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Hyperbolic Elements of Semisimple Symmetric Pairs and Their Orbits

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§1. The motivation

We start this paper with explaining the motivation of the present study.

Let $G = SL(2,\mathbf{R})$ and $\mathfrak{g} = \mathfrak{sl}(2,\mathbf{R})$. There are two kinds of G-orbits of non-zero semisimple elements of $\mathfrak{g}$.

(A) Take $x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $\mathrm{ad}_{\mathfrak{g}}(x)$ has eigenvalues $-2i$, 0, $2i$ and the centralizer $Z_G(x)$ of x is $SO(2)$. The G-orbit of x is identified with $G/Z_G(x) = SL(2,\mathbf{R})/SO(2)$. Therefore $\mathrm{Ad}(G)\cdot x$ is regarded as the upper half plane H_+. Or equivalently, $\mathrm{Ad}(G)\cdot x$ is imbedded in $SL(2,\mathbf{C})/B$, where B is the Borel subgroup of $SL(2,\mathbf{C})$ consisting of upper triangular matrices.

(B) Take $x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\mathrm{ad}_{\mathfrak{g}}(x)$ has eigenvalues -2, 0, 2 and the centalizer $Z_G(x)$ of x is $\left\{\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}; a \in \mathbf{R}-\{0\}\right\}$. The G-orbit of x is identified with $G/Z_G(x)$ which is imbedded into the product $G/P \times G/\overline{P}$, where P (resp. $\overline{P}$) is the parabolic subgroup of G consisting of upper (resp. lower) triangular matrices.

In each case, the G-orbit of x is imbedded in a flag manifold as an open subset. The purpose of this paper is to generalize the results above to the case of semisimple symmetric pairs.

Remark 1 *The main result of this paper is already obtained by T. Kobayashi (RIMS, Kyoto University). The author thanks to T. Kobayashi for pointing out his result as well as for showing the notes of his lectures at Harvard University.*

§2. Hyperbolic elements of semisimple Lie algebras

Before entering into the main subject, we introduce some notation. Let $\mathfrak{g}$ be a real semisimple Lie algebra and let σ be its involution. Then there is a direct sum decomposition $\mathfrak{g} = \mathfrak{h}+\mathfrak{q}$ with respect to σ, namely, $\mathfrak{h} = \{x \in \mathfrak{g}; \sigma(x) = x\}$ and $\mathfrak{q} = \{x \in \mathfrak{g}; \sigma(x) = -x\}$. The pair $(\mathfrak{g},\mathfrak{q})$ is called a semisimple symmetric pair. There exists a Cartan involution θ of $\mathfrak{g}$ commuting with σ. Let $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition.

Let G be the group of inner automorphisms of $\mathfrak{g}$. Then σ is liftable to G and the lifting is denoted by the same letter for brevity. We denote by H the identity component of the fixed subgroup of σ.

Definition 1 *An element x of $\mathfrak{q}$ is hyperbolic if x is semisimple and $\mathrm{ad}_{\mathfrak{g}}(x)$ has only real eigenvalues.*

We now give examples of semisimple symmetric pairs and hyperbolic elements.

Example 1 *Assume that $\mathfrak{g}$ is a complex semisimple Lie algebra with real form $\mathfrak{g}_R$ and σ is a conjugation with respect to $\mathfrak{g}_{\mathbf{R}}$. In this case $\mathfrak{h} = \mathfrak{g}_R$ and $\mathfrak{q} = \sqrt{-1}\mathfrak{g}_{\mathbf{R}}$.*

An element $x \in \mathfrak{q} = \sqrt{-1}\mathfrak{g}_{\mathbf{R}}$ is hyperbolic if and only if $\mathrm{ad}_{\mathfrak{g}}(x)$ is semisimple and has only real eigenvalues, or equivalently, $\mathrm{ad}_{\mathfrak{g}_{\mathbf{R}}}(\sqrt{-1}x)$ is elliptic as an element of $\mathfrak{g}_{\mathbf{R}}$.

Example 2 *Assume that $\mathfrak{g}$ is a direct sum of two copies of a semisimple Lie algebra $\mathfrak{g}_1$, that is, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1$. Take an involution σ defined by $\sigma(x_1, y_1) = (y_1, x_1)$ for all $x_1, y_1 \in \mathfrak{g}_1$. In this case, $\mathfrak{h} = \Delta\mathfrak{g}_1 = \{(x_1, x_1);\ x_1 \in \mathfrak{g}_1\}$ and $\mathfrak{q} = \{(x_1, -x_1);\ x_1 \in \mathfrak{g}_1\}$.*

An element $x = (x_1, -x_1) \in \mathfrak{q}$ with $x_1 \in \mathfrak{g}_1$ is hyperbolic if and only if $\mathrm{ad}_{\mathfrak{g}_1}(x_1)$ is semisimple and has only real eigenvalues.

It is known [2] that the orbit of an elliptic element of a real simple Lie algebra has a pseudo-Kähler structure and it is holomorphically imbedded in a complex flag manifold as an open subset. (A typical case is (A).) On the other hand, the orbit of a hyperbolic element of a real semisimple Lie algebra has a double foliation, called a parakähler structure [3], [5] and it is imbedded in the product of two flag manifolds as a dense open orbit. (A typical case is (B).) These two results imply at least the claim that if $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair of Examples 1, 2 and $x \in \mathfrak{q}$ is hyperbolic, there is a parabolic subgroup P of G such that the H-orbit of x is H-equivariantly imbedded in the flag manifold G/P as an open subset. The purpose of this paper is, therefore, to imply a theorem which shows an H-equivariant imbedding of the H-orbit of every hyperbolic element of $\mathfrak{q}$ into a flag manifold as an open subset.

§3. The main theorem

The starting point of our study is the following lemma.

Lemma 1 *If $x \in \mathfrak{q}$ is hyperbolic, there is $h \in H$ such that $\mathrm{Ad}(h)x$ is contained in $\mathfrak{p}$.*

For a proof of this lemma, refer to [9].

Take a hyperbolic element $x \in \mathfrak{q}$. Then, by Lemma 1, we may assume that $\theta(x) = -x$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ containing x. A non-zero linear form λ on $\mathfrak{a}$ is a root (of $(\mathfrak{g}, \mathfrak{a})$) if $\mathfrak{g}_\lambda = \{y \in \mathfrak{g};\ [z, y] = \lambda(z)y\ \ (\forall z \in \mathfrak{a})\} \neq \{0\}$. Let $\Sigma(\mathfrak{a})$ be the totality of the roots of $(\mathfrak{g}, \mathfrak{a})$. Then it is known ([11]) that $\Sigma(\mathfrak{a})$ is a root system. Moreover, the Cartan involution θ induces a linear transformation on $\Sigma(\mathfrak{a})$, namely, for any $\lambda \in \Sigma(\mathfrak{a})$, $\theta(\lambda)$ is the root defined by $\mathfrak{g}_{\theta(\lambda)} = \theta(g_\lambda)$. Let $\mathrm{Spec}(x)$ be the set of eigenvalues of $\mathrm{ad}(x)$ and we put

$$\mathfrak{g}(c) = \{y \in \mathfrak{g};\ [x, y] = cy\}$$

for any eigenvalue $c \in \mathrm{Spec}(x)$. Since $[x, \theta(y)] = -c\theta(y)$ for any $y \in \mathfrak{g}(c)$, it follows that $\theta(\mathfrak{g}(c)) = \mathfrak{g}(-c)$ which implies that if $c \in \mathrm{Spec}(x)$, then $-c \in \mathrm{Spec}(x)$. Put

$$\mathfrak{l}_x = \sum_{c \in \mathrm{Spec}(x), c \geq 0} \mathfrak{g}(c), \quad \mathfrak{n}_x = \sum_{c \in \mathrm{Spec}(x), c > 0} \mathfrak{g}(c).$$

Then $\mathfrak{l}_x$ is a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{n}_x$ is its nilradical. Let L_x be the parabolic subgroup of G with Lie algebra l_x.

Let $\mathcal{O}_H(x)$ be the H-orbit of x. Then $\mathcal{O}_H(x)$ is identified with $H/Z_H(x)$, where $Z_H(x)$ denotes the centralizer of x in H. Let $\mathfrak{z}_{\mathfrak{h}}(x)$ be the Lie algebra of $Z_H(x)$. Then $\mathfrak{z}_{\mathfrak{h}}(x)$ is contained in $\mathfrak{g}(0)$.

Put $\mathfrak{h}_{[c]} = \langle y + \sigma(y);\ y \in \mathfrak{g}(c)\rangle$. Then, since $\sigma(\mathfrak{g}(c)) = \mathfrak{g}(-c)$, it follows that $\mathfrak{h}_{[c]} = \mathfrak{h}_{[-c]}$. Moreover, the following lemma holds.

Lemma 2

$$\mathfrak{h} = \mathfrak{z}_{\mathfrak{h}}(x) \oplus (\oplus_{c\in \mathrm{Spec}(x), c>0}\mathfrak{h}_{[c]}).$$

Proof. Take $y \in \mathfrak{h}$ and write

$$y = \sum_{c\in \mathrm{Spec}(x)} y_c.$$

where $y_c \in \mathfrak{g}(c)$ for all $c \in \mathrm{Spec}(x)$. Since $\sigma(y) = y$ and $\sigma(y_c) \in \mathfrak{g}(-c)$, it follows that $\sigma(y_c) = y_{-c}$. Then

$$y = y_0 + \sum_{c\in \mathrm{Spec}(x), c>0} (y_c + \sigma(y_c))$$

and the lemma follows.[]

Theorem 1 *Retain the notation above. The map φ of $\mathcal{O}_H(x)$ to G/L_x defined by*

$$\varphi(hZ_H(x)) = hL_x \quad (\forall h \in H)$$

is well-defined and is injective.

Proof. Let $Z_G(x)$ be the centralizer of x in G and let N_x be the analytic subgroup of G corresponding to $\mathfrak{n}_x$. Then $L_x = Z_G(x)N_x$. Note that $\mathfrak{g}(0)$ is the Lie algebra of $Z_G(x)$. Lemma 2 shows that $\mathfrak{h} \cap \mathfrak{l}_x = \mathfrak{z}_{\mathfrak{h}}(x)$. On the other hand, clearly $Z_H(x)$ is contained in $Z_G(x)$. These imply the theorem. []

Remark 2 *By the theorem, the closure of $\mathcal{O}_H(x)$ in G/L_x is a compactification of $\mathcal{O}_H(x)$.*

In special cases as in Examples 1, 2, the orbit $\mathcal{O}_H(x)$ has a pseudo-Kähler structure or a parakähler structure. It is interesting to study what kind of geometric structure $\mathcal{O}_H(x)$ has in general.

We now mention elliptic elements which is the counterpart of hyperbolic elements. An element $x \in \mathfrak{q}$ is said to be elliptic if x is semisimple as an element of $\mathfrak{g}$ and $\mathrm{ad}_{\mathfrak{g}}(x)$ has only pure imaginary eigenvalues. We recall the definitions of dual and associated pairs to $(\mathfrak{g}, \mathfrak{q})$ (cf. [11]). Let $(\mathfrak{g}, \mathfrak{h})^d$ (resp. $(\mathfrak{g}, \mathfrak{h})^a$) be the dual (resp. associated) pair to $(\mathfrak{g}, \mathfrak{h})$. Then $(\mathfrak{g}, \mathfrak{h})^{ada} = (\mathfrak{g}, \mathfrak{h})^{dad}$. Define $\mathfrak{g}^{ada}$ and $\mathfrak{h}^{ada}$ by $(\mathfrak{g}^{ada}, \mathfrak{h}^{ada}) = (\mathfrak{g}, \mathfrak{h})^{ada}$. It is known ([11]) that $\mathfrak{g}^{ada} = \mathfrak{h} + \sqrt{-1}\mathfrak{q}$, $\mathfrak{h}^{ada} = \mathfrak{h}$, $\mathfrak{q}^{ada} = \sqrt{-1}\mathfrak{q}$. This implies in particular that $x \in \mathfrak{q}^{ada}$ is elliptic if and only if $\sqrt{-1}x$ is contained in $\mathfrak{q}$ and is hyperbolic. Take $x \in \mathfrak{q}$ as in Theorem 1. Then $x' = \sqrt{-1}x$ is contained in $\mathfrak{q}^{ada}$ and is elliptic and H^{ada}-orbit $\mathcal{O}_{H^{ada}}(x')$ of x' is identified with the H-orbit $\mathcal{O}_H(x)$ in $\mathfrak{q}$. Here $H^{ada} = H$.

§4. Hyperbolic orbits as symmetric spaces

In the sequel, we treat special cases of x. We first consider the case where $\mathrm{Spec}(x)$ is contained in $\mathbf{Z}$.

Example 3 *There are $e, f \in \mathfrak{q}$ such that*

$$[x, e] = 2e, \quad [x, f] = -2f, \quad [e, f] = x.$$

Then (e, x, f) is said to be a TDS. It is known that $\mathrm{Spec}(x) \subset \mathbf{Z}$ *in this case.*

We define a linear automorphism τ of $\mathfrak{g}$ depending on x by

$$\tau(y) = (-1)^c y \quad (\forall y \in \mathfrak{g}(c),\ \forall c \in \mathrm{Spec}(x).)$$

It is clear from the definition that τ is an involution and commutes with both σ and θ. In particular, if $\mathrm{Spec}(x)$ is contained in $2\mathbf{Z}$, then τ is trivial.

We next treat a special case where $\mathrm{Spec}(x)$ has just three elements $0, \pm c$ ($c \neq 0$). Then we may assume that $\mathrm{Spec}(x) = \{-1, 0, 1\}$ by changing x with x/c. We define an automorphism τ of $\mathfrak{g}$ by

$$\tau(g) = \exp(\sqrt{-1}\pi x) y \exp(-\sqrt{-1}\pi x) \quad (\forall g \in G).$$

Then clearly τ defines an involution of G commuting with both σ and θ. We denote by the same letter the involution of $\mathfrak{g}$ induced from τ. Then

$$\tau(y) = (-1)^c y \quad (\forall y \in \mathfrak{g}(c),\ c \in \mathrm{Spec}(x)).$$

Proposition 1 *The pair $(\mathfrak{h}, \mathfrak{z}_{\mathfrak{h}}(x))$ is a semisimple symmetric pair with respect to the involution τ.*

Proof. It follows from the definition that $\mathfrak{h}$ is preserved by the action of τ and the fixed point set of τ in $\mathfrak{h}$ coincides with $\mathfrak{z}_{\mathfrak{h}}(x)$. []

Remark 3 *By Theorem 1 and Proposition 1, we obtain a compactification of every semisimple symmetric space corresponding to the pair $(\mathfrak{h}, \mathfrak{z}_{\mathfrak{h}}(x))$ for a hyperbolic element $x \in \mathfrak{q}$. The cases (A), (B) in §1 are typical examples of such compactifications. We already obtained a compactification of each semisimple symmetric space of the form $G^\sigma/G^{\sigma,\theta}$ by a different idea in [8, Theorem 4].*

We now propose problems concerning the symmetric pairs in Proposition 1 and its imbedding.

Problem 1 *Classify semisimple symmetric pairs $(\mathfrak{h}, \mathfrak{z}_{\mathfrak{h}}(x))$ with the condition* $\mathrm{Spec}(x) = \{0, 1, -1\}$.

Problem 2 *Determine the H-orbital structure of G/L_x concretely for such hyperbolic elements x that* $\mathrm{Spec}(x) = \{0, 1, -1\}$.

Problem 2 is already solved for the case of semisimple symmetric pairs of Example 2 by Kaneyuki (cf. [4]). The H-orbital structure of general flag manifolds is obtained by T. Matsuki [7]. What we ask in Problem 2 is to show a more concrete description of the H-orbital structure as in [4].

§5. Examples of imbeddings.

In this section, we restrict our attention to such symmetric pairs of the so-called $\mathfrak{k}_\varepsilon$-type. We first recall some notation in [10]. Let $\mathfrak{g}$ be a semisimple Lie algebra and let θ be its Cartan involution. Then we obtain the Cartan decomposition: $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Let $\Sigma(\mathfrak{a})$ be the restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. For a signature ε of $\Sigma(\mathfrak{a})$ (cf. [10], p.5), we define an involution of $\mathfrak{g}$ denoted by θ_ε. Then the symmetric pair $(\mathfrak{g}, \mathfrak{h})$ with respect to the involution $\sigma = \theta_\varepsilon$ is called a symmetric pair of $\mathfrak{k}_\varepsilon$-type. In [10], the subgroup H for the involution σ is denoted by K_ε.

We take a fundamental system $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ of $\Sigma(\mathfrak{a})$. For each i $(1 \le i \le l)$, let ε_i be the signature of $\Sigma(\mathfrak{a})$ with the condition:

$$\varepsilon_i(\alpha_i) = -1, \quad \varepsilon_i(\alpha_j) = 1 \; (j \neq i)$$

Moreover let x_i be the element of $\mathfrak{a}$ such that

$$\alpha_i(x_i) = 1, \quad \alpha_j(x_i) = 0 \; (j \neq i)$$

We now take $\sigma = \varepsilon_i$ and fix it for the moment. Clearly each x_j is a hyperbolic element of $\mathfrak{q}$ and $\mathrm{Spec}(x_i) = \{0, 1, -1\}$. This implies that the K_{ε_i}-orbit $\mathcal{O}_{K_{\varepsilon_i}}(x_j) = K_{\varepsilon_i} \cdot x_j$ of x_j is a symmetric space realized as an open subset of the flag manifold G/L_{x_j}. The following lemma is a direct consequence of the definition of the involution θ_ε.

Lemma 3 *The orbit $\mathcal{O}_{K_{\varepsilon_i}}(x_i)$ is a Riemannian symmetric space.*

¿From now on, we focus our attention to the case where $\mathfrak{g}$ is one of $\mathfrak{g}_2^{\mathbf{C}}$, $\mathfrak{f}_4^{\mathbf{C}}$, $\mathfrak{e}_6^{\mathbf{C}}$ and their real forms. The ordering of fundamental roots is same as [1] (see also [10], Table 1). Then the orbits of the form $\mathcal{O}_{K_{\varepsilon_i}}(x_j)$ are given in the following table.

Explanation of the columns of the tables:

1. The cloumn (a) $\Longrightarrow$ Lie algebra $\mathfrak{g}$
2. The cloumn (b) $\Longrightarrow$ The type of the root system $\Sigma(\mathfrak{a})$
3. The cloumn (c) $\Longrightarrow$ The signature ε of $\Sigma(\mathfrak{a})$
4. The cloumn (d) $\Longrightarrow$ The symmetric pair for θ_ε
5. The cloumn (e) $\Longrightarrow$ The element $x_j \in \mathfrak{a}$
6. The cloumn (f) $\Longrightarrow$ The symmetric pair corresponding to the orbit $\mathcal{O}_{K_{\varepsilon_i}}(x_j)$

(a)	(b)	(c)	(d)	(e)	(f)
$\mathfrak{g}_{2(2)}$	G_2	ε_1	$(\mathfrak{g}_{2(2)}, \mathfrak{sl}(2,\mathbf{R}) \oplus \mathfrak{sl}(2,\mathbf{R}))$	x_1	$(\mathfrak{sl}(2,\mathbf{R}), \mathfrak{so}(2)) \oplus (\mathfrak{sl}(2,\mathbf{R}), \mathfrak{so}(2))$
				x_2	$(\mathfrak{sl}(2,\mathbf{R}), \mathfrak{so}(2)) \oplus (\mathfrak{sl}(2,\mathbf{R}), \mathfrak{so}(1,1))$
$\mathfrak{g}_2^{\mathbf{C}}$	G_2	ε_1	$(\mathfrak{g}_2^{\mathbf{C}}, \mathfrak{g}_{2(2)})$	x_1	$(\mathfrak{g}_{2(2)}, \mathfrak{so}(4))$
				x_2	$(\mathfrak{g}_{2(2)}, \mathfrak{su}(2) \oplus \mathfrak{sl}(2,\mathbf{R}))$
$\mathfrak{f}_{4(4)}$	F_4	ε_1	$(\mathfrak{f}_{4(4)}, \mathfrak{sp}(3,\mathbf{R}) \oplus \mathfrak{sl}(2,\mathbf{R}))$	x_1	$(\mathfrak{sp}(3,\mathbf{R}), \mathfrak{u}(3)) \oplus (\mathfrak{sl}(2,\mathbf{R}), \mathfrak{so}(2))$
				x_2	$(\mathfrak{sp}(3,\mathbf{R}), \mathfrak{gl}(3,\mathbf{R})) \oplus (\mathfrak{sl}(2,\mathbf{R}), \mathfrak{so}(1,1))$
				x_3	$(\mathfrak{sp}(3,\mathbf{R}), \mathfrak{sp}(2,\mathbf{R}) \oplus (\mathfrak{sp}(1,\mathbf{R}))$
				x_4	$(\mathfrak{sp}(3,\mathbf{R}), \mathfrak{sp}(2,\mathbf{R}) \oplus (\mathfrak{sp}(1,\mathbf{R}))$
$\mathfrak{f}_{4(4)}$	F_4	ε_4	$(\mathfrak{f}_{4(4)}, \mathfrak{sp}(2,1) \oplus \mathfrak{su}(2))$	x_1	$(\mathfrak{sp}(2,1), \mathfrak{u}(2,1)) \oplus (\mathfrak{su}(2), \mathfrak{so}(2))$
				x_2	$(\mathfrak{sp}(2,1), \mathfrak{u}(2,1)) \oplus (\mathfrak{su}(2), \mathfrak{so}(2))$
				x_3	$(\mathfrak{sp}(2,1), \mathfrak{u}(2,1))$
				x_4	$(\mathfrak{sp}(2,1), \mathfrak{sp}(2) \oplus \mathfrak{sp}(1))$
$\mathfrak{f}_4^{\mathbf{C}}$	F_4	ε_1	$(\mathfrak{f}_4^{\mathbf{C}}, \mathfrak{f}_{4(4)})$	x_1	$(\mathfrak{f}_{4(4)}, \mathfrak{sp}(3) \oplus \mathfrak{su}(2))$
				x_2	$(\mathfrak{f}_{4(4)}, \mathfrak{sp}(3,\mathbf{R}) \oplus \mathfrak{sp}(1,\mathbf{R}))$
				x_3	$(\mathfrak{f}_{4(4)}, \mathfrak{so}(5,4))$
				x_4	$(\mathfrak{f}_{4(4)}, \mathfrak{so}(5,4))$
$\mathfrak{f}_4^{\mathbf{C}}$	F_4	ε_1	$(\mathfrak{f}_4^{\mathbf{C}}, \mathfrak{f}_{4(-20)})$	x_1	$(\mathfrak{f}_{4(-20)}, \mathfrak{sp}(2,1) \oplus \mathfrak{su}(2))$
				x_2	$(\mathfrak{f}_{4(-20)}, \mathfrak{sp}(2,1) \oplus \mathfrak{su}(2))$
				x_3	$(\mathfrak{f}_{4(-20)}, \mathfrak{so}(8,1))$
				x_4	$(\mathfrak{f}_{4(-20)}, \mathfrak{so}(9))$
$\mathfrak{e}_{6(6)}$	E_6	ε_2	$(\mathfrak{e}_{6(6)}, \mathfrak{sp}(4,\mathbf{R}))$	x_1	$(\mathfrak{sp}(4,\mathbf{R}), \mathfrak{sp}(2,\mathbf{C}))$
				x_2	$(\mathfrak{sp}(4,\mathbf{R}), \mathfrak{u}(4))$
				x_3	$(\mathfrak{sp}(4,\mathbf{R}), \mathfrak{u}(2,2))$
				x_4	$(\mathfrak{sp}(4,\mathbf{R}), \mathfrak{gl}(4,\mathbf{R}))$
$\mathfrak{e}_{6(6)}$	E_6	ε_6	$(\mathfrak{e}_{6(6)}, \mathfrak{sp}(2,2))$	x_1	$(\mathfrak{sp}(2,2), \mathfrak{sp}(1,1) \oplus \mathfrak{sp}(1,1))$
				x_2	$(\mathfrak{sp}(2,2), \mathfrak{su}^*(4) \oplus \mathbf{R})$
				x_3	$(\mathfrak{sp}(2,2), \mathfrak{u}(2,2))$
				x_4	$(\mathfrak{sp}(2,2), \mathfrak{u}(2,2))$
				x_5	$(\mathfrak{sp}(2,2), \mathfrak{su}^*(4) \oplus \mathbf{R})$
				x_6	$(\mathfrak{sp}(2,2), \mathfrak{sp}(2) \oplus \mathfrak{sp}(2))$
$\mathfrak{e}_6^{\mathbf{C}}$	E_6	ε_2	$(\mathfrak{e}_6^{\mathbf{C}}, \mathfrak{e}_{6(2)})$	x_1	$(\mathfrak{e}_{6(2)}, \mathfrak{so}^*(10) \oplus \mathfrak{so}(2))$
				x_2	$(\mathfrak{e}_{6(2)}, \mathfrak{su}(6) \oplus \mathfrak{su}(2))$
				x_3	$(\mathfrak{e}_{6(2)}, \mathfrak{su}(4,2) \oplus \mathfrak{su}(2))$
				x_4	$(\mathfrak{e}_{6(2)}, \mathfrak{su}(3,3) \oplus \mathfrak{sl}(2,\mathbf{R}))$
$\mathfrak{e}_6^{\mathbf{C}}$	E_6	ε_6	$(\mathfrak{e}_6^{\mathbf{C}}, \mathfrak{e}_{6(-14)})$	x_1	$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(8,2) \oplus \mathfrak{so}(2))$
				x_2	$(\mathfrak{e}_{6(-14)}, \mathfrak{su}(5,1) \oplus \mathfrak{sl}(2,\mathbf{R}))$
				x_3	$(\mathfrak{e}_{6(-14)}, \mathfrak{su}(4,2) \oplus \mathfrak{su}(2))$
				x_4	$(\mathfrak{e}_{6(-14)}, \mathfrak{su}(4,2) \oplus \mathfrak{su}(2))$
				x_5	$(\mathfrak{e}_{6(-14)}, \mathfrak{su}(5,1) \oplus \mathfrak{sl}(2,\mathbf{R}))$
				x_6	$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) \oplus \mathfrak{so}(2))$

(a)	(b)	(c)	(d)	(e)	(f)
$\mathfrak{e}_{6(2)}$	F_4	ε_1	$(\mathfrak{e}_{6(2)}, \mathfrak{su}(3,3) \oplus \mathfrak{sl}(2,\mathbf{R}))$	x_1	$(\mathfrak{su}(3,3), \mathfrak{s}(\mathfrak{u}(3) \oplus \mathfrak{u}(3))) \oplus (\mathfrak{sl}(2,\mathbf{R}), \mathfrak{so}(2))$
				x_2	$(\mathfrak{su}(3,3), \mathfrak{so}(3,3)) \oplus (\mathfrak{sl}(2,\mathbf{R}), \mathfrak{so}(1,1))$
				x_3	$(\mathfrak{su}(3,3), \mathfrak{sl}(3,\mathbf{C}) \oplus \mathbf{R})$
				x_4	$(\mathfrak{su}(3,3), \mathfrak{sl}(3,\mathbf{C}) \oplus \mathbf{R})$
$\mathfrak{e}_{6(2)}$	F_4	ε_4	$(\mathfrak{e}_{6(2)}, \mathfrak{su}(4,2) \oplus \mathfrak{su}(2))$	x_1	$(\mathfrak{su}(4,2), \mathfrak{so}(4,2)) \oplus (\mathfrak{su}(2), \mathfrak{so}(2))$
				x_2	$(\mathfrak{su}(4,2), \mathfrak{so}(4,2)) \oplus (\mathfrak{su}(2), \mathfrak{so}(2))$
				x_3	$(\mathfrak{su}(4,2), \mathfrak{s}(\mathfrak{u}(3) \oplus \mathfrak{u}(1,2)))$
				x_4	$(\mathfrak{su}(4,2), \mathfrak{s}(\mathfrak{u}(4) \oplus \mathfrak{u}(2)))$
$\mathfrak{e}_{6(-14)}$	BC_2	ε_1	$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(8,2) \oplus \mathfrak{so}(2))$	x_1	$(\mathfrak{so}(8,2), \mathfrak{so}(8) \oplus \mathfrak{so}(2))$
				x_2	$(\mathfrak{so}(8,2), \mathfrak{so}(6,2) \oplus \mathfrak{so}(2))$
$\mathfrak{e}_{6(-14)}$	BC_2	ε_2	$(\mathfrak{e}_{6(-14)}, \mathfrak{so}^*(10) \oplus \mathfrak{so}(2))$	x_1	$(\mathfrak{so}^*(10), \mathfrak{u}(4,1))$
				x_2	$(\mathfrak{so}^*(10), \mathfrak{u}(5)))$
$\mathfrak{e}_{6(-26)}$	A_2	ε_1	$(\mathfrak{e}_{6(-26)}, \mathfrak{f}_{4(-20)})$	x_1	$(\mathfrak{f}_{4(-20)}, \mathfrak{so}(9))$
				x_2	$(\mathfrak{f}_{4(-20)}, \mathfrak{so}(8,1))$

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