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Applications of computer algebra to some bifurcation problems in nonlinear vibrations

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1. Introduction

We studied some bifurcation problems in nonlinear vibrations ([K1-3]). In this article, we will explain mainly how we use computer algebra in establishing our results. In [K1-3] we did not explain it in detail. The computer algebra actually plays, however, a very important role in our study. We show that we can obtain quickly the computation results with good precision if we appropriately use the computer algebra. Though we mainly mention a bifurcation problem in forced vibration, our method works well for the problem in self-excited vibration (see Section 5).

We design our article in the following way. In Section 2 we summarize our problem and result in nonlinear forced vibration. In Section 3 we mention how to use the computer algebra in our computer simulations. In Section 4, we explain our numerical verification method with the computer algebra. In Section 5 we consider the self-excited vibration. This study is now in progress. We explain that we can prove the existence of period doubling bifurcation points essentially in the same way as in the forced vibration case. Therefore, for this case we can use the computer algebra extensively.

2. Our problem and result

Let $f(\lambda,u) := u_{tt} - c^2 u_{xx} + \mu u_t + u^3 - \lambda \cos t \sin x$. Here, $c,\mu > 0$ are constants and $\lambda > 0$ is a parameter. We consider the bifurcation phenomena of periodic solutions for the following dissipative semilinear wave equation:

$$\text{(W)} \quad \begin{cases} f(\lambda,u) = 0 \quad \text{in} \quad (0,\pi)\times \mathbf{R}^+, \\ u(0,t) = u(\pi,t) = 0 \quad \text{for} \quad t \geq 0. \end{cases}$$

This problem has some deep relations to the ordinary differential equation called the Duffing equation:

$$\text{(D)} \quad g(\lambda,y) := \frac{d^2y}{dt^2} + \mu\frac{dy}{dt} + y^3 - \lambda\cos t = 0.$$

By some numerical simulations (see Section 3) we can observe rich bifurcation phenomena (such as the existence of turning points, symmetry-breaking bifurcation, chaos) for our problem (W) and (D). The system (W) has some symmetry. Let S be the transformation defined by

$$S: \ u(x,t) \longmapsto -u(x,t+\pi). \tag{2.1}$$

Then we have $f(\lambda, Su) = Sf(\lambda, u)$. The symmetric periodic solution (resp. the asymmetric periodic solution) is a solution satisfying $Su = u$ (resp. $Su \neq u$).

In what follows, we will consider (W) with $c := 1.5$, $\mu := 0.05$. (The values of these constants have no special meaning.) Let us move the value of λ gradually larger from 0. Then we can observe by numerical simulations that a branch of asymmetric 2π-periodic solutions bifurcates from a branch of symmetric 2π-periodic solutions at a certain value $\lambda = \Lambda_0 \in (2.8, 2.9)$. We can give a mathematically rigorous proof to this observation.

Proposition 2.1. *Let $c = 1.5$, $\mu = 0.05$. Then, (W) has a symmetry-breaking bifurcation point (Λ_0, U_0) where a branch of 2π-symmetric solutions and a branch of 2π-asymmetric solutions intersect with each other. The bifurcation point (Λ_0, U_0) satisfies*

$$|\Lambda_0 - \lambda_0|^2 + \|U_0 - u_0\,;\, H^1(D)\|^2 \leq (0.000708)^2.$$

Here, $D := (0,\pi) \times (0, 2\pi)$, $\lambda_0 := 2.8828613$ and $u_0 := 1.2897865 \cos t \sin x + \cdots + 0.14470778 \times 10^{-7} \sin 5t \sin 9x$ has the form of a finite Fourier expansion consisting of 55 terms. We omit here the complete form of u_0.

In what follows, we give the outline of the proof. We refer [K1-3] for the details. Let X be a closed linear subspace in $H^1(D)$ defined by

$$X := \{ \sum_{\substack{m \in \mathbf{Z} \\ n \in 2\mathbf{N}-1}} a_{mn}\phi_{mn}\,;\ \sum_{\substack{m \in \mathbf{Z} \\ n \in 2\mathbf{N}-1}} (m^2+n^2+1)|a_{mn}|^2 < \infty \}.$$

Here, we set $\phi_{mn} := e^{imt} \sin nx$. Let S be a transformation defined by (2.1). We define the symmetric subspace X_s and the anti-symmetric subspace X_a:

$$X_s := \{u \in X; Su = u\} = \{ \sum_{\substack{m \in 2\mathbf{Z}-1 \\ n \in 2\mathbf{N}-1}} a_{mn}\phi_{mn}\,;\ \sum_{\substack{m \in 2\mathbf{Z}-1 \\ n \in 2\mathbf{N}-1}} (m^2+n^2+1)|a_{mn}|^2 < \infty \},$$

$$X_a := \{u \in X; Su = -u\} = \{ \sum_{\substack{m \in 2\mathbf{Z} \\ n \in 2\mathbf{N}-1}} a_{mn}\phi_{mn}\,;\ \sum_{\substack{m \in 2\mathbf{Z} \\ n \in 2\mathbf{N}-1}} (m^2+n^2+1)|a_{mn}|^2 < \infty \}.$$

Then, we have $X = X_s \oplus X_a$. We also define

$$Y := \overline{X}^{L^2(D)} = \{ \sum_{\substack{m \in \mathbf{Z} \\ n \in 2\mathbf{N}-1}} a_{mn}\phi_{mn}\,;\ \sum_{\substack{m \in \mathbf{Z} \\ n \in 2\mathbf{N}-1}} |a_{mn}|^2 < \infty \},$$

$Y_s := \overline{X_s}^{L^2(D)}$ and $Y_a := \overline{X_a}^{L^2(D)}$. We define two Hilbert spaces $\mathcal{V} := \mathbf{R} \times X_s \times X_a$ and $\mathcal{W} := \mathbf{R} \times Y_s \times Y_a$. Let $\mathcal{D}_0 := \{h \in X\,;\, h_{tt} - c^2 h_{xx} \in L^2(D)\}$. We define an extended system:

$$F\begin{pmatrix} \lambda \\ u \\ \phi \end{pmatrix} := \begin{pmatrix} l\phi - 1 \\ f(\lambda, u) \\ D_u f(\lambda, u)\phi \end{pmatrix} = 0.$$

Here, $F : \mathcal{V} \to \mathcal{W}$ with $\mathcal{D}(F) := \mathbf{R} \times \mathcal{D}_0$ and $l \in X_a^*$ is a functional defined by

$$l\phi := \frac{2}{\pi^2}(\phi, \sin 2t \sin x) \quad \text{for} \quad \phi \in X_a,$$

i.e. $l\,\cdot$ is Fourier coefficient of $\sin 2t \sin x$. To obtain Proposition 2.1 it suffices to prove the following (2.2) and (2.3) in view of our bifurcation theorem [K2, Theorem 3.1].

(2.2) $F(\lambda, u, \phi) = 0$ has an isolated solution (Λ_0, U_0, Φ_0) in a neighborhood of (λ_0, u_0, ϕ_0),

(2.3) $f_u(\Lambda_0, U_0)(\mathcal{D}_0 \cap X_s) = Y_s$.

Here, $\phi_0 \in X_a$ is a function satisfying $l\phi_0 = 1$ and approximately $D_u f(\lambda_0, u_0)\phi_0 = 0$. We can apply the convergence theorem of Newton's method ([K2, Theorem 1.1]) to obtain (2.2). For this purpose, we show the existence of $DF(\Lambda_0, U_0, \Phi_0)^{-1}$ and estimate its operator norm. To obtain (2.3) we show the existence of $f_u(\Lambda_0, U_0)^{-1}$.

3. Numerical simulations

3.1. Derivation of a truncated ordinary differential equation

We set $\phi_k(x) = \sin(2k-1)x$ $(k \in \mathbf{N})$ and

$$u_n(x,t) = \sum_{k=-n}^{n} a_k(t)\phi_k(x).$$

We constructed a truncated ordinary differential system of (W) with respect to a_k $(k = 1, \cdots, n)$. We use the Galerkin method. By using computer algebra, we can obtain the Fourier sine expansion of $f(\lambda, u_n)$:

$$f(\lambda, u_n) = \sum_k A_k \phi_k(x).$$

Here, A_k is a polynomial of $a_i(t)$, $a_j'(t)$ and $a_k''(t)$ $(1 \le i, j, k \le n)$. We regard the following system as a truncated system of (W):

(3.1) $$A_k = 0 \quad (k = -n, \cdots, n).$$

If we set $n = 5$, it is sufficient to observe our symmetry-breaking bifurcation phenomena in Section 2 by using our truncated system. Of course, we can use another method (e.g. the finite difference method) to observe our bifurcation phenomena. From our experience, however, our truncation method seems to be better in precision and in computation time for the simulation of our problem than the other methods.

3.2. Construction of approximate solutions with high precision

By using a truncation method in Section 3.1 and the digital Fourier analysis, we can obtain an approximate solution of (W) for each λ. We explain how to find another approximate solution with much higher precision. Here, we describe the method for (D) for simplicity. (For (W) the algorithm is essentially same but is more complicated.) Let $y_n^0 = \sum_{k=-n}^{n} c_k^0 e^{ikt}$ be an approximate solution of (D). We use the Galerkin method to obtain another approximate solution y_n with much better precision:

$$y_n = \sum_{k=-n}^{n} c_k e^{ikt}. \tag{3.2}$$

Let $g(\lambda, y_n) = \sum_k H_k e^{ikt}$ be the Fourier expansion of $g(\lambda, y_n)$. Here, H_k ($k \in \mathbf{Z}$) are polynomials of c_l ($l = -n, \cdots, n$). We have

$$\frac{\partial g(\lambda, y_n)}{\partial c_l} = \sum_k \frac{H_k}{\partial c_l} e^{ikt} \quad (-n \le l \le n). \tag{3.3}$$

We solve the system:

$$H_k = 0 \quad (k = -n, \cdots, n) \tag{3.4}$$

by the Newton's method. We set $\mathbf{c} := (c_{-n}, \cdots, c_n)$ and $\mathbf{H} := (H_{-n}, \cdots, H_n)$. Then, we compute

$$\mathbf{c}_1 = \mathbf{c}_0 - \frac{D\mathbf{H}}{D\mathbf{c}}(\mathbf{c}_0)^{-1}\mathbf{H}(\mathbf{c}_0). \tag{3.5}$$

Here, we simply write $\mathbf{H}(\mathbf{c}_0) := \mathbf{H}|_{\mathbf{c}=\mathbf{c}_0}$ and so on. We see that (3.2) with $\mathbf{c} = \mathbf{c}_1$ is in general our approximate solution with higher precision. We need not find $\mathbf{H}$ explicitly. (It takes too long time!) Actually, it suffices to find $\mathbf{H}(\mathbf{c}_0)$ and $\frac{D\mathbf{H}}{D\mathbf{c}}(\mathbf{c}_0)$. We easily expand $g(\lambda, y_n^0)$ by computer algebra and find the Fourier coefficients $\mathbf{H}(\mathbf{c}_0)$. In the same way, we easily find $\frac{D\mathbf{H}}{D\mathbf{c}}(\mathbf{c}_0)$ by using (3.3). It is also possible to find the approximate Fourier coefficients of $g(\lambda, y_n^0)$ without using computer algebra (e.g. see [UR]). However, it needs the complicated procedure and the answers contain the approximate errors.

Remark 3.1. We actually use a kind of the least square method in finding an approximate solution with high precision (see [K3]). It is, however, similar to the Galerkin method case with respect to how to use the computer algebra. Therefore, we described the latter case to which the readers are familiar. □

4. Numerical verification

In this section we briefly write how to control our numerical computations and to estimate the norms of functions.

4.1. Control of numerical computations

We approximate $x \in \mathbf{R}$ by finite decimal numbers in some fashions. First we approximate a number by an integer plus n-digit decimal number of the decimal form:

$$m.a_1a_2\cdots a_n,$$

Here, $m \in \mathbf{Z}$ and $0 \le a_j \le 9$ is a figure $(1 \le j \le n)$. Let $\mathbf{Z}_+ := \mathbf{N} \cup \{0\}$ and $n \in \mathbf{Z}_+$. For $x \ge 0$ we define

$$\mathrm{ceil}(x,n) := \min\{m \in \mathbf{Z}_+\,;\, m \ge 10^n x\} \times 10^{-n},$$

$$\mathrm{floor}(x,n) := \max\{m \in \mathbf{Z}_+\,;\, m \le 10^n x\} \times 10^{-n},$$

$$\mathrm{round}(x,n) := \begin{cases} \mathrm{floor}(x,n) & \text{if} \quad x - \mathrm{floor}(x,n) < 0.5 \times 10^{-n}, \\ \mathrm{ceil}(x,n) & \text{if} \quad x - \mathrm{floor}(x,n) \ge 0.5 \times 10^{-n}. \end{cases}$$

Next, we approximate $x \ge 0$ by n-digit floating point form:

$$0.a_1a_2\cdots a_n \times 10^m \quad \text{with} \quad 1 \le a_1 \le 9,$$

i.e. $0.a_1a_2\cdots a_n$ is the mantissa with length n. We set $\varepsilon_0 := 10^{-25}$. We define

$$\mathrm{float}(x,n) := \begin{cases} \mathrm{round}(10^{n-m}x, 0) \times 10^{m-n} & \text{if} \quad |x| \ge \varepsilon_0, \\ 0 \quad \text{if} \quad |x| < \varepsilon_0, \end{cases}$$

where $m := \max\{k \in \mathbf{Z}\,;\, k > \log_{10}|x|\}$. We expand the domain of $\mathrm{ceil}(\cdot,n)$, $\mathrm{floor}(\cdot,n)$, $\mathrm{round}(\cdot,n)$ and $\mathrm{float}(\cdot,n)$ so that they are odd functions. We can realize these functions on the computer without difficulty.

In our proof of Proposition 2.1 we construct big matrices to show the existence of inverses for linearized operators. For this purpose, we need to show explicitly the way of *unique* construction of an approximate inverse matrix for a given big square matrix. In [K1] we use classical Gauss-Jordan method with partial pivot selection. We realize the complete control of numerical computations by using the function $\mathrm{float}(\cdot,\cdot)$.

4.2. Estimate of norms

Let $h(t,x)$ be a 2π-periodic function with respect to t-variable and x-variable which has the form of finite Fourier series:

$$h(t,x) = \sum_{\substack{m \in \mathbf{Z} \\ n \in I}} C_{mn} e^{imt+inx} \quad \text{with} \quad I = 2\mathbf{N}-1 \quad \text{or} \quad I = 2\mathbf{N}.$$

Then, by Parseval equality, we have

$$\|h\|_{L^2(D)} = \sqrt{2}\pi\Big(\sum_{\substack{m \in \mathbf{Z} \\ n \in I}} |C_{mn}|^2\Big)^{1/2}.$$

We define

$$|h|_{2,n} := \sqrt{2}\pi\Big[\sum_{\substack{m \in \mathbf{Z} \\ n \in I}} \mathrm{ceil}(|C_{mn}|^2,\ n)\Big]^{1/2}.$$

Then, we have $\|h\|_{L^2(D)} \le |h|_{2,n}$. By using the computer algebra, we can easily find the explicit value of $|h|_{2,n}$. We also define and use L^∞-version of $|\cdot|_{2,n}$.

5. Analysis for self-excited vibration

We briefly mention how we can prove the existence of bifurcation points in self-excited vibrations. Though our method also works well for partial differential systems, we consider here the following self-excited ordinary differential system for the simplicity of description:

$$\dot{\mathbf{y}} = \mathbf{f}(\lambda, \mathbf{y}) \quad \text{with} \quad \mathbf{y},\ \mathbf{f}(\lambda,\mathbf{y}) \in \mathbf{R}^n. \tag{5.1}$$

In this case, the period of a solution varies as the value of λ changes. Since we have the difficulty in treating (5.1) directly, we study the following transformed extended system: $F(\lambda,\omega,\mathbf{z}) = 0$. We define $F : \mathbf{R} \times X \to Y$ by

$$F : (\lambda, \begin{pmatrix} \omega \\ \mathbf{z} \end{pmatrix}) \longmapsto \begin{pmatrix} l\mathbf{z} \\ \dot{\mathbf{z}} - \omega \mathbf{f}(\lambda, \mathbf{z}) \end{pmatrix}. \tag{5.2}$$

Here, we set $X := \mathbf{R} \times \mathbf{H}^1_{\mathrm{per}}(0,2\pi)$ and $Y := \mathbf{R} \times \mathbf{L}^2(0,2\pi)$, and assume that $l : \mathbf{H}^1_{\mathrm{per}}(0,2\pi) \to \mathbf{R}$ is an appropriate functional. We need l to normalize $\mathbf{z}$. Indeed, if $\mathbf{z}(t)$ is a solution of $\dot{\mathbf{z}} - \omega\mathbf{f}(\lambda,\mathbf{z}) = 0$ then $\mathbf{z}(t+\tau)$ also satisfies the same equation for a fixed $\tau \in \mathbf{R}$. We verify that $(\lambda,\omega,\mathbf{z})$ is a solution of $F = 0$ if and only if $(\lambda,\mathbf{y})$ with $\mathbf{y}(t) = \mathbf{z}(t/\omega)$ is a periodic solution of (5.1) with the period

$2\pi\omega$. As an important case, we will consider the period doubling bifurcation. We set $l : \mathbf{z} = (z_1, \cdots, z_n) \mapsto (z_1, \cos 2t)_{L^2(0,2\pi)}$. Then, F has the following symmetry:

$$F(\lambda, S\begin{pmatrix}\omega\\ \mathbf{z}\end{pmatrix}) = SF(\lambda, \begin{pmatrix}\omega\\ \mathbf{z}\end{pmatrix}) \quad \text{with} \quad S\begin{pmatrix}\omega\\ \mathbf{z}(t)\end{pmatrix} := \begin{pmatrix}\omega\\ \mathbf{z}(t+\pi)\end{pmatrix}. \tag{5.3}$$

A period doubling bifurcation point of (5.1) corresponds to a symmetry-breaking bifurcation point of $F = 0$. We can find the latter in the same way as in Section 2. As an application to a concrete example, our method guarantees the existence of a period doubling bifurcation point in self-excited vibration described by a truncated Navier-Stokes system in [BF]. We will write the details in a near future work ([K4]).

References

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