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確定特異点型の可換微分作用素系と完全積分可能量子系

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1. Introduction

Consider the Shrödinger operator

$$P = \sum_{j=1}^{n} \partial_j^2 + R(x), \tag{1.1}$$

where
$(x_1, \ldots, x_n)$: natural coordinate of $\mathbb{R}^n$ (or $\mathbb{C}^n$),
$\partial_j = \dfrac{\partial}{\partial x_j}$ $(j = 1, \ldots, n)$.
Problem: Study P with a commuting differential operator

$$Q = \sum_{1 \le i < j \le n} \partial_i^2 \partial_j^2 + \text{(lower order)}, \tag{1.2}$$

i.e.
$$PQ = QP.$$

We have the following interesting examples of such P.

Example 1.1. 1) Equations satisfied by zonal spherical functions.
an extension of the root multiplicity to a continuous parameters by J. Sekiguchi (type A, [Sj]),
Heckman-Opdam's hypergeometric equations (type BC etc. [HO]).
2) (cf. [OP1], [OP2]) Calogero-Moser, Sutherland systems (completely integrable systems),
one dimensional (quantum) n-body problems,
completely integrable quantum systems $\longrightarrow$ classical integrable systems.
3) Equations satisfied by a Whittaker vector.
4) Toda finite chains (associated to (extended) Dynkin diagrams).

In this note we will study
Problem. Classify P (and Q and higher order commuting operators)!.

Note that this problem is solved as follows when P is B_n-invariant, i.e. $R(x)$ is symmetric with respect to the coordinate $(x_1, \ldots, x_n)$ and even for any coordinate x_i for $i = 1, \ldots, n$.
Invariant case ([OOS], P is B_n-invariant.):
<u>Case $n \ge 3$</u> (Ellip-B_n, [OS]):

$$\begin{aligned} R(x) &= \sum_{1 \le i < j \le n} \big(u(x_i + x_j) + u(x_i - x_j)\big) + \sum_{k=1}^{n} v(x_k), \\ u(t) &= C_5 \wp(x) + C_6, \\ v(t) &= \frac{\text{a polynomial of } \wp \text{ with degree } \le 4}{(\wp')^2} \\ &= \sum_{i=1}^{4} C_i \wp(t + \omega_i) + C_0 \quad \text{if } \omega_1 \text{ and } \omega_2 \text{ are finite.} \end{aligned} \tag{1.3}$$

Here $\wp(x)$ is the Weierstrass elliptic function with fundamental periods $2\omega_1$ and $2\omega_2$ which may be infinite (cf. [WW]):

$$\wp(z) = \wp(z; 2\omega_1, 2\omega_2) = \frac{1}{z^2} + \sum_{\omega \neq 0} \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \Big),$$

$$(1.4) \qquad \wp(z; \sqrt{-1}\lambda^{-1}\pi, \infty) = \lambda^2 \sinh^{-2} \lambda z + \frac{1}{3}\lambda^2,$$

$$\wp(z; \infty, \infty) = z^{-2},$$

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3),$$

$$e_\nu = \wp(\omega_\nu) \text{ for } \nu = 1,2,3,\ \omega_3 = -\omega_1 - \omega_2 \text{ and } \omega_4 = 0,$$

Case $n = 2$ ([OO]): the answer is more complicated:

(1.5)
$$\begin{cases} \text{The same solution as above (Ellip-}B_2\text{, 5 parameters),} \\ \text{Its } \textit{dual} \text{ (cf. Lemma 2.7, Ellip}^d\text{-}B_2\text{, 5 parameters),} \\ \text{A special self-dual solution (Ellip-}B_2\text{-S, 4 parameters) :} \\ \begin{cases} u(t) = A_2 \dfrac{(\wp(\frac{t}{2}) - e_3)^2}{\wp'(\frac{t}{2})^2} + A_1\wp(t) + A_0, \\ v(t) = \dfrac{C_2\wp(t)^2 + C_1\wp(t) + C_0}{\wp(t) - e_3}. \end{cases} \end{cases}$$

2. Toric coordinate

Introducing the coordinates

$$(2.1) \qquad t_j = e^{-(x_j - x_{j+1})} \quad (j = 1, \ldots, n-1), \quad t_n = e^{-x_n},$$

we assume the following in this section.

Assumption 2.1. $R(x)$ is locally defined and meromorphic at $t = (t_1, \ldots, t_n) = 0$.

Definition 2.2. If $R(x)$ is holomorphic at $t = 0$, P is said to have a *regular singularity* at $t = 0$.

Remark 2.3. i) Heckman-Opdam's hypergeometric system has a regular singularity at every infinite point. In fact, this property characterizes Heckman-Opdam's system (cf. Theorem 2.6).

ii) The equations satisfied by a Whittaker vector have a regular singularity at an infinite point.

The root system $\Sigma = \Sigma(B_n)$ of type B_n is realized in $\mathbb{R}^n$ by

$$(2.2) \qquad \begin{cases} \Sigma(A_{n-1})^+ &= \{e_i - e_j;\ 1 \le i < j \le n\}, \\ \Sigma(D_n)^+ &= \{e_i \pm e_j;\ 1 \le i < j \le n\}, \\ \Sigma(B_n)_S^+ &= \{e_k;\ 1 \le k \le n\}, \\ \Sigma(B_n)^+ &= \Sigma(D_n)^+ \cup \Sigma(B_n)_S^+, \\ \Sigma(B_n) &= \{\alpha, -\alpha;\ \alpha \in \Sigma(B_n)^+\}. \end{cases}$$

We will use the following notation.

$$(\partial_v\phi)(x) = \frac{d\phi(x+tv)}{dt}\Big|_{t=0} \quad \text{for } v \in \mathbb{R}^n,$$

$$(2.3) \qquad w_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{\langle \alpha, \alpha\rangle}\alpha,$$

$$W(B_n) = \text{the Weyl group of } \Sigma(B_n) \text{ generated by } w_\alpha \quad (\alpha \in \Sigma(B_n)).$$

Let $F \subset \Sigma(B_n)$.

W_F: The subgroup of $W(B_n)$ generated by w_α $(\alpha \in F)$.

$\bar{F} := W_F F$, which we call the root system generated by F.

Lemma 2.4. $R(x) = \sum_{\alpha\in\Sigma(B_n)^+} u_\alpha(\langle\alpha, x\rangle)$ *with functions* u_α *of one variable.*

Put $u_{-\alpha}(t) = u_\alpha(-t)$ for $\alpha \in \Sigma(B_n)^+$ and define

$$\Delta = \{\alpha \in \Sigma(B_n)^+;\ u'_\alpha \neq 0\},$$

$$\bar{\Delta} = \bar{\Delta}_1 \cup \cdots \cup \bar{\Delta}_N \quad \text{(irreducible decomposition)}, \tag{2.4}$$

$$\Delta_j = \bar{\Delta}_j \cap \Delta.$$

Remark 2.5. $\mathbb{C}[\partial_1 + v_1(x_1), \ldots, \partial_n + v_n(x_n)]$ is commutative for any $v_j(t)$ $(j = 1, \ldots, n)$.

Theorem 2.6. *Under* Assumption 2.1 *the potential function* $\sum_{\alpha\in\Delta_j} u_\alpha(\langle\alpha, x\rangle)$ *is a transformation (by a translation + by* $W(B_n)$*) of one of the following functions.*
Δ_m is of Type B_m, D_m or A_{m-1} with $m \geq 3$:
There exist the following $5+1$ *cases.*

(Trig-B_m): **Trigonometric potential of type B_m**

$$C_0 \sum_{1\leq i<j\leq m} \left(\sinh^{-2}\lambda(x_i + x_j) + \sinh^{-2}\lambda(x_i - x_j)\right)$$
$$+ \sum_{k=1}^{m} \left(C_1 \sinh^{-2} 2\lambda x_k + C_2 \sinh^{-2}\lambda x_k + C_3 \cosh 2\lambda x_k + C_4 \cosh 4\lambda x_k\right),$$

· $C_3 = C_4 = 0 \Rightarrow$ (Trig-B_m-reg): *Heckman-Opdam's hypergeometric system*

(Trig-A_{m-1}-bry): **Trigonometric potential of type A_{m-1} with boundary terms**

$$\sum_{1\leq i<j\leq m} C_0 \sinh^{-2}\lambda(x_i - x_j)$$
$$+ \sum_{k=1}^{m} \left(C_1 e^{-2\lambda x_k} + C_2 e^{-4\lambda x_k} + C_3 e^{2\lambda x_k} + C_4 e^{4\lambda x_k}\right),$$

· $C_3 = C_4 = 0 \Rightarrow$ (Trig-A_{m-1}-bry-reg):Trigonometric potential of type A_{m-1} *with boundary terms and with regular singularity*

· $C_1 = C_2 = C_3 = C_4 = 0 \Rightarrow$ (Trig-A_{m-1}):Trigonometric potential of type A_{m-1}

(Toda-$B_m^{(1)}$-bry): **Toda potential of type $B_m^{(1)}$ with boundary terms**

$$C_0 \sum_{i=1}^{m-1} e^{-2\lambda(x_i - x_{i+1})} + C_0 e^{-2\lambda(x_{m-1}+x_m)} + C_1 e^{2\lambda x_1} + C_2 e^{4\lambda x_1}$$
$$+ C_3 \sinh^{-2}\lambda x_m + C_4 \sinh^{-2} 2\lambda x_m,$$

· $C_3 = C_4 = 0 \Rightarrow$ (Toda-$B_m^{(1)}$):Toda potential of type $B_m^{(1)}$

· $C_1 = C_2 = 0 \Rightarrow$ (Toda-D_m-bry):Toda potential of type D_m *with boundary terms*

· $C_1 = C_2 = C_3 = C_4 = 0 \Rightarrow$ (Toda-D_m):Toda potential of type D_m

(Toda-$C_m^{(1)}$): **Toda potential of type $C_m^{(1)}$**

$$C_0 \sum_{i=1}^{m-1} e^{-2\lambda(x_i - x_{i+1})} + C_1 e^{2\lambda x_1} + C_2 e^{4\lambda x_1} + C_3 e^{-2\lambda x_m} + C_4 e^{-4\lambda x_m},$$

· $C_1 = C_2 = 0 \Rightarrow$ (Toda-BC_m):Toda potential of type BC_m

· $C_1 = C_2 = C_3 = C_4 = 0 \Rightarrow$ (Toda-A_{m-1}): Toda potential of type A_{m-1}

(Toda-$D_m^{(1)}$-bry): **Toda potential of type $D_m^{(1)}$ with boundary terms**

$$C_0 \sum_{i=1}^{m-1} \left(e^{-2\lambda(x_i - x_{i+1})} + e^{-2\lambda(x_{m-1}+x_m)} + e^{2\lambda(x_1+x_2)}\right)$$
$$+ C_1 \sinh^{-2} \lambda x_m + C_2 \sinh^{-2} 2\lambda x_m + C_3 \sinh^{-2} \lambda x_1 + C_4 \sinh^{-2} 2\lambda x_1,$$

· $C_1 = C_2 = C_3 = C_4 = 0 \Rightarrow$ (Toda-$D_m^{(1)}$): Toda potential of type $D_m^{(1)}$

(Toda-$A_{m-1}^{(1)}$): **Toda potential of type $A_{m-1}^{(1)}$**

$$C_0 \sum_{i=1}^{m-1} e^{-2\lambda(x_i - x_{i+1})} + C_0 e^{2\lambda(x_1 - x_m)}.$$

<u>Δ_m is of Type B_2</u>:

Lemma 2.7. (duality) *Put* $(x, y) = (x_1, x_2)$. *If the potential function*

$$R(x, y) = u^+(x + y) + u^-(x - y) + v(x) + w(y)$$

admits the commuting differential operator Q, *so is*

$$R^d(x, y) := v(x + y) + w(x - y) + u^+(2x) + u^-(2y).$$

(u^+, u^-, v, w) *is a transformation of one of the followings or its dual:*
<u>Case 1</u>: $u^+ = u^-$, $v = w$ *and* $(u^+; v)$ *is in the following list.*

(Trig-B_2) $(\langle \sinh^{-2} 2\lambda t\rangle;\ \langle \sinh^{-2} 2\lambda t,\ \sinh^{-2} \lambda t,\ \cosh 2\lambda t,\ \cosh 4\lambda t\rangle)$,

(Trig-B_2-S) $(\langle \sinh^{-2} \lambda t,\ \sinh^{-2} 2\lambda\rangle,\ \langle \sinh^{-2} 2\lambda t,\ \cosh 4\lambda t\rangle)$.

<u>Case 2</u>: $u^+ = u^-$, $(u^+; v, w)$ *is in the following list.*

(Toda-$D_2^{(1)}$-bry) $(\langle \cosh 2\lambda t\rangle;\ \langle \sinh^{-2} \lambda t,\ \sinh^{-2} 2\lambda t\rangle,\ \langle \sinh^{-2} \lambda t,\ \sinh^{-2} 2\lambda t\rangle)$,

(Toda-$D_2^{(1)}$-S-bry) $(\langle \cosh \lambda t,\ \cosh 2\lambda t\rangle;\ \langle \sinh^{-2} \lambda t\rangle,\ \langle \sinh^{-2} \lambda t\rangle)$,

(Toda-$B_2^{(1)}$-bry) $(\langle e^{-2\lambda t}\rangle;\ \langle e^{2\lambda t},\ e^{4\lambda t}\rangle,\ \langle \sinh^{-2} \lambda t,\ \sinh^{-2} 2\lambda t\rangle)$,

(Toda-$B_2^{(1)}$-S-bry) $(\langle e^{-\lambda t},\ e^{-2\lambda t}\rangle;\ \langle e^{2\lambda t}\rangle,\ \langle \sinh^{-2} \lambda t\rangle)$.

<u>Case 3</u>: $v = w$, $(u^+, u^-; v)$ *is in the following list.*

(Trig-A_1-bry) $(0,\ \langle \sinh^{-2} \lambda t\rangle;\ \langle e^{-2\lambda t}, e^{-4\lambda t}, e^{2\lambda t}, e^{4\lambda t}\rangle)$,

(Trig-A_1-S-bry) $(0,\ \langle \sinh^{-2} \lambda t,\ \sinh^{-2} 2\lambda t\rangle;\ \langle e^{-4\lambda t},\ e^{4\lambda t}\rangle)$.

<u>Case 4</u>: (u^+, u^-, v, w) *is in the following list.*

(Toda-$C_2^{(1)}$) $(0,\ \langle e^{-\lambda t}\rangle,\ \langle e^{\lambda t},\ e^{2\lambda t}\rangle,\ \langle e^{-\lambda t},\ e^{-2\lambda t}\rangle)$,

(Toda-$C_2^{(1)}$-S) $(0,\ \langle e^{-\lambda t},\ e^{-2\lambda t}\rangle,\ \langle e^{2\lambda t}\rangle,\ \langle e^{-2\lambda t}\rangle)$.

Here, for example, (Trig-A_1-bry) *means*

$$\begin{cases} u^+(t) = 0, \ u^-(t) = C_1 \sinh^{-2}\lambda t + C_6, \\ v(t) = w(t) = C_2 e^{-2\lambda t} + C_3 e^{-4\lambda t} + C_4 e^{2\lambda t} + C_5^{4\lambda t} + C_7. \end{cases}$$

Outline of the proof of Theorem 2.6: We reduce the equations satisfied by $R(x)$ to the functional equation of type A_2

$$(2.5) \qquad \big(U_1(x) + U_2(y) + U_3(z)\big)^2 = F_1(x) + F_2(y) + F_3(z) \quad \text{for } x + y + z = 0$$

and that of type B_2

$$(2.6) \quad V(x)\big(U^+(x+y) + U^-(x-y)\big) + W(y)\big(U^+(x+y) - U^-(x-y)\big) \\ = F_1(x+y) + F_2(x-y) + G_1(x) + G_2(y).$$

Remark 2.8. 1) The above functional equation of type A_2 is solved by [BP], [BB]. It is quite easy to solve it under our assumption. The solution (U_1, U_2, U_3) is a translation of

$$(C\coth\lambda t, C\coth\lambda t, C\coth\lambda t) \text{ or } (e^{\lambda t}, e^{\lambda t}, \epsilon e^{\lambda t}) \text{ with } C \in \mathbb{C}, \ \epsilon = 0 \text{ or } 1.$$

2) That of type B_2 is solved by [Oc] if at least two of u^+, u^-, v and w are not entire functions on $\mathbb{C}$. It is also proved by Ochiai that these functions are meromorphically extended to $\mathbb{C}$.

3) If $u_\alpha(t)$ and $u_\beta(t)$ are (a sum of) exponential functions and $\lim_{t\to+\infty}|u_\alpha(t)| = |u_\beta(t)| = \infty$, then $\langle\alpha,\beta\rangle \le 0$. Hence the set of roots

$$\{\alpha;\ u_\alpha(t) \text{ is a (sum of) exponential function(s) and } \lim_{t\to+\infty}|u_\alpha(t)| = \infty\}$$

forms an (extended) Dynkin diagram.

3. Hierarchy

Example 3.1. Suppose $\operatorname{Re}\lambda > 0$. Then

$$\lim_{R\to+\infty} e^{2\lambda R}\cdot\sinh^{-2}\lambda(t+R) = \lim_{R\to\infty}\frac{4}{(e^{\lambda t} - e^{-\lambda(t+2R)})^2} = 4e^{-\lambda t}.$$

(Trig-A_{n-1}) $\to$ (Toda-A_{n-1}):

$$\lim_{R\to+\infty} e^{2\lambda R} C \sum_{1\le i<j\le n} \sinh^{-2}\lambda\big((x_i - iR) - (x_j - jR)\big) = 4C\sum_{i=1}^{n-1} e^{-2\lambda(x_i - x_{i+1})},$$

(Trig-A_{n-1}) $\to$ (Rat-A_{n-1}):

$$\lim_{\lambda\to 0}\lambda^2 \sum_{1\le i<j\le n}\sinh^{-2}\lambda(x_i - x_j) = C\sum_{1\le i<j\le n}\frac{1}{(x_i - x_j)^2},$$

(Ellip-A_{n-1}) $\to$ (Toda-$A_{n-1}^{(1)}$):

$$\lim_{\omega_2\to\infty} e^{\frac{4}{n}\lambda\omega_2} C \sum_{1\le i<j\le n} \wp_0\big((x_i - \frac{2i\omega_2}{n}) - (x_j - \frac{2j\omega_2}{n}); 2\omega_1, 2\omega_2\big) \\ = 4\lambda^2 C\Big(\sum_{i=1}^{n-1} e^{-2\lambda(x_i - x_{i+1})} + e^{2\lambda(x_1 - x_n)}\Big).$$

Note that

$$
\begin{aligned}
\wp(z;2\omega_1,2\omega_2) &= -\frac{\eta_1}{\omega_1} + \lambda^2\sinh^{-2}\lambda z + \sum_{n=1}^{\infty}\frac{8\lambda^2 e^{-4n\lambda\omega_2}}{1-e^{-4n\lambda\omega_2}}\cosh 2n\lambda z, \\
\eta_1 &= \zeta(\omega_1;2\omega_1,2\omega_2), \quad \tau = \frac{\omega_2}{\omega_1} \quad \text{and} \quad \lambda = \frac{\pi}{2\sqrt{-1}\omega_1}.
\end{aligned} \tag{3.1}
$$

Fix ω_1 with $\sqrt{-1}\omega_1 < 0$ and let $\omega_2 \in \mathbb{R}$ with $\omega_2 > 0$. Put

$$
\wp_0(z;2\omega_1,2\omega_2) = \wp(z;2\omega_1,2\omega_2) + \frac{\zeta(\omega_1;2\omega_1,2\omega_2)}{\omega_1}. \tag{3.2}
$$

Then $\operatorname{Re}\lambda > 0$ and since

$$
\lim_{\omega_2\to\infty} e^{2(2-r)\lambda\omega_2}\frac{e^{-4\lambda\omega_2}}{1-e^{-4\lambda\omega_2}}\cosh 2\lambda(z+r\omega_2) = \frac{e^{2\lambda z}}{2} \quad \text{if } r>0, \tag{3.3}
$$

we have easily

$$
\lim_{\omega_2\to+\infty}\wp_0(z;2\omega_1,2\omega_2) = \lambda^2\sinh^{-2}\lambda z, \tag{3.4}
$$

$$
\lim_{\omega_2\to+\infty}\wp_0(z+\omega_1;2\omega_1,2\omega_2) = -\lambda^2\cosh^{-2}\lambda z, \tag{3.5}
$$

$$
\lim_{\omega_2\to\infty} e^{2r\lambda\omega_2}\wp_0(z+r\omega_2;2\omega_1,2\omega_2) = 4\lambda^2 e^{-2\lambda z} \quad \text{if } 0<r<1, \tag{3.6}
$$

$$
\lim_{\omega_2\to\infty} e^{2\lambda\omega_2}\wp_0(z+\omega_2;2\omega_1,2\omega_2) = 8\lambda^2\cosh 2\lambda z, \tag{3.7}
$$

$$
\lim_{\omega_2\to\infty} e^{2(2-r)\lambda\omega_2}\wp_0(z+r\omega_2;2\omega_1,2\omega_2) = 4\lambda^2 e^{2\lambda z} \quad \text{if } 1<r<2. \tag{3.8}
$$

In general, we get most known integrable systems as suitable limits from the A_n/B_n-invariant integrable system with the potential function expressed by the elliptic functions. We will show how they are obtained in the following.

For example, "Ellip-$A_{n-1} \to$ Toda-A_{n-1}^d" means that (Toda-A_{n-1}^d) is obtained from (Ellip-A_{n-1}) by taking a suitable limit, which is explained in Example 3.1, and "Trig-$B_n \overset{5:3}{\Rightarrow}$ Trig-B_n-reg" means that 2 parameters out of 5 in the potential function (Trig-B_n) are specialized to get the potential function (Trig-B_n-reg). Here we do not count the parameter corresponding to the periods of functions.

Note that we do not show all the relations in the following.

Hierarchy of Integrable Potentials with 5 parameters ($n \geq 2$)

$$
\begin{array}{ccccc}
 & & \text{Toda-}D_n^{(1)}\text{-bry} & \to & \text{Toda-}B_n^{(1)}\text{-bry} \\
 & \nearrow & & \nearrow\!\!\!\!\searrow & \\
\text{Ellip-}B_n & \to & \text{Trig-}B_n & \to & \text{Toda-}C_n^{(1)} \\
 & & \downarrow & \searrow & \\
 & & \text{Trig-}A_{n-1}\text{-bry} & & \text{Rat-}B_n \\
 & & \downarrow & & \\
 & & \text{Rat-}A_{n-1}\text{-bry} & &
\end{array}
$$

ierarchy of Elliptic-Trigonometric-Rational Integrable Potentials ($n \geq$

$$\begin{array}{ccccccc}
 & & & & \text{Ellip-}D_n & & \\
 & & & & \downarrow & & \\
 & & \text{Rat-}B_n & & \text{Trig-}D_n & \rightarrow & \text{Rat-}D_n \\
 & & \uparrow & & \Uparrow_{3:1} & & \\
\text{Ellip-}B_n & \rightarrow & \text{Trig-}B_n & \stackrel{5:3}{\Rightarrow} & \text{Trig-}BC_n\text{-reg} & & \text{Ellip-}A_{n-1} \\
 & & \downarrow & & & & \downarrow \\
 & & \text{Trig-}A_{n-1}\text{-bry} & \stackrel{5:3}{\Rightarrow} & \text{Trig-}A_{n-1}\text{-bry-reg} & \stackrel{3:1}{\Rightarrow} & \text{Trig-}A_{n-1} \\
 & & \downarrow & & & & \downarrow \\
 & & \text{Rat-}A_{n-1}\text{-bry} & & \stackrel{5:1}{\Rightarrow} & & \text{Rat-}A_{n-1}
\end{array}$$

Hierarchy of Toda Integrable Potentials ($n \geq 3$)

$$\begin{array}{ccccccc}
 & & \text{Trig-}BC_n\text{-reg} & \rightarrow & \text{Toda-}D_n\text{-bry} & \stackrel{3:1}{\Rightarrow} & \text{Toda-}D_n \\
 & & \Uparrow_{5:3} & & \Uparrow_{5:3} & & \Uparrow_{3:1} \\
 & & \text{Trig-}B_n & \rightarrow & \text{Toda-}B_n^{(1)}\text{-bry} & \stackrel{5:3}{\Rightarrow} & \text{Toda-}B_n^{(1)} \\
 & \nearrow & & \nearrow & & & \\
\text{Ellip-}B_n & \rightarrow & \text{Toda-}D_n^{(1)}\text{-bry} & \stackrel{5:1}{\Rightarrow} & \text{Toda-}D_n^{(1)} & & \\
 & \searrow & & \searrow & & & \\
 & & \text{Trig-}B_n & \rightarrow & \text{Toda-}C_n^{(1)} & \stackrel{5:3}{\Rightarrow} & \text{Toda-}BC_n \\
 & & & & & & \Downarrow_{3:1} \\
 & & \text{Ellip-}A_{n-1} & \rightarrow & \text{Toda-}A_{n-1}^{(1)} & \rightarrow & \text{Toda-}A_{n-1} \\
 & & & \searrow & & \nearrow & \\
 & & & & \text{Trig-}A_{n-1} & &
\end{array}$$

Hierarchy of Normal Integrable Potentials of type B_2

$$\begin{array}{ccccccc}
 & & \text{Trig-}BC_2\text{-reg} & \rightarrow & \text{Toda-}D_2\text{-bry} & & \\
 & & \Uparrow_{5:3} & & \Uparrow_{5:3} & & \\
 & & \text{Trig-}B_2 & \rightarrow & \text{Toda-}B_2^{(1)}\text{-bry} & \stackrel{5:3}{\Rightarrow} & \text{Toda-}B_2^{(1)} \\
 & \nearrow & & \nearrow & & & \\
\text{Ellip-}B_2 & \rightarrow & \text{Toda-}D_2^{(1)}\text{-bry} & \rightarrow & \text{Rat-}D_2^{(1)}\text{-bry} & & \\
 & \searrow & & \searrow & & & \\
 & & \text{Trig-}B_2 & \rightarrow & \text{Toda-}C_2^{(1)} & \stackrel{5:3}{\Rightarrow} & \text{Toda-}BC_2 \\
 & \swarrow & \downarrow & & & & \\
\text{Rat-}B_2 & & \text{Trig-}A_1\text{-bry} & \stackrel{5:3}{\Rightarrow} & \text{Trig-}A_1\text{-bry-reg} & & \\
 & & \downarrow & & & & \\
 & & \text{Rat-}A_1\text{-bry} & & & &
\end{array}$$

Hierarchy of Special Integrable Potentials of type B_2

$$\begin{array}{ccccccc}
 & & \text{Trig}^{(d)}\text{-}B_2\text{-S-reg} & \rightarrow & \text{Toda}^{(d)}\text{-}D_2\text{-S-bry} & & \\
 & & \Uparrow_{4:3} & & \Uparrow_{4:3} & & \\
 & & \text{Trig}^{(d)}\text{-}B_2\text{-S} & \rightarrow & \text{Toda}^{(d)}\text{-}B_2^{(1)}\text{-S-bry} & \stackrel{4:3}{\Rightarrow} & \text{Toda}^{(d)}\text{-}B_2^{(1)}\text{-S} \\
 & \nearrow & & \nearrow & & & \\
\text{Ellip-}B_2\text{-S} & \rightarrow & \text{Toda}^{(d)}\text{-}D_2^{(1)}\text{-S-bry} & \rightarrow & \text{Rat}^{(d)}\text{-}D_2^{(1)}\text{-S-bry} & & \\
 & \searrow & & \searrow & & & \\
 & & \text{Trig}^{(d)}\text{-}B_2\text{-S} & \rightarrow & \text{Toda}^{(d)}\text{-}C_2^{(1)}\text{-S} & \stackrel{4:3}{\Rightarrow} & \text{Toda}^{(d)}\text{-}B_2\text{-S} \\
 & \swarrow & \downarrow & & & & \\
\text{Rat-}B_2\text{-S} & & \text{Trig}^{(d)}\text{-}A_1\text{-bry-S} & \stackrel{4:3}{\Rightarrow} & \text{Trig}^{(d)}\text{-}A_1\text{-S-bry-reg} & &
\end{array}$$

Identity

$$(3.9)\qquad (\text{Trig-}BC_2\text{-reg}) = (\text{Trig}^d\text{-}B_2\text{-S-reg}),$$
$$(3.10)\qquad (\text{Toda-}D_2\text{-bry}) = (\text{Trig}^d\text{-}A_1\text{-S-bry-reg}),$$
$$(3.11)\qquad (\text{Trig-}A_1\text{-bry-reg}) = (\text{Toda}^d\text{-}D_2\text{-S-bry}),$$
$$(3.12)\qquad (\text{Toda-}BC_2) = (\text{Toda}^d\text{-}B_2\text{-S}).$$

Remark 3.2. 1) The superfix ^{d} means the dual in the above.
2) [I], [Ru] and [vD2] etc. considered hierarchies.

Conjecture 3.3. The above is the list of all the completely integrable system without Assumption 2.1.

4. Higher order integrals

Higher order integrals are generators of the commuting family whose highest order terms are the $W(\bar{\Delta})$-invariants of $\mathbb{C}[\partial_1, \ldots, \partial_n]$.

Type A_{n-1}([OS]): Elliptic-Trigonometric-Rational-(cyclic) Toda.

$$(4.1)\qquad \begin{aligned} P_k := \sum_{0\le j\le [\frac{k}{2}]} \sum_{w\in\mathfrak{S}_n/\mathbb{Z}_2^j\times\mathfrak{S}_j\times\mathfrak{S}_{k-2j}} & w\big(v_{e_1-e_2}(x)\cdot v_{e_3-e_4}(x)\cdots \\ & \cdot v_{e_{2j-1}-e_{2j}}(x)\partial_{e_{2j+1}}\cdots\partial_{e_k}\big) \quad (k=1,\ldots,n),\end{aligned}$$

and

$$(4.2)\qquad \begin{aligned} & v_\alpha(x) = -\frac{1}{2}u_\alpha(\langle\alpha,x\rangle) \quad \text{for } \alpha\in\Sigma(A_{n-1})^+, \\ & P = P_1^2 - 2P_2 = \sum_{k=1}^n \partial_k^2 + \sum_{1\le i<j\le n} u_{e_i-e_j}(x_i-x_j), \\ & [P_i,P_j] = 0 \quad \text{for } 1\le i<j\le n. \end{aligned}$$

Type B_2([Oc]):

$$(4.3)\qquad \begin{aligned} & V(x)\big(U^+(x+y)+U^-(x-y)\big) + W(y)\big(U^+(x+y)-U^-(x-y)\big) \\ & \quad = F_1(x+y)+F_2(x-y)+G_1(x)+G_2(y), \\ & u^\pm(t) = \frac{d}{dt}U^\pm(t),\ v(t) = \frac{d}{dt}V(t) \text{ and } w(t) = \frac{d}{dt}W(t), \\ & T(x,y) = \frac{1}{2}\big(\partial_x^2-\partial_y^2\big)\Big(V(x)\big(U^+(x+y)+U^-(x-y)\big) - G_1(x)\Big), \\ & P = \partial_x^2+\partial_y^2+u^+(x+y)+u^-(x-y)+v(x)+w(y), \\ & Q = \left(\partial_x\partial_y + \frac{u^+(x+y)-u^-(x-y)}{2}\right)^2 + w(y)\partial_x^2 + v(x)\partial_y^2 \\ & \quad + v(x)w(y) + T(x,y), \end{aligned}$$

Type B_n: Invariant elliptic case ([O]).
Define a differential operator

$$P(u,T) = \sum_{k=0}^n \sum_{w\in\mathfrak{S}_n/\mathfrak{S}_k\times\mathfrak{S}_{n-k}} w\Big(q_{\{1,\ldots,k\}}\Delta^2_{\{k+1,\ldots,n\}}\Big)$$

$$\Delta_{\{1,\dots,k\}} = \sum_{0\le j\le[\frac{k}{2}]} \sum_{w\in W(B_k)/\mathbb{Z}_2^j\times\mathfrak{S}_j\times\mathfrak{S}_{k-2j}} \varepsilon(w)w\Big(u(x_1-x_2)\cdot u(x_3-x_4)\cdots u(x_{2j-1}-x_{2j})\partial_{2j+1}\partial_{2j+2}\cdots\partial_k\Big),$$

$$q_{\{1,\dots,k\}} = \sum_{I_1\amalg\cdots\amalg I_\nu=\{1,\dots,k\}} T_{I_1}\cdots T_{I_\nu},$$

where

$$q_\varnothing = 1,\ q_{\{1\}} = T_{\{1\}},\ q_{\{12\}} = T_{\{1\}}T_{\{2\}} + T_{\{1,2\}},\dots$$

$$T_{w(\{1,\dots,k\})} = w(T_{\{1,\dots,k\}}),\quad \Delta_{w(\{1,\dots,k\})} = w(\Delta_{\{1,\dots,k\}}) \quad \text{for} \quad w\in\mathfrak{S}_n,$$

$$\varepsilon(w) = \begin{cases} 1 & \text{if } w\in W(D_n),\\ -1 & \text{if } w\notin W(D_n).\end{cases}$$

Put

$$u(t) = C_5\wp(t),$$

$$v(t) = \sum_{j=1}^{4} C_j\wp(t+\omega_j) - \frac{C_0}{2}.$$

Define $P_n(C_0) = P(u,T)$ by

$$T_{\{1,\dots,k\}} = (-C_5)^{k-1}\Big(\frac{C_0}{2}T^o_{\{1,\dots,k\}}(1) - \sum_{j=1}^{4} C_j T^o_{\{1,\dots,k\}}(\wp(t+\omega_j))\Big),$$

$$T^o_{\{1,\dots,k\}}(\psi) = \sum_{I_1\amalg\cdots\amalg I_\nu=\{1,\dots,k\}} (-1)^{\nu-1}(\nu-1)!S_{I_1}(\psi)\cdots S_{I_\nu}(\psi),$$

$$S_{\{1,\dots,k\}}(\psi) = \sum_{w\in W(B_k)} w\Big(\psi(x_1)\wp(x_1-x_2)\wp(x_2-x_3)\cdots\wp(x_{k-1}-x_k)\Big).$$

Then

$$[P_n(C), P_n(C')] = 0$$

for C, $C'\in\mathbb{C}$ and

$$P_n = P_n(0),$$

$$P_{n-k} = \sum_{i=k}^{n}\sum_{j=i}^{n} \sum_{w\in\mathfrak{S}_n/\mathfrak{S}_i\times\mathfrak{S}_{j-i}\times\mathfrak{S}_{n-j}} \sum_{I_1\amalg\cdots\amalg I_k=\{1,\dots,i\}} w\big((-C_5)^{i-k}2^{-k}T^o_{I_1}(1)\cdots T^o_{I_k}(1)q_{\{i+1,\dots,j\}}\Delta^2_{\{j+1,\dots,n\}}\big)$$

for $k=1,\dots,n-1$, where $q_{\{i+1,\dots,j\}}$ are defined by putting $C_0=0$.

Remark 4.1. Replacing ∂_i by ξ_i for $i=1,\dots,n$ in the definition of $\Delta_{\{1,\dots,k\}}$ and $P(u,T)$, we define functions $\bar{\Delta}_{\{1,\dots,k\}}$ and $\bar{P}(u,T)$ of (x,ξ), respectively, and we have classical completely integrable system.

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