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Enbeddings of derived functor modules into degenerate principal series

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§ 1. Formulation of the problem

Let G be a real linear reductive Lie group and let $G_{\mathbb{C}}$ its complexification. We denote by $\mathfrak{g}_0$ (resp. $\mathfrak{g}$) the Lie algebra of G (resp. $G_{\mathbb{C}}$) and denote by σ the complex conjugation on $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. We fix a maximal compact subgroup K of G and denote by θ the corresponding Cartan involution. We denote by $\mathfrak{k}$ the complexified Lie algebra of K.

We fix a prabolic subgroup P of G with θ-stable Levi part M. We denote by N the nilradical of P. We denote by $\mathfrak{p}$, $\mathfrak{m}$, and $\mathfrak{n}$ the complexified Lie algebras of P, M, and N, respectively. We denote by $P_{\mathbb{C}}$, $M_{\mathbb{C}}$, and $N_{\mathbb{C}}$ the analytic subgroups in $G_{\mathbb{C}}$ with respect to $\mathfrak{p}$, $\mathfrak{m}$, and $\mathfrak{n}$, respectively.

For $X \in \mathfrak{m}$, we define

$$\delta(X) = \frac{1}{2}\mathrm{tr}\left(\mathrm{ad}_{\mathfrak{g}}(X)|_{\mathfrak{n}}\right).$$

Then, δ is a one-dimesional representation of $\mathfrak{m}$. We see that 2δ lifts to a holomorphic group homomorphism $\xi_{2\delta} : M_{\mathbb{C}} \to \mathbb{C}^{\times}$. Defining $\xi_{2\delta}|_{N_{\mathbb{C}}}$ trivial, we may extend $\xi_{2\delta}$ to $P_{\mathbb{C}}$. We put $X = G_{\mathbb{C}}/P_{\mathbb{C}}$. Let $\mathcal{L}$ be the holomorphic line bundle on X corresponding to the canonical divisor. Namely, $\mathcal{L}$ is the $G_{\mathbb{C}}$-homogeneous line bundle on X associated to the character $\xi_{2\delta}$ on $P_{\mathbb{C}}$. We denote the restriction of $\xi_{2\delta}$ to P by the same letter.

For a character $\eta : P \to \mathbb{C}^{\times}$, we consider the unnormalized parabolic induction ${}^{u}\mathrm{Ind}_P^G(\eta)$. Namely, ${}^{u}\mathrm{Ind}_P^G(\eta)$ is the K-finite part of the space of the C^{∞}-sections of the G-homogeneous line bundle on G/P associated to η. ${}^{u}\mathrm{Ind}_P^G(\eta)$ is a Harish-Chandra $(\mathfrak{g}, K)$-module.

If G/P is orientable, then the trivial G-representation is the unique irreducible quotient of ${}^{u}\mathrm{Ind}_P^G(\xi_{2\delta})$. If G/P is not orientable, there is a character ω on P such that ω is trivial on the identical componnent of P and the trivial G-representation is the unique irreducible quotient of ${}^{u}\mathrm{Ind}_P^G(\xi_{2\delta} \otimes \omega)$.

Let $\mathcal{O}$ be an open G-orbit on X. We put the following assumption:

Assumption 1.1 There is a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ such that $\mathfrak{q} \in \mathcal{O}$.

Under the above assumption, $\mathfrak{q}$ has a Levi decompsition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ such that $\mathfrak{l}$ is a θ and σ-stable Levi part. In fact $\mathfrak{l}$ is unique, since we have $\mathfrak{l} = \sigma(\mathfrak{q}) \cap \mathfrak{q}$.

For each open G-orbit $\mathcal{O}$ on X, we put

$$\mathcal{A}_{\mathcal{O}} = \mathrm{H}^{\dim \mathfrak{u} \cap \mathfrak{k}}(\mathcal{O}, \mathcal{L})_{K\text{-finite}}.$$

Namely, in the termilogy in [Vogan-Zuckerman 1984], we have $\mathcal{A}_{\mathcal{O}} = \mathcal{A}_{\mathfrak{q}} = \mathcal{A}_{\mathfrak{q}}(0)$.

We consider the following problem:

Problem 1.2 Is there an embedding: $\mathcal{A}_{\mathcal{O}} \hookrightarrow {}^u\mathrm{Ind}_P^G(\xi_{2\delta})$ or $\mathcal{A}_{\mathcal{O}} \hookrightarrow {}^u\mathrm{Ind}_P^G(\xi_{2\delta} \otimes \omega)$?

§ 2. Complex groups

Let G be a connected real split reductive linear Lie group. Here, we consider Problem 1.2 for the complexification $G_{\mathbb{C}}$ rather than G itself. Embedding $G_{\mathbb{C}}$ into $G_{\mathbb{C}} \times G_{\mathbb{C}}$ via $g \rightsquigarrow (g, \sigma(g))$, we may regard $G_{\mathbb{C}} \times G_{\mathbb{C}}$ as a complexification of $G_{\mathbb{C}}$. Each parabolic subgroup of $G_{\mathbb{C}}$ is the complexification of a parabolic subgroup of G. Let P be a parabolic subgroup of G. Then, the complexification of $P_{\mathbb{C}}$ can be identified with $P_{\mathbb{C}} \times P_{\mathbb{C}}$ via the above embedding $G_{\mathbb{C}} \hookrightarrow G_{\mathbb{C}} \times G_{\mathbb{C}}$. Hence, the complex generalized flag variety for $G_{\mathbb{C}}$ is $X \times X$. We fix a θ and σ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h} \subseteq \mathfrak{p}$. We denote by w_0 (resp. $w_{\mathfrak{p}}$) the longest element of the Weyl group with respect to $(\mathfrak{g}, \mathfrak{h})$ (resp. $(\mathfrak{m}, \mathfrak{h})$).

We easily have:

Proposition 2.1. *$X \times X$ has a unique $G_{\mathbb{C}}$-orbit (say $\mathcal{O}_0$). $\mathcal{O}_{\mathbb{C}}$ satisfies the Assumption 1.1 if and only if $w_0 w_{\mathfrak{p}} = w_{\mathfrak{p}} w_0$.*

We consider "$\xi_{2\delta}$" for G. Then the character $\xi_{2\delta} \boxtimes \xi_{2\delta}$ on $P_{\mathbb{C}} \times P_{\mathbb{C}}$ is the "$\xi_{2\delta}$" for $G_{\mathbb{C}}$. For characters μ and ν of $P_{\mathbb{C}}$, we denote the restriction of $\mu \boxtimes \nu$ to $P_{\mathbb{C}}$ realized as a real form of $P_{\mathbb{C}} \times P_{\mathbb{C}}$ as above by the same letter.

For the complex case, we have :

Theorem 2.2. *([Vogan-Zuckerman 1984])*

$$\mathcal{A}_{\mathcal{O}_0} \cong {}^u\mathrm{Ind}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}}(\xi_{2\delta} \boxtimes 1) \cong {}^u\mathrm{Ind}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}}(1 \boxtimes \xi_{2\delta}).$$

Therefore, Problem 1.2 reduced to the problem of the existence of intertwining operators.

For $t \in \mathbb{C}$, we define the following generalized Verma module:

$$M_{\mathfrak{p}}(t\delta) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \xi_{t\delta}.$$

The following result is well-known.

Proposition 2.3. *For $t_1, t_2 \in 2\mathbb{Z}$,*

$${}^u\mathrm{Ind}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}}(\xi_{t_1\delta} \boxtimes \xi_{t_2\delta}) \cong (M_{\mathfrak{p}}(-t_1\delta) \boxtimes M_{\mathfrak{p}}(-t_2\delta))^*_{K_{\mathbb{C}}-finite}$$

So, our Problem 1.2 is seriouly related to the existence of homomorphisms between generalized Verma modules. In fact, the following result is known.

Theorem 2.4. *([Matumoto 1993])*
Let t be a non-negative even integer. Then we have

$$M_{\mathfrak{p}}(-(t+2)\delta) \hookrightarrow M_{\mathfrak{p}}(t\delta)$$

if and only if $w_0 w_{\mathfrak{p}}$ is a Duflo involution in the Weyl group for $(\mathfrak{g}, \mathfrak{h})$.

If $w_0 w_{\mathfrak{p}}$ is a Duflo involution, using Propostion 2.2 we have:

$$\begin{array}{ccc} {}^u\mathrm{Ind}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}}(1 \boxtimes 1) & \longrightarrow & {}^u\mathrm{Ind}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}}(1 \boxtimes \xi_{2\delta}) \\ \downarrow & \not\circlearrowleft & \downarrow \\ {}^u\mathrm{Ind}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}}(\xi_{2\delta} \boxtimes 1) & \longrightarrow & {}^u\mathrm{Ind}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}}(\xi_{2\delta} \boxtimes \xi_{2\delta}). \end{array}$$

In fact, we have :

Theorem 2.5. $\mathcal{A}_{\mathcal{O}_0} \hookrightarrow {}^u\mathrm{Ind}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}}(\xi_{2\delta} \boxtimes \xi_{2\delta})$ *if and only if $w_0 w_{\mathfrak{p}}$ is a Duflo involution in the Weyl group for $(\mathfrak{g}, \mathfrak{h})$.*

§ 3. Type A case

As we seen in the case of complex groups, the statement in Problem 1.2 is not correct in general. However, for type A groups, we have affirmative answers.

3.1 GL$(n, \mathbb{C})$

We retain the notation in §2. We fix a Borel subalgebra $\mathfrak{b}$ such that $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{p}$. We denote by Π the basis of the root system with respect to $(\mathfrak{g}, \mathfrak{h})$ corresponding to $\mathfrak{b}$. We denote by S the subset of Π corresponding to $\mathfrak{p}$. Assumption 1.1 holds if and only if S is compatible with the symmetry of the Dynkin diagram. For a Weyl group of the type A, each involution is a Duflo involution. Hence, we have:

Theorem 3.6. *Under Assumption 1.1, we have* $\mathcal{A}_{\mathcal{O}_0} \hookrightarrow {}^u\mathrm{Ind}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}}(\xi_{2\delta} \boxtimes \xi_{2\delta})$.

3.2 GL$(n, \mathbb{R})$

Speh proved any derived functor module of GL$(n, \mathbb{R})$ is parabolically induced from the external tensor product of some so-called Speh representations and possibly a one-dimensional representation. Using this fact, we can reduce Problem 1.2 to embedding Speh representations into degenerate principal series. More pricisely, we consider $G = \mathrm{GL}(2n, \mathbb{R})$ and let P be a maximal parabolc subgroup whose Levi part is isomorphic to $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$. Then, $X = G_{\mathbb{C}}/P_{\mathbb{C}}$ contains a unique open G-orbit (say $\mathcal{O}$). In this setting, Assuption 1.1 holds. The fine structure of degenerate principal series for P has already been studied precisely. ([Sahi 1995], [Zhang 1995], [Howe-Lee 1999],[Barbasch-Sahi-Speh 1988]) From their results, we have:

$$\begin{array}{lll} \mathcal{A}_{\mathcal{O}} & \hookrightarrow {}^u\mathrm{Ind}_P^G(\xi_{2\delta}) & \text{if } n \text{ is odd,} \\ \mathcal{A}_{\mathcal{O}} & \hookrightarrow {}^u\mathrm{Ind}_P^G(\xi_{2\delta} \otimes \omega) & \text{if } n \text{ is even.} \end{array}$$

We can deduce an affirtive answer to Problem 1.2 from this.

3.3 $\mathrm{GL}(n, \mathbb{H})$

In this case, we also have an affirmative answer to Problem 1.2. The argument is similar to (and easier than) the case of $\mathrm{GL}(n, \mathbb{R})$.

3.4 $\mathrm{U}(m, n)$

Let $G = \mathrm{U}(m, n)$ and let P be an arbitrary prarabolic subgroup of G. In this case, Assumption 1.1 automatically holds. We denote by $\mathcal{V}$ the set of open G-orbits on $X = G_{\mathbb{C}}/P_{\mathbb{C}}$. In fact, we have:

$$\mathrm{Socle}({}^u\mathrm{Ind}_P^G(\xi_{2\delta})) = \bigoplus_{\mathcal{O}\in\mathcal{V}} \mathcal{A}_{\mathcal{O}}.$$

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