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Singular solutions of Nonlinear Fuchsian Equations and Applications to Normal Form Theory

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Motivation and Examples

Vector fields with an isolated singular point

Let us consider the following vector field with an isolated singular point at the origin

$$\mathcal{X}(x)=\sum_{j=1}^{n}a_j(x)\frac{\partial}{\partial x_j}, \tag{3}$$

where $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$ or $\mathbb{C}^n$, and $a_j(x)$ is smooth in x. Namely we assume

$$\mathcal{X}(0)=0, \tag{4}$$

and $\mathcal{X}$ does not vanish in some neighborhood of $x=0$ except for the origin.

Linearization and Homology Equation

We want to linearize $\mathcal{X}(x)$ by a change of variables

$$x=y+v(y),\quad v=O(|y|^2). \tag{5}$$

We write $\mathcal{X}(x)$ in the form

$$\mathcal{X}(x)=x\Lambda\frac{\partial}{\partial x}+R(x)\frac{\partial}{\partial x}\equiv X(x)\frac{\partial}{\partial x}, \tag{6}$$

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$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right),$$

$$X(x) = x\Lambda + R(x), \tag{7}$$

where

$$R(x) = (R_1(x), \dots, R_n(x)), \quad R(x) = O(|x|^2), \tag{8}$$

and Λ is an $n \times n$ constant matrix.

Noting that

$$\begin{aligned} X(x)\frac{\partial}{\partial x} &= X(y+v(y))\frac{\partial y}{\partial x}\frac{\partial}{\partial y} \\ &= X(y+v(y))\left(\frac{\partial x}{\partial y}\right)^{-1}\frac{\partial}{\partial y}, \end{aligned}$$

the **linearization condition** can be written in the following form

$$X(y+v)(1+\partial_y v)^{-1} = y\Lambda.$$

Therefore

$$(y+v)\Lambda + R(y+v) = y\Lambda(1+\partial_y v) = y\Lambda + y\Lambda\partial_y v. \tag{9}$$

Hence v satisfies the so-called **homology equation**

$$\mathcal{L}v \equiv y\Lambda\partial_y v - v\Lambda = R(y+v(y)), \quad v = (v_1, \dots, v_n). \tag{$*$}$$

Summing up we obtain

The necessary and sufficient condition for that $(*)$ *has a solution* v *is that* $\mathcal{X}$ *is linearized by the change of substitution* $x = y + v(y)$.

Expression of a homology equation

We assume that Λ is in a diagonal matrix, namely

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}. \tag{10}$$

Noting that

$$y\Lambda\partial_y = \sum_{k=1}^{n} \lambda_k y_k \frac{\partial}{\partial y_k}$$

we obtain

$$\mathcal{L}v$$

$$(11) \qquad = \begin{pmatrix} \sum \lambda_k y_k \frac{\partial}{\partial y_k} - \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \sum \lambda_k y_k \frac{\partial}{\partial y_k} - \lambda_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

In the following, for the sake of simplicity we always assume that a homology equation has the above expression.

Non-resonant condition

The **indicial polynomial** of $\mathcal{L}$ is given by

$$(12) \qquad \sum_{k=1}^{n} \lambda_k \zeta_k - \lambda_j, \quad (j = 1, \ldots, n).$$

$\mathcal{L}$ is said to be **non-resonant** if

$$(13) \qquad \sum_{k=1}^{n} \lambda_k \alpha_k - \lambda_j \neq 0$$

for $\forall \alpha \in (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$, $|\alpha| \geq 2$, and $j = 1, \ldots, n$.

If (13) does not hold we say that $\mathcal{L}$ is **resonant**. The set of y^α with α not satisfying (13) for some j is called a **resonance**. We have

Under non-resonant condition there exists a formal power series solution.

Indeed, $\mathcal{L}v = f$ is written in

$$\mathcal{L}(\sum_\alpha v_\alpha y^\alpha) = \sum_\alpha (\sum_{k=1}^{n} \lambda_k \alpha_k - \Lambda) v_\alpha y^\alpha = \sum_\alpha f_\alpha y^\alpha.$$

Because $(\sum_{k=1}^{n} \lambda_k \alpha_k - \Lambda)$ is invertible $\mathcal{L}^{-1}$ exists. Because$R(x) = O(|x|^2)$ we can determine a formal power series solution by a method of indeterminate coefficients.

Two theorems for the solvability of a homology equation

Poincaré introduced a famous **Poincaré condition**

$$\text{Re } \lambda_j > 0, \quad j = 1, \dots, n$$

and showed the solvability of $(*)$ in a class of analytic functions.

Solvability of $(*)$ in a real domain

Theorem (Sternberg) *Assume the hyperbolic condition*

$$\text{Re } \lambda_k \neq 0, \quad k = 1, \dots, n. \tag{14}$$

Moreover, suppose the non-resonant condition. Then $(*)$ has a smooth solution.

If resonance occurs we have

Theorem (Grobman- Hartman) *Assume the hyperbolicity. Then $(*)$ has a continuous solution.*

Remark A continuous solution of $(*)$ is defined as a **weak solution.** The definition of a weak solution is standard. There are extensions of this result to the C^k $(k \geq 0)$ case by Blitskiy et. al for a certain class of vector fields with resonances.

Object of Study

We want to solve $(*)$ in the case of resonances in a class of functions with a "log" type singularity. We also want to solve $(*)$ in a class of functions holomorphic in the domain which is a product of sectors with vertex at the origin.

Statement of the results

Singular solutions

Theorem 1. *Assume the Poincaré condition and*

$$\forall\, i, j, k, \qquad \lambda_i + \lambda_j \neq \lambda_k.$$

Then Eq. $(*)$ has a solution v of the form

$$v(y) = \sum_{|\alpha| \geq 2, \alpha \geq \beta} v_{\alpha\beta} y^\alpha (\log y)^\beta,$$

where $(\log y)^\beta = \prod_{j=1}^n (\log y_j)^{\beta_j}$. $v(y)$ *converges in*

$$\{y \in \mathbb{C}^n; |y| < \exists\varepsilon, \ |y_j \log y_j| < \varepsilon(j = 1, \ldots, n)\}.$$

Remark. If there is no resonance the above solution is a classical solution constructed by Poincaré.

If we restrict the solution v to the real domain we obtain a finitely smooth solution of $(*)$. Hence **a finite smoothness occurs because of the log type singularity caused by the resonance.**

Example Consider the case $n = 2$. Let $m \geq 2$ be an integer. Let us consider

$$\mathcal{L}_1 = x_1\partial_1 + mx_2\partial_2 - 1, \quad \mathcal{L}_2 = x_1\partial_1 + mx_2\partial_2 - m.$$

The only resonance is $(\alpha_1, \alpha_2) = (m, 0)$. The solution v has singularity of $\log x_1$ type.

Indeed, the resonance $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ satisfies $\alpha_1 + \alpha_2 \geq 2$ and

$$\alpha_1 + m\alpha_2 - 1 = 0, \qquad \text{or} \qquad \alpha_1 + m\alpha_2 = m.$$

Since $\alpha_1 + m\alpha_2 - 1 \neq 0$ by assumption we obtain $\alpha_1 + m\alpha_2 = m$ and $\alpha_1 + \alpha_2 \geq 2$. It follows that $(\alpha_1, \alpha_2) = (m, 0)$.

Sketch of the proof of Theorem 1. For the sake of simplicity we will prove the above example. We will construct a formal solution of $(*)$ in the following form

$$u_j(x) = \sum_{\alpha \in \mathbb{Z}_+^2, |\alpha| \geq 2, k} u_{\alpha,k}^j x^\alpha (\log x_1)^k, \quad j = 1, 2.$$

The equation $(*)$ can be writtten in the following form

$$(*) \qquad \mathcal{L}_j u_j = R_j(x_1 + u_1, x_2 + u_2), \quad j = 1, 2.$$

We set $u_{\alpha,k} = (u_{\alpha,k}^1, u_{\alpha,k}^2)$. We determine $u_{\alpha,k}$ $k = 0, 1, 2, \ldots$ inductively. We determine $u_{\alpha,0}$. By comparing the coefficients we can determine $u_{\alpha,0}$ for $|\alpha| \leq m$, $\alpha \neq (m, 0)$. On the other hand we note

$$\mathcal{L}_2(x_1^m) = 0, \quad \mathcal{L}_2(x_1^m \log x_1) = x_1^m.$$

Hence we set $u^2_{(m,0),0} = 0$, $u_{(m,0),0} = (u^1_{(m,0),0}, 0)$. We note that we can determine $u^1_{(m,0),0}$ and $u^2_{(m,0),1}$ by comparing the coefficients of x_1^m in $(*)$ since $\mathcal{L}_1$ has the nonresonance property. It is clear that we can determine $u_{\alpha,0}$ for $|\alpha| > m$ from $(*)$ because there is no resonance for $|\alpha| > m$.

We next determine $u_{\alpha,1}$. We have already determined $u_{(m,0),1} = (0, u^2_{(m,0),1})$. By the nonresonance property we can determine $u_{\alpha,1}$ for $|\alpha| > m$. Inductively, $u_{\alpha,2}$ ($|\alpha| = 2m$) can be determined by comparing the coefficients of $x_1^{2m}(\log x_1)^2$. The terms $u_{\alpha,2}$ ($|\alpha| > 2m$) can be determined inductively by the nonresonance property. Inductively, we can determine $u_{\alpha,k}$ ($k = 0, 1, 2, \ldots$). Hence we can determine a formal power series solution. The convergence can be proved by the method of majorant series. This ends the proof.

Solvability in the sectorial domain

Let S_0 be a sector in the complex plane, $S_0 := \{z; |\arg z| < \theta\}$, where $\theta > 0$ is a given small number and the branch of $\arg z$ is taken so that the argument is zero on the real axis. We define a sectorial domain S in $\mathbb{C}^n$ as the product of n copies of S_0, $S = S_0 \times \cdots \times S_0$. In the following we consider the solvability of the equation $(*)$ in the sectorial domain S. The typical example of the nonlinear term $R(x)$ is the following:

$$R(x) = A \prod_{j=1}^{n} \frac{x_j^{\alpha_j}}{(x_j - c_j)^{\beta_j}},$$

where A, $c_j \in \mathbb{C} \setminus \overline{S}$, $0 < \alpha_j < \beta_j$ ($j = 1, \ldots, n$) are constants. We set $\lambda := (\lambda_1, \ldots, \lambda_n)$. Then we have

Theorem 2. *Suppose that*

$$\lambda_j \in \mathbb{R} \setminus 0 \qquad (j = 1, \ldots, n).$$

Let $\Gamma \subset \mathbb{R}^n$ be an open set such that $0 \in \Gamma$ and

$$\Gamma \cap \{\eta; \langle \lambda, \eta \rangle = \lambda_j\} = \emptyset,$$

for every $j = 1, \ldots, n$, where $\langle \lambda, \eta \rangle = \sum_{k=1}^n \lambda_k \eta_k$. Suppose that, for every $\eta \in \Gamma$,

$$R(x) = O(x^{-\eta}), \text{ (when } x \to 0 \text{ or } x \to \infty, x \in S).$$

Then there exists $\varepsilon > 0$ such that if $\sup_{x\in S}|R(x)| < \varepsilon$ *the equation* $(*)$ *has a solution u holomorphic in S. Moreover, for every $\eta \in \Gamma$, u behaves like $O(x^{-\eta})$ when $x \to 0$ or $x \to \infty$ $x \in S$.*

Example. For $R(x)$ in the above example the conditions in the theorem are fulfilled if Γ is a sufficiently small neighborhood of the origin and A is sufficiently small.

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