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Integral local systems

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§1. Rigid local systems and integral representations of their sections.

Let $t_1, \ldots, t_p$ be p points in $\mathbf{C}$, and set $X = \mathbf{P}^1 \setminus \{t_1, \ldots, t_p, \infty\}$. A local system $\mathcal{F}$ on X of rank n can be specified by a $(p+1)$-tuple $(A_1, \ldots, A_p, A_{p+1})$ of matrices in $\mathrm{GL}(n, \mathbf{C})$ satisfying $A_{p+1} \cdots A_1 = I_n$, if we associate $\gamma_j \in \pi_1(X, x_0)$ given by Figure 1 with the matrix A_j for $j = 1, \ldots, p$. Thus we denote $\mathcal{F} = (A_1, \ldots, A_p, A_{p+1})$.

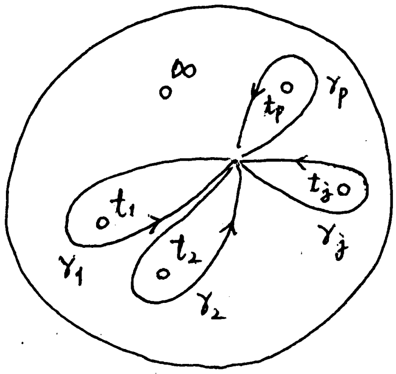

Figure 1.

A local system $\mathcal{F} = (A_1, \ldots, A_{p+1})$ is said to be *rigid* if it is determined by the conjugacy classes of the A_j's uniquely up to isomorphisms of local systems. In other words $\mathcal{F}$ is rigid if, for any local system $\mathcal{G} = (B_1, \ldots, B_{p+1})$ with $B_{p+1} \cdots B_1 = I_n$ such that there exists $C_j \in \mathrm{GL}(n, \mathbf{C})$ such that $B_j = C_j A_j {C_j}^{-1}$ for each j, there exists $D \in \mathrm{GL}(n, \mathbf{C})$ such that $B_j = D A_j D^{-1}$ for all j. Local systems over X corresponds to Fuchsian differential equations over X. In this correspondence rigid local systems correspond to Fuchsian differential equations *without accessory parameters.*

It is easy to see whether a local system $\mathcal{F} = (A_1, \ldots, A_{p+1})$ is rigid. Define the *index of rigidity* $\iota(\mathcal{F})$ by

$$\iota(\mathcal{F}) = (2 - (p+1))n^2 + \sum_{j=1}^{p+1} \dim Z(A_j),$$

where $Z(A)$ denotes the centralizer of A. If $\mathcal{F}$ is irreducible, then $\iota(\mathcal{F}) \leq 2$ holds, and in this case $\mathcal{F}$ is rigid if and only if $\iota(\mathcal{F}) = 2$ ([Ka]). Katz [Ka] gave an algorithm for constructing all rigid local systems, and Detteweiler and Reiter [DR] reformulated the algorithm into down-to-earth way. In this section we give another algorithm for constructing rigid local systems, and show that there exist integral representations of their sections.

Let $\mathcal{F} = (A_1, \ldots, A_{p+1})$ be an irreducible rigid local system. Kostov [Ko] showed that there exists a Fuchsian system of differential equations

$$\frac{dw}{dx} = \left(\sum_{j=1}^{p} \frac{B_j}{x - t_j} \right) w \tag{1.1}$$

whose monodromy representation coincides with the local system $\mathcal{F}$. Note that $\exp B_j \sim A_j$ for $1 \le j \le p$ and $\exp(-\sum_{j=1}^p B_j) \sim A_{p+1}$. We set $B_{p+1} = -\sum_{j=1}^p B_j$. From now on we assume that every A_j is semi-simple (i.e. diagonalizable). We call such local system $\mathcal{F}$ and the corresponding system (1.1) *of semi-simple type.*

We are going to construct the Fuchsian system (1.1). Suppose for a moment that the tuple $(B_1, \dots, B_p)$ is given. Set

$$\hat{T} = \begin{pmatrix} t_1 I_n & & \\ & \ddots & \\ & & t_p I_n \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} B_1 & B_2 & \cdots & B_p \\ B_1 & B_2 & \cdots & B_p \\ \vdots & \vdots & & \vdots \\ B_1 & B_2 & \cdots & B_p \end{pmatrix},$$

and consider the system of differential equations

$$(xI_{pn} - \hat{T})\frac{dU}{dx} = (\hat{B} + \lambda I_{pn})U \tag{1.2}$$

with a parameter λ. It is shown that the system (1.2) is free from accessory parameters (we call such system also *rigid*) and irreducible for generic values of λ. The system (1.2) is of *Okubo normal form* (ONF, for short) ([O]). Yokoyama [Y] gave an algorithm for constructing all irreducible rigid systems of ONF of semi-simple type: He defined two kinds of operations — the extension and the restriction — for systems of ONF of semi-simple type, and showed that every irreducible rigid system of ONF of semi-simple type can be obtained from a system $(x-t)du/dx = au$ of rank 1 by a finite iteration of these operations. The solutions of the systems obtained by these operations can be represented by using the solutions of the original system ([H]). Here we roughly sketch how to obtain the solutions.

Consider a system (#) of ONF of semi-simple type with regular singular points $t_1, \dots, t_p, \infty$, and let $u(x)$ be its solution. We define a function $\hat{u}(x,y)$ in two variables by the integral

$$\hat{u}(x,y) = \int_0^1 t^{\rho_1}(1-t)^{-\rho_2-1}u(x+(y-x)t)dt.$$

It is shown that $\hat{u}(x,y)$ satisfies a Pfaffian system with singular locus $\bigcup_{j=1}^p(\{x=t_j\}\cup\{y=t_j\})\cup\{x=y\}$.

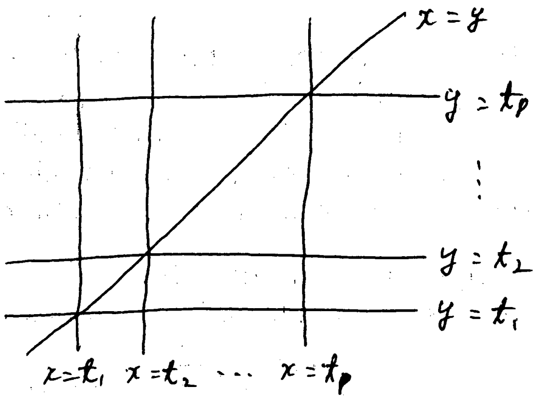

Figure 2.

Then the restrcition of $\hat{u}(x,y)$ to a regular locus $y = y_0$ $(y_0 \neq t_j)$ gives a solution of the extension of (#), and the restriction of $\hat{u}(x,y)$ to a singular locus $y = t_j$ gives a solution of the restriction of the extension of (#). Thus the solutions of these systems can be represented by the integrals whose integrand contain $u(x)$. Since we start from the system $(x-t)du/dx = au$ which has a solution $u(x) = (x-t)^a$, we conclude that the solutions of every rigid system of ONF of semi-simple type have an integral representation of Euler type. Thus the system (1.2) is constructed by Yokoyama's algorithm, and its solutions have an integral representation of Euler type.

The system (1.1) is obtained from (1.2) by the specialization $\lambda = -1$.

Proposition 1.1. *We assume that* $\det B_{p+1} \neq 0$, *and set*

$$Q := \begin{pmatrix} {B_{p+1}}^{-1}B_1 & {B_{p+1}}^{-1}B_2 & {B_{p+1}}^{-1}B_3 & \cdots & {B_{p+1}}^{-1}B_p \\ I_n & -I_n & & & \\ I_n & & -I_n & & \\ \vdots & & & \ddots & \\ I_n & & & & -I_n \end{pmatrix}.$$

If $\lambda = -1$, *the system (1.2) becomes reducible. In fact, if we set*

$$W(x) := Q(xI_{pn} - \hat{T})U(x), \tag{1.3}$$

the system (1.2) has a solution $U(x)$ *such that*

$$W(x) = \begin{array}{r} n\{ \\ pn-n\{ \end{array} \begin{pmatrix} w(x) \\ 0 \end{pmatrix},$$

and $w(x)$ *satisfies the system (1.1).*

We illustrate the above process in Figure 3.

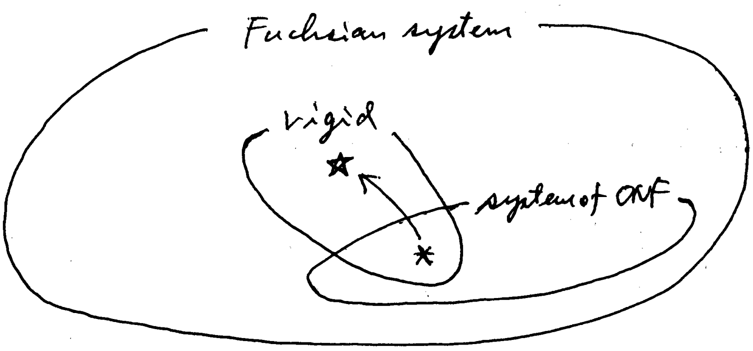

Figure 3.

An integral representation of the solutions of the system (1.1) is derived from one of the system (1.2) also by the specialization $\lambda = -1$. Let

$$U(x) = \int_\Delta \prod_{j=1}^{m} P_j(s)^{\alpha_j}\eta \tag{1.4}$$

be an integral representation of the solution of (1.2), where $P_j(s)$ is a polynomial in the integral variables $s = (s_1, \ldots, s_k)$ and η is a vector of twisted cocycles. We note that the exponents α_j's are linear functions of the eigenvalues of the residue matrices at the singular points of (1.2). Then the parameter λ, which is an eigenvalue of the residue matrix at ∞, appears linearly in the exponents α_j's. If none of the exponents α_j becomes a negative integer when we put $\lambda = -1$, the integral representation (1.4) with $\lambda = -1$ gives an integral representation of solutions of (1.2) via the transformation (1.3). Suppose that some of the α_j's become negative integers by the specialization $\lambda = -1$. In this case the integral (1.4) has a pole at $\lambda = -1$ as a function in λ, and by taking the residue we still get an integral representation of the system (1.2) with $\lambda = -1$ and hence of the system (1.1). Thus we get the following theorem.

Theorem 1.2. *Every Fuchsian system of differential equations over* $\mathbf{P}^1$ *whose monodromy representation is irreducible, rigid and of semi-simple type can be obtained from a rank 1 system by a finite iteration of Yokoyama's operations together with a specialization of an exponent. The solutions of such system have an integral representation of Euler type.*

We call the Fuchsian systems of differential equations which have integral representations of solutions *integral.* We also call the corresponding local systems integral. Then we can sum up what we have shown in the following figure.

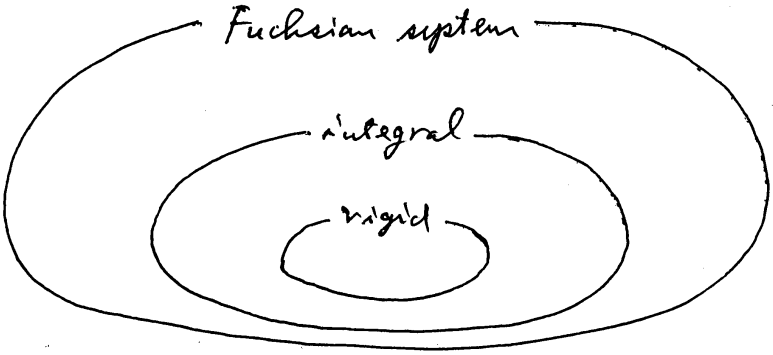

Figure 4.

Example. Let B_1, B_2 be 5×5-matrices such that

$$B_1 \sim \begin{pmatrix} a_1 I_2 & & \\ & a_2 I_2 & \\ & & a_3 \end{pmatrix}, \quad B_2 \sim \begin{pmatrix} b_1 I_2 & & \\ & b_2 I_2 & \\ & & b_3 \end{pmatrix},$$

$$B_1 + B_2 \sim \begin{pmatrix} \mu_1 I_2 & & \\ & \mu_2 I_2 & \\ & & \mu_3 \end{pmatrix}.$$

It is shown that the system

$$\frac{dw}{dx} = \left(\frac{B_1}{x - t_1} + \frac{B_2}{x - t_2} \right) w \tag{1.5}$$

is rigid and irreducible for generic values of the parameters. Following Theorem 1.2, we get the integral representation of the solutions of (1.5)

$$
\begin{aligned}
w(x) = (x-t_1)^{a_1-1}(x-t_2)^{b_2} \int_\Delta \left(1-\frac{x-t_1}{t_2-t_1}s_4\right)^{\mu_1-a_1-b_1} & {s_4}^{a_1+b_1-\mu_2} \\
& \times (s_3+s_4-s_3s_4)^{\mu_2-a_2-b_1}(s_2+s_3-s_2s_3)^{\mu_1-a_1-b_2} \\
& \times {s_2}^{a_2+b_2-\mu_1}(s_1-s_2)^{\mu_2-a_2-b_2} \\
& \times {s_1}^{a_1+a_2+b+1+b+2-\mu_2-\mu_3}(1-s_1)^{a_1+a_2+b_2+b_3-\mu_1-\mu_3}\eta,
\end{aligned}
$$

where η is a 5-vector of twisted cocycles.

§2. Non-rigid integral local systems.

It will be much interesting to study non-rigid integral local systems. It may be very hard to see whether a given non-rigid local system is integral or not, however, it is very easy to obtain non-rigid integral local systems if we start from integral representations. In this section we give one such example from [DF].

Let Φ be the following product of power functions in s_1, s_2:

$$\Phi := {s_1}^a(s_1-1)^b(s_1-x)^c{s_2}^a(s_2-1)^b(s_2-x)^c(s_1-s_2)^g.$$

We consider the vector $Y(x)$ of functions given by the integral

$$Y(x) = \int_\Delta \Phi \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}, \tag{2.1}$$

where

$$\varphi_1 = \frac{ds_1 \wedge ds_2}{s_1 s_2},\quad \varphi_2 = \frac{ds_1 \wedge ds_2}{(s_1-1)(s_2-1)},\quad \varphi_3 = \frac{ds_1 \wedge ds_2}{s_1(s_2-1)} + \frac{ds_1 \wedge ds_2}{(s_1-1)s_2}.$$

Then $Y(x)$ satisfies the system of differential equations

$$\frac{dY}{dx} = \left(\frac{A}{x} + \frac{B}{x-1}\right)Y \tag{2.2}$$

with

$$A = \begin{pmatrix} 2a+2c+g & 0 & b \\ 0 & 0 & 0 \\ 0 & 2b+g & a+c \end{pmatrix},\quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2b+2c+g & 0 \\ 2a+g & 0 & b+c \end{pmatrix}.$$

It is easy to see that

$$A \sim \begin{pmatrix} 2a+2c+g & & \\ & a+c & \\ & & 0 \end{pmatrix}, \quad B \sim \begin{pmatrix} 2b+2c+g & & \\ & b+c & \\ & & 0 \end{pmatrix},$$
$$A+B \sim \begin{pmatrix} 2a+2b+2c+g & & \\ & a+b+2c+g & \\ & & 2c \end{pmatrix}.$$

Since the monodromy matrices at $x = 0, 1, \infty$ have the same spectral types as $A, B, A+B$, respectively, the index of rigidity of the corresponding local system is calculated to be 0. Then the system (2.2) is non-rigid, and has one accessory parameter. Precisely speaking, the *integral* system (2.2) is obtained from a system containing one accessory parameter by putting some special value into the accessory parameter. We are going to see what is the special value.

It is convenient to consider a single differential equation instead of the system (2.2). The differential equation satisfied by the first element $y_1(x)$ of $Y(x) = {}^t(y_1(x), y_2(x), y_3(x))$ is calculated as

$$x^2(x-1)^2y''' + \bar{p}(x)y'' + \bar{q}(x)y' + \bar{r}(x)y = 0, \tag{2.3}$$

where

$$\begin{aligned}
\bar{p}(x) &= x(x-1)[(3-3a-3b-6c-2g)x - (3-3a-3c-g)],\\
\bar{q}(x) &= (1-3a+2b^2-3b+4ab+2b^2-6c+12ac+12bc+12c^2-2g\\
&\quad +3ag+3bg+8cg+g^2)x^2+(-2+6a-4a^2+4b-4ab+10c\\
&\quad -16ac-8bc-12c^2+3g-4ag-2bg-8cg-g^2)x\\
&\quad +(a+c-1)(2a+2c+g-1),\\
\bar{r}(x) &= -2c(a+b+2c+g)(2a+2b+2c+g)x\\
&\quad +c(2a+2c+g-1)(2a+2b+2c+g).
\end{aligned}$$

Then we get the Riemann scheme of the equation (2.3):

$$\left\{\begin{matrix} x=0 & x=1 & x=\infty \\ 0 & 0 & -2c \\ a+c & b+c+1 & -(a+b+2c+g) \\ 2a+2c+g & 2b+2c+g+2 & -(2a+2b+2c+g) \end{matrix}\right\} \tag{2.4}$$

Conversely we shall start from the Riemann scheme

$$\left\{\begin{matrix} x=0 & x=1 & x=\infty \\ 0 & 0 & \lambda \\ \alpha & \gamma & \mu \\ \beta & \delta & \nu \end{matrix}\right\}, \tag{2.5}$$

and determine the corresponding differential equation. Then we get

$$x^2(x-1)^2y''' + p(x)y'' + q(x)y' + r(x)y = 0, \tag{2.6}$$

where

$$\begin{aligned} p(x) &= p_1x + p_2x^2 + p_3x^3, \\ q(x) &= q_0 + q_1x + q_2x^2, \\ r(x) &= r_0 + r_1x, \end{aligned}$$

and the coefficients of these polynomials are given by

$$\begin{aligned} p_1 &= 3 - \alpha - \beta, \\ p_2 &= 2\alpha + 2\beta + \gamma + \delta - 9, \\ p_3 &= \lambda + \mu + \nu + 3, \\ q_0 &= (\alpha - 1)(\beta - 1), \\ q_1 &= -\alpha\beta + \gamma\delta - \lambda\mu - \mu\nu - \nu\lambda + 2\alpha + 2\beta - 4, \\ q_2 &= \lambda\mu + \mu\nu + \nu\lambda + \lambda + \mu + \nu + 1, \\ r_1 &= \lambda\mu\nu. \end{aligned}$$

Note that the value of the coefficient r_0 is arbitrary, which means that r_0 is the accessory parameter.

Now we put the values of $\alpha, \ldots, \nu$ so that the Riemann schemes (2.4) and (2.5) coincide (i.e. $\alpha = a + c$, etc.), and compare the coefficients of the differential equations (2.3) and (2.6). Then we see that the differential equation (2.3) is obtained from (2.6) by taking the value of the accessory parameter

$$r_0 = c(2a + 2c + g - 1)(2a + 2b + 2c + g). \tag{2.7}$$

We think that the differential equation (2.6) does not have an integral representation of solutions for generic values of the accessory parameter r_0. Then it will be a very interesting problem how we can determine the values of accessory parameters so that the differential equation becomes integral. I think p-adic approach and the deformation theory of differential equations will be helpful.

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