been intensive study of Besov spaces, (which are roughly speaking, fractional extensions of Sobolev spaces) on d-sets, which correspond to regular fractals (see [11], [24], [25]). In this survey paper, we summarize several recent works to connect these two research areas, i.e., functional spaces and stochastic processes.

In Section 2, we first review basic facts in the Dirichlet form theory which connects functional spaces and stochastic processes. We will also give a definition of d-sets, the state space we work on. In Section 3, we discuss on function spaces appear as domains of local regular Dirichlet forms on fractals, whose corresponding generators are so called Laplacians on fractals. The characterization of the domain is possible through heat kernel estimates (3.1) of the Laplacian. In Section 4, we introduce other types of function spaces appear as domains of

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non-local Dirichlet forms on d-sets, whose corresponding processes are stable-like jump processes. Three natural jump-type processes are introduced on d-sets and the corresponding three forms are shown to be equivalent. It turns out that one of the forms (and the corresponding operator) corresponds to the form (and the operator) studied by Triebel in [24]. In Section 5, we will summarize the results on heat kernel estimates for the stable-like jump processes.

Throughout the paper, we only consider compact fractals, but most of the results hold for unbounded fractals with suitable modifications of the statements. We do not give any proof except Theorem 3.2. The proof and further details are given in the references cited.

2 Dirichlet forms and d-sets

In this section, we will briefly review the definition of Dirichlet forms and the correspondence to processes following [6]. We will also introduce d-sets.

Let X be a locally compact separable metric space and ν de a positive Radon measure on X whose support is X. Let $\mathcal{E}$ be a symmetric bilinear closed form on $\mathbf{L}^2(X,\nu)$ with domain $\mathcal{F}$. $(\mathcal{E},\mathcal{F})$ is called a Dirichlet form if it is Markovian, i.e., for each $u \in \mathcal{F}$, $v := (0 \vee u) \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(v,v) \le \mathcal{E}(u,u)$. A Dirichlet form $(\mathcal{E},\mathcal{F})$ is regular if there exists $C \subset \mathcal{F} \cap C_0(X)$ such that C is dense in $\mathcal{F}$ with $\mathcal{E}_1$-norm and C is dense in $C_0(X)$ under the uniform norm, where $C_0(X)$ is a space of continuous compact supported functions on X and $\mathcal{E}_1(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + \|\cdot\|^2_{\mathbf{L}^2}$. $(\mathcal{E},\mathcal{F})$ is local if for each $u, v \in \mathcal{F}$ whose supports are disjoint compact sets, $\mathcal{E}(u,v) = 0$. There is a one to one correspondence between a regular Dirichlet form on $\mathbf{L}^2(X,\nu)$ and a ν-symmetric Hunt process (i.e., a strong Markov process whose paths are right continuous and quasi-left continuous w.r.t. some filtration) on X except for some exceptional set of starting points. Further, if the regular Dirichlet form is local, then the corresponding process is a diffusion process (i.e. Hunt process with continuous paths).

We next introduce our state space. Let G be a compact d-set in $\mathbf{R}^n$ $(n \geq 2, 0 < d \leq n)$. That is, $G \subset \mathbf{R}^n$ and there exists $c_{2.1}, c_{2.2} > 0$ such that

$$c_{2.1} r^d \leq \mu(B(x,r)) \leq c_{2.2} r^d \qquad \text{for all } \ x \in G, \ 0 < r < 1, \tag{2.1}$$

where $B(x,r)$ is a ball centered at x and radius r w.r.t. the Euclidean metric. Thus d is the Hausdorff dimension of G and μ the Hausdorff measure on G. We normalize the size of G so that the diameter of G is 1.

3 Lipschitz spaces and domains of Dirichlet forms

For several sub-classes of d-sets, diffusion processes, corresponding Laplace operators and Dirichlet forms have been studied extensively (see [2], [12], [13], [17] etc). The most typical example is the Sierpinski gasket which we will define later. In this section, we will consider a class of d-sets which has a *fractional diffusion* in the sense of Barlow [2] and show that the domain of the Dirichlet form is the Lipschitz space. We first give a definition of the fractional diffusion.

Definition 3.1 *Let (G, ρ) be a complete compact metric space. (G, ρ) is called a* fractional metric space *and $\{B_t^G\}_{t \geq 0}$ is called a* fractional diffusion *if the following holds.*

1) ρ has the midpoint property; for each $x, y \in G$, there exists $z \in G$ such that $\rho(x,y) = \rho(x,z)/2 = \rho(z,y)/2$. Further, there exists a Borel measure μ which satisfies (2.1) w.r.t. ρ.

2) $\{B_t^G\}_{t \geq 0}$ is a μ-symmetric conservative Feller diffusion on G which has a sym-

metric jointly continuous transition density (fundamental solution of the heat equation) $p_t(x,y)$ $(t>0, x,y\in G)$ *satisfying the following estimate,*

$$\begin{aligned} & c_{3.1}t^{-d_s/2}\exp(-c_{3.2}(\rho(x,y)^{d_w}t^{-1})^{1/(d_w-1)}) \le p_t(x,y) \\ \le\ & c_{3.3}t^{-d_s/2}\exp(-c_{3.4}(\rho(x,y)^{d_w}t^{-1})^{1/(d_w-1)}) \quad \textit{for all } \ 0<t<1,\ x,y\in G, \end{aligned} \tag{3.1}$$

with some constants $d_s>0, d_w\ge 2$.

We note that in the original definition of the fractional diffusions in [2], G is not necessarily compact. For simplicity, we further assume that $\rho(\cdot,\cdot)$ is equivalent to the Euclidean metric, i.e.,

$$c_{3.5}|x-y|\le\rho(x,y)\le c_{3.6}|x-y| \qquad \text{for all} \quad x,y\in G. \tag{3.2}$$

In this case, $d_s/2=d/d_w$ holds.

Example: Sierpinski gasket

Let D_n be a n-dimensional simplex whose vertices are $\{p_0,p_1,\cdots,p_n\}$. For $i=1,2,\cdots,n+1$, let $F_i(z)=(z-p_i)/2+p_i$, $z\in\mathbf{R}^n$. Then, there exists a unique non-void compact set G such that $G=\cup_{i=1}^{n+1}F_i(G)$. This G is called a (n*-dimensional*) *Sierpinski gasket.* It is known that the Sierpinski gasket has a fractional diffusion with $d=\log(n+1)/\log 2$, $d_s=2\log(n+1)/\log(n+3)$, $d_w=\log(n+3)/\log 2$. (Note that $d_s<2$ for this example.)

In general, (affine) nested fractals, (which is a class of fractals including Sierpinski gaskets) and Sierpinski carpets have fractional diffusions with $c_1|x-y|^{d_c}\le\rho(x,y)\le c_2|x-y|^{d_c}$ for some $d_c\ge 1$ instead of (3.2).

We now introduce Lipschitz spaces. For $1\le p<\infty$, $1\le q\le\infty$, $\beta\ge 0$ and $m\in\mathbf{N}\cup\{0\}$, set

$$a_m(\beta,f):=L^{m\beta}(L^{md}\int\int_{|x-y|<c_0L^{-m}}|f(x)-f(y)|^p d\mu(x)d\mu(y))^{1/p} \quad f\in\mathbf{L}^p(G,\mu),$$

where $1 < L < \infty$, $0 < c_0 < \infty$. Define a *Lipschitz space* $\mathrm{Lip}(\beta, p, q)(G)$ as a set of $f \in \mathbf{L}^p(G, \mu)$ such that $\bar{a}(\beta, f) := \{a_m(\beta, f)\}_{m=0}^{\infty} \in l^q$. $\mathrm{Lip}(\beta, p, q)(G)$ is a Banach space with the norm $\|f\|_{\mathrm{Lip}} := \|f\|_{\mathbf{L}^p} + \|\bar{a}(\beta, f)\|_{l^q}$. Note that the Lipschitz space is determined independently of the choice of L and c_0 as long as the former is greater than 1 and the latter is positive.

Theorem 3.2 ([9], [14], [19], [8]) *Let $(\mathcal{E}, \mathcal{F})$ be a local regular Dirichlet form on G which corresponds to a fractional diffusion. Then, the following holds.*

$$\mathcal{F} = Lip(\frac{d_w}{2}, 2, \infty)(G).$$

PROOF. We will follow the argument in [19]. We first prove $\mathcal{F} \subset \mathrm{Lip}$. For $f \in \mathbf{L}^2(G, \mu)$, let $\mathcal{E}_t(f, f) := (f - P_t f, f)_{\mathbf{L}^2}/t$, where P_t is a semigroup corresponding to $(\mathcal{E}, \mathcal{F})$. Then,

$$\begin{aligned}
\mathcal{E}_t(f, f) &= \frac{1}{2t} \int\int_{G \times G} (f(x) - f(y))^2 p_t(x, y) \mu(dx) \mu(dy) \\
&\geq \frac{1}{2t} \int\int_{|x-y| \leq c_0 t^{1/d_w}} (f(x) - f(y))^2 p_t(x, y) \mu(dx) \mu(dy) \\
&\geq \frac{c_1}{2t} \int\int_{|x-y| \leq c_0 t^{1/d_w}} t^{-d_s/2} (f(x) - f(y))^2 \mu(dx) \mu(dy), \qquad (3.3)
\end{aligned}$$

where we use the lower bound of (3.1) in the last inequality. Taking $t = L^{-m d_w}$ and using the fact $d_s/2 = d/d_w$, we see that (3.3) is equal to $c_1 a_m(d_w/2, f)^2$. It is well known that $\mathcal{E}_t(f, f) \nearrow \mathcal{E}(f, f)$ as $t \downarrow 0$ ([6], Lemma 1.3.4). We thus obtain $\sup_m a_m(d_w/2, f)^2 \leq c_2 \sqrt{\mathcal{E}(f, f)}$ and the result holds.

We next prove $\mathcal{F} \supset \mathrm{Lip}$. Set $\gamma = 1/(d_w - 1)$. Since the diameter of G is 1, we have for each $g \in \mathrm{Lip}$,

$$\begin{aligned}
\mathcal{E}_t(g, g) &= \frac{1}{2t} \int\int_{\substack{x, y \in G \\ |x-y| \leq 1}} (g(x) - g(y))^2 p_t(x, y) \mu(dx) \mu(dy) \\
&\leq \frac{1}{2t} \sum_{m=1}^{\infty} c_3 t^{-d_s/2} e^{-c_4 (t L^{m d_w})^{-\gamma}} \int\int_{L^{-m} < |x-y| \leq L^{-m+1}} (g(x) - g(y))^2 \mu(dx) \mu(dy)
\end{aligned}$$

$$\leq \; c_3 t^{-(1+d_s/2)} \sum_{m=1}^{\infty} e^{-c_4 (tL^{md_w})^{-\gamma}} L^{-m(d_w+d)} a_{m-1}(d_w/2, g)^2, \tag{3.4}$$

where we use the upper bound of (3.1) in the first inequality. For $0 < t$ and $0 \leq x$, let $\Phi_t(x) = e^{-c_4 (tL^{xd_w})^{-\gamma}} L^{-x(d_w+d)}$. By elementary calculation, we see that $\Phi_t(0) > 0$, $\lim_{x\to\infty} \Phi_t(x) = 0$ and $\int_0^\infty \Phi_t(x)dx = c_5 t^{1+d_s/2}$. Further, there exists $x_t > 0$ such that $\Phi_t(x)$ is increasing for $0 \leq x < x_t$, decreasing for $x_t < x < \infty$ and $\Phi_t(x_t) = c_6 t^{1+d_s/2}$. Thus, $\sum_{m=1}^{\infty} \Phi_t(m) \leq \int_0^\infty \Phi_t(x)dx + 2\Phi_t(x_t) \leq c_7 t^{1+d_s/2}$. Since (3.4) is less than or equal to $c_3 t^{-(1+d_s/2)} \|g\|_{\mathrm{Lip}}^2 \sum_{m=1}^{\infty} \Phi_t(m)$, we conclude that $\sup_{t>0} \mathcal{E}_t(g,g) = \lim_{t\to 0} \mathcal{E}_t(g,g) \leq c_8 \|g\|_{\mathrm{Lip}}^2$ and the result holds. ∎

4 Dirichlet forms and jump type processes on d-sets

In [15], three natural non-local regular Dirichlet forms are introduced, whose corresponding processes are stable-like jump type processes on compact d-sets. We will survey the results here.

4.1 Jump process as a Besov space on a d-set

We first introduce Besov spaces on G and their trace theory within the scope of our use (see [11], [24] etc. for details).

For $0 < \alpha < 1$, we introduce a Besov space $B_\alpha^{2,2}(G)$ as follows,

$$\|u|B_\alpha^{2,2}(G)\| \;=\; \|u\|_{L^2(G,\mu)} + \Big(\int\int_{G\times G} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\alpha}} \mu(dx)\mu(dy)\Big)^{1/2} \tag{4.1}$$

$$B_\alpha^{2,2}(G) \;=\; \{u : u \text{ is measurable}, \|u|B_\alpha^{2,2}(G)\| < \infty\}. \tag{4.2}$$

In [11], it is shown that for $0 < \alpha < 1$, $B_\alpha^{2,2}(G) = \mathrm{Lip}(\alpha, 2, 2)(G)$ and the two norms are equivalent (Chapter V, Proposition 3).

For each $f \in \mathbf{L}^1_{loc}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$, define

$$Rf(x) = \lim_{r \downarrow 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)dy,$$

if the limit exists, where m is the Lebesgue measure in $\mathbf{R}^n$. It is well-known that the limit exists quasi-everywhere (i.e. except a set of zero capacity) in $\mathbf{R}^n$ with respect to the Newtonian capacity if $n \geq 3$ or logarithmic capacity if $n = 2$ and coincides with $f(x)$ almost everywhere in $\mathbf{R}^n$. For each $\beta > 0$, denote by $B^{2,2}_\beta(\mathbf{R}^n)$ the classical Besov space on $\mathbf{R}^n$ (see Remark 4.2 below for its definition). The following trace theorem plays an important role in the study of Besov spaces on d-sets (see, for instance, Chapters V and VI in [11] or Section 20 in [24]).

Proposition 4.1 *For $0 < s < 1$, the trace operator $Tr_G : f \mapsto Rf$ is a bounded linear surjection from $B^{2,2}_{s+(n-d)/2}(\mathbf{R}^n)$ onto $B^{2,2}_s(G)$ and it has a bounded linear right inverse operator E_G (which is called the extension operator in literature) so that $Tr_G \circ E_G$ is the identity map on $B^{2,2}_s(G)$.*

Remark 4.2 Note that for $\beta > 0$ with integer $k < \beta \leq k+1$, the classical Besov space $B^{2,2}_\beta(\mathbf{R}^n)$ is defined to be

$$B^{2,2}_\beta(\mathbf{R}^n) = \left\{ u \in C^k(\mathbf{R}^n) : \|u\|_{B^{2,2}_\beta} := \sum_{0 \leq |j| \leq k} \|D^j u\|_2 + \sum_{|j|=k} \left(\int_{\mathbf{R}^n} \frac{\|\Delta_h D^j f\|_2^2}{|h|^{n+2(\beta-k)}} dh \right)^{1/2} < \infty \right\}$$

where for $j = (j_1, j_2, \cdots, j_n) \in \mathbf{Z}^n_+$, $|j| = \sum_{k=1}^n j_k$ and $D^j = \frac{\partial^{|j|}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}$, Δ_h is the difference operator so that for $h \in \mathbf{R}^n$, $(\Delta_h f)(x) = f(x+h) - f(x)$, and $\|\cdot\|_2$ denotes the $\mathbf{L}^2$-norm in $\mathbf{L}^2(\mathbf{R}^n, m)$ (see, for instance, section I.1.5 in [11]). It is known (cf. Section V.1.1 in [11]) that when $0 < \beta < 1$, the norm $\|u\|_{B^{2,2}_\beta}$ is equivalent to $\left\| u \,|\, B^{2,2}_\beta(\mathbf{R}^n) \right\|$ defined by (4.2) with $G = \mathbf{R}^n$, and therefore $B^{2,2}_\beta(\mathbf{R}^n)$ is the same as the space defined by (4.1) with $G = \mathbf{R}^n$. Furthermore, the space $B^{2,2}_{\beta}(\mathbf{R}^n)$ coincides with the classical Bessel potential space on $\mathbf{R}^n$ (also called the fractional Sobolev space or the Liouville space); see, for instance, p. 8 in Section I.1.5 of [11].

Now, for $0 < \alpha < 1$ and $u, v \in B^{2,2}_{\alpha}(G)$, define

$$\mathcal{E}_{Y^{(\alpha)}}(u,v) = \int\int_{G\times G} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2\alpha}} \mu(dx)\mu(dy).$$

By standard properties of Besov spaces, it is easy to check that $(\mathcal{E}_{Y^{(\alpha)}}, B^{2,2}_{\alpha}(G))$ is a regular Dirichlet space on $\mathbf{L}^2(G,\mu)$ (a detailed proof is given, for instance, in Theorem 3 of [21]). We denote $\{Y_t^{(\alpha)}\}_{t\geq 0}$ the corresponding Hunt process on G. We note that when $G = \mathbf{R}^n$, this is a (2α)-stable process on $\mathbf{R}^n$.

4.2 Jump process as a subordination of a diffusion

In this subsection, we assume that there exists a fractional diffusion on G.

For $0 < \alpha < 1$, let $\{\xi_t\}_{t>0}$ be the strictly α-stable subordinator, i.e., it is a one dimensional non-negative Lévy process independent of $\{B_t^G\}_{t\geq 0}$ with the generating function $E[\exp(-u\xi_t)] = \exp(-tu^{\alpha})$. Let $\{\eta_t(u) : t > 0, u \geq 0\}$ be the distribution density of $\{\xi_t\}_{t>0}$. Using $p_t(x,y)$ in (3.1), we define

$$q_t(x,y) = \int_0^{\infty} p_u(x,y)\eta_t(u)du \qquad \text{for all } \ t > 0, \ x, y \in G.$$

Then, by a general theory, $q_t(x,y)$ is a transition density of some Markov process which we denote by $\{X_t^{(\alpha)}\}_{t\geq 0}$, called the subordinate process (see [3], [20]). In our case, $\{X_t^{(\alpha)}\}_{t\geq 0}$ is a μ-symmetric Hunt process and we denote the corresponding Dirichlet form on $\mathbf{L}^2(G,\mu)$ as $(\mathcal{E}_{X^{(\alpha)}}, \mathcal{F}_{X^{(\alpha)}})$.

Remark 4.3 *The argument here can be extended to a class of diffusions wider than fractional diffusions. Indeed, by checking the proof of* [15], [21] *carefully,* *we see that the same results hold for diffusions whose transition densities satisfy estimates similar to (3.1), but with different orders* $0 < \gamma_1, \gamma_2 < \infty$ *on the shoulders of* $\rho(x,y)^{d_w} t^{-1}$ *(instead of* $1/(d_w - 1)$*). If we weaken the condition as above, then we can include all diffusions on p.c.f. self-similar sets (which roughly corresponds to finitely ramified fractals) as mentioned in* [16].

4.3 Jump process as a time change of a stable process on $\mathbf{R}^n$

We first briefly give a result by Triebel. In [24], Triebel define a Besov space on d-set G as follows.

$$\begin{aligned} \|u|\hat{B}^{2,2}_\alpha(G)\| &= \inf_{Tr_G g = u} \|g|B_{\alpha+(n-d)/2}(\mathbf{R}^n)\|, \\ \hat{B}^{2,2}_\alpha(G) &= Tr_G\, B_{\alpha+(n-d)/2}(\mathbf{R}^n), \end{aligned}$$

where $\alpha > 0$. Here Tr_G is the trace operator given in Proposition 4.1 and the norm of $B_{\alpha+(n-d)/2}(\mathbf{R}^n)$ is the one given in Remark 4.2. In general, this Besov space is different from the one defined by Jonsson-Wallin ([11]), but it is known that for $0 < \alpha < 1$, the two spaces coincide and the two norms are equivalent. Let H_α be the corresponding self-adjoint operator on G so that

$$(H_\alpha^{1/2} u, H_\alpha^{1/2} u)_{\mathbf{L}^2(G,\mu)} = \|u|\hat{B}^{2,2}_\alpha(G)\|^2, \quad \text{Dom}\,(H_\alpha^{1/2}) = \hat{B}^{2,2}_\alpha(G).$$

Theorem 4.4 ([24]; Theorem 25.2) *H_α is a positive definite self-adjoint operator on $\mathbf{L}^2(G,\mu)$ with pure point spectrum. Let μ_k be its k-th eigenvalue (including multiplicities). Then there exist $c_{4.1}, c_{4.2} > 0$ such that the following holds.*

$$c_{4.1} k^{2\alpha/d} \le \mu_k \le c_{4.2} k^{2\alpha/d} \qquad \text{for all} \qquad k \in \mathbf{N}.$$

We now give the third jump process on d-sets ([15]). It turns out that the corresponding operator of the process is the one given by Triebel when $0 < \alpha < 1$.

We construct a jump process though a time change of (2α)-stable process on $\mathbf{R}^n$. Let G be a d-set on $\mathbf{R}^n$ with a Radon measure μ which satisfies (2.1). Also, let $\{B_t^{(\alpha)}\}_{t\geq 0}$ $(0<\alpha\leq 1)$ be a rotation invariant (2α)-stable process on $\mathbf{R}^n$ (when $\alpha=1$ it is a Brownian motion). Then, it is proved in [15] that when $2\alpha > n-d$, then μ is a smooth measure w.r.t. $\{B_t^{(\alpha)}\}_t$, i.e. μ charges no set of zero capacity w.r.t. the form corresponding to $\{B_t^{(\alpha)}\}_t$. Thus, by a general theory (see [6]), there exists a unique positive continuous additive functional $\{A_t^{(\alpha)}\}_{t\geq 0}$ which is in Revuz correspondence with μ (thus, $A_t^{(\alpha)}$ increases only when $B_t^{(\alpha)}\in G$). Set $\tau_t=\inf\{s>0 : A_s^{(\alpha)}>t\}$ and define $Z_t^{(\alpha)}=B_{\tau_t}^{(\alpha)}$. Then, again by a general theory, $\{Z_t^{(\alpha)}\}_{t\geq 0}$ is a μ-symmetric jump process whose corresponding regular Dirichlet form we denote by $(\mathcal{E}_{Z^{(\alpha)}},\mathcal{F}_{Z^{(\alpha)}})$. Since the corresponding form is a trace of the Dirichlet form of the (2α)-stable process on $\mathbf{R}^n$ (see Theorem 6.2.1 in [6]), we can check that the corresponding operator is $H_{\alpha-(n-d)/2}$ given above.

4.4 Comparison of the forms and heat kernel bounds

Define $\bar{\alpha}=\alpha d_w/2$ and $\hat{\alpha}=\alpha-(n-d)/2$. We then have the following.

Proposition 4.5 ([21],[15])
Let G be a d-set. For $(n-d)/2<\alpha<1$ or $\alpha=1, n-2<d<n$,

$$c_{4.3}\mathcal{E}_{Y^{(\hat{\alpha})}}(f,f)\leq \mathcal{E}_{Z^{(\alpha)}}(f,f)\leq c_{4.4}\mathcal{E}_{Y^{(\hat{\alpha})}}(f,f)\quad \textit{for all}\quad f\in \mathbf{L}^2(G,\mu). \tag{4.3}$$

Assume further that there exists a fractional diffusion on G. For $0<\alpha<1$,

$$c_{4.5}\mathcal{E}_{Y^{(\bar{\alpha})}}(f,f)\leq \mathcal{E}_{X^{(\alpha)}}(f,f)\leq c_{4.6}\mathcal{E}_{Y^{(\bar{\alpha})}}(f,f)\quad \textit{for all}\quad f\in \mathbf{L}^2(G,\mu). \tag{4.4}$$

In particular, under the conditions,

$$\mathcal{F}_{Z^{(\alpha)}}=B_{\hat{\alpha}}^{2,2}(G),\quad \mathcal{F}_{X^{(\alpha)}}=B_{\bar{\alpha}}^{2,2}(G).$$

Remark 4.6 *1) In [15], (4.4) is stated for $0 < \alpha < 2/d_w$. But the proof there shows that it actually holds for $0 < \alpha < 1$. From this fact, we see that when G is a d-set on which there exists a fractional diffusion, $(\mathcal{E}_{Y^{(\alpha)}}, B^{2,2}_{\alpha}(G))$ is a regular Dirichlet form for $0 < \alpha < d_w/2$.*

2) Note that in general the three-type Dirichlet forms introduced are different and the corresponding processes cannot be obtained by time changes of others by positive continuous additive functionals (see [15]).

5 Heat kernel estimates for stable-like processes on d-sets

In [4], detailed estimates of heat kernels for $\{Y^{(\alpha)}\}$ are obtained. There exists a non-negative bounded heat kernel $p_t(x,y)$ on $(t,x,y) \in (0,\infty) \times G \times G$ with

$$P_t^{Y^{(\alpha)}} f(x) = \int_G p_t(x,y) f(y) \mu(dy) \qquad \text{for all} \quad x \in G, \ f \in \mathbf{L}^2(G,\mu),$$

(where $P_t^{Y^{(\alpha)}}$ is the heat semigroup w.r.t. $\mathcal{E}_{Y^{(\alpha)}}$), satisfying the following.

Theorem 5.1 ([4]) *For $0 < \alpha < 1$, the following holds.*

1) For all $x, y \in G$, $\ 0 < t < 1$,

$$c_{5.1}\left(t^{-\frac{d}{2\alpha}} \wedge \frac{t}{|x-y|^{d+2\alpha}}\right) \le p_t(x,y) \le c_{5.2}\left(t^{-\frac{d}{2\alpha}} \wedge \frac{t}{|x-y|^{d+2\alpha}}\right).$$

2) There are constants $c_{5.3} > 0$ and $\beta > 0$ such that for any $0 < t, s < 1$ and $(x_i, y_i) \in G \times G$ with $i = 1, 2$,

$$|p_s(x_1,y_1) - p_t(x_2,y_2)| \le c_{5.3}\,(t \wedge s)^{-\frac{d+\beta}{2\alpha}} \left(|t-s|^{\frac{1}{2\alpha}} + |x_1 - x_2| + |y_1 - y_2|\right)^{\beta}.$$

Theorem 5.2 ([4]) *For every $x \in F$, $\mathbf{P}^x$-a.s., the Hausdorff dimension of $Y[0,1] := \{Y_t : 0 \le t \le 1\}$ is $d \wedge (2\alpha)$.*

Note that above theorems hold under a wider framework, i.e. when the form is expressed as

$$\mathcal{E}(f,f) = \int\int_{G\times G}(f(x)-f(y))^2 n(x,y)d\mu(x)d\mu(y),$$

where $n(x,y)$, $x,y \in G$ is a jointly measurable function such that $n(x,y) = n(y,x)$ for all $x,y \in G$ and satisfies

$$\frac{c_{5.4}}{|x-y|^{d+2\alpha}} \le n(x,y) \le \frac{c_{5.5}}{|x-y|^{d+2\alpha}}.$$

In [4], it is also proved that all the parabolic functions (two variable functions which satisfy the heat equation) satisfy the parabolic Harnack inequality.

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