数理解析研究所講究録 1296 巻 2002 年 29-34

On the definitions of the Painlevé equations

パリ第 VI 大学, Daniel Bertrand
Département de mathématiques
Paris VI
名古屋大学 大学院多元数理科学研究科 梅村 浩 (Hiroshi Umemura)
Graduate school of mathematics
Nagoya University

1 Introduction

There many ways of defining the Painlevé equations.

(1) Historically the origin of the Painlevé equations goes back to the pursuit of special functions defined by algebraic differential equations of the second order. Around 1900 Painlevé succeeded in classifying algebraic differential equations $y" = f(t, y, y')$ without movable singular points, where F is a rational function of t, y and y' and t is the independent variable so that $y' = dy/dt$ and $y" = d^2y/dt^2$. Painlevé then threw away those equations that he could integrate by the so far known functions and thus he arrived at the list of the Painlevé equations. This is the first definition of the Painlevé equations. It is, however, very lucky that he could discover the Painlevé equations in this manner.

(2) In 1907, R. Fuchs discovered that the sixth Painlevé equation describes a monodromy preserving deformation of a second order linear equation $y" = p(x)y$. Later R. Garnier generalized this for the other Painlevé equations.

(3) In our former work [2], we showed that we can recover the second Painlevé equation form a raional surface with a rational double point. We can regard this as an algebro-geometric definition of the second Painlevé equation.

(4) Masatoshi Noumi and Yasuhiko Yamada interpreted theory of Painlevé equations form the view point of Kač-Moody Lie algebra. They not only unifromly reviewed the threory of τ function of the Painlevé equations but also generalized the Painlevé equations in a natural frame work.

(5) There is another deinition due to J. Drach [1] in 1914. He asserts the equivalence of the following two conditions for a function $\lambda(t)$.

(i) $\lambda(t)$ satisfies the sixth Painlevé equation.

(ii) The dimension of the Galois group of a non-linear differential equation

$$\frac{dy}{dt} = \frac{y(y-1)(t-\lambda)}{t(t-1)(y-\lambda)}$$

is finite.

In the second assertion, the Galois group of general algebraic differential equation is involved. Namely the second assertion depends on his infinite dimensional differential Galois theory, which has been an object of discussion.

In this note, we apply our infinite dimensional Galois theory of differential equations [3] to study the result of J. Drach. It is difficult to imagine the equivalence of the assertions. We prove that (i) implies (ii) for the first Painlevé equation.

Theorem 1 *Let $\lambda(t)$ be a function satisfying the first Painlevé equation $\lambda" = 6\lambda^2 + t$. Then the Galois group $Infgal(L/K) = \widehat{SL_2}$, where*

$$K = \mathbf{C}(t, \lambda(t), \lambda'(t)), \; L = K(y)$$

such that y is transcendental over K satisfying

$$\frac{dy}{dt} = \frac{1}{2}\frac{1}{y - \lambda(t)}.$$

Then .

$$Infgal(L/K) \simeq \widehat{SL}_{2L^\natural}.$$

Why is the theorem interesting? Because the Galois group, which is a formal group of infinite dimension, is very difficult to calculate. We have only two types of examples where we can calculate the Galois group. (1) If L/K is strongly normal, then $Infgal(L/K) = \hat{G}$, where G is the Galois group $Gal(L/K)$ of the extension and (2) the Galois group of a Riccati equation coincides with the formal completion of the Galois group of the linearization of the Riccati equation.

2 Review of R. Fuchs' paper

R. Fuch studied a monodromy preserving deformation of a linear differential equation $d^2y/dx^2 = p(x)y$. Namely he considered a system of linear equations

$$(1) \qquad \begin{cases} \frac{\partial^2 y_i}{\partial x^2} &= p y_i, \\ \frac{\partial y_i}{\partial t} &= B y_i - A \frac{\partial y_i}{\partial x}, \end{cases} \qquad \text{for } i = 1, 2,$$

where

$$p = \frac{a}{x^2} + \frac{b}{(x-1)^2} + \frac{c}{(x-t)^2} + \frac{e}{(x-\lambda)^2} + \cdots$$

and we assume that λ is not a function of t but it is a function of x, i.e. $\partial\lambda/\partial x = 0$. y_1 and y_2 are linearly independent solutions. The integrability of the system ( ) implies

$$A(x, t) = \frac{x(x-1)(t-\lambda)}{t(t-1)(x-\lambda)} \quad \text{and} \quad B(x, t) = \frac{1}{2}\frac{\partial A}{\partial x}$$

and $\lambda(t)$ satisfies the sixth Painlevé equation P_{VI}.

Where comes the non-linear differential equation

$$\frac{dy}{dt} = \frac{y(y-1)(y-\lambda)}{t(t-1)(t-\lambda)}$$

from?

Lemma 1 *We may assume that the Wronskian*

$$W_r = \begin{vmatrix} y_1 & y_2 \\ \frac{\partial y_1}{\partial x} & \frac{\partial y_2}{\partial x} \end{vmatrix} = 1.$$

Proof. It is an exercise to check $\partial W_r/\partial t = \partial W_r/\partial x = 0$.

From now on we write T for t, W for x so that we consider the system

$$(2) \qquad \begin{cases} \frac{\partial^2 y_i}{\partial W^2} &= p y_i, \\ \frac{\partial y_i}{\partial T} &= B(W, T) y_i - A(W, T) \frac{\partial y_i}{\partial W}, \end{cases} \qquad \text{for } i = 1, 2,$$

Lemma 2 *If we set $y = y_1/y_2$, then we have*

$$\begin{cases} \frac{\partial y}{\partial W} &= \frac{1}{y_1^2}, \\ \frac{\partial y}{\partial T} &= -A \frac{1}{y_1^2}. \end{cases}$$

We are working in the differebtial field

$$(3) \qquad \mathbf{C}(W,T)\langle\lambda(T)\rangle = \mathbf{C}(W,T,\lambda(T),\lambda'(T),\cdots)(y_1,y_2,\frac{\partial y_1}{\partial T},\frac{\partial y_2}{\partial T})$$

with derivations $\{\partial/\partial W,\ \partial/\partial T\}$. The differential field extension

$$\mathbf{C}(W,T)\langle\lambda(T)\rangle(y_1,y_2,\frac{\partial y_1}{\partial T},\frac{\partial y_2}{\partial T})/\mathbf{C}(W,T)\langle\lambda(T)\rangle$$

is defined by the adjunction of the solutions y_i, y_2 of the system (1) of linear equations.

Now we introduce differential operators

$$\begin{cases} D_t &= \frac{\partial}{\partial T}+\frac{\partial}{\partial W}, \\ D_w &= y_1^2\frac{\partial}{\partial W}. \end{cases}$$

so that the field (??) is a differntial field with derivations $\{D_t,\ D_w\}$. If we regard the the field (??) as a differential field with derivations $\{D_t,\ D_w\}$, then it involvs non-linear differential equations.

Lemma 3 $D_tW = A(W,\ T)$.

Proof. This follows from the definition of the operator D_t.

Lemma 4

$$\frac{\partial y}{\partial T}+A\frac{\partial y}{\partial W}=0 \qquad \textit{so that } D_ty=0.$$

Proof. This is a consequaence of Lemma 2.

Lemma 4 shows that y is a first integral of $dY/dT = A(Y,\ T)$.

It follows from Lemma 2 $D_w(W) = y_1^2$ and hence y_1 is algebraic of degree (at most 2) over $\mathbf{C}(t)\langle\lambda\rangle\langle y\rangle\langle W\rangle$. Here $\langle\,\rangle$ should be interpreted in the differential field (3) with derivations $\{D_t,\ D_w\}$. Since $y_2 = yy_1$,

$$(\mathbf{C}(W,\ T)\langle\lambda\rangle(y_1,y_2,\frac{\partial y_1}{\partial W},\frac{\partial y_2}{\partial W}:\mathbf{C}(t)\langle\lambda\rangle\langle y\rangle\langle W\rangle)=2.$$

3 Infinite dimensional differential Galois theory

We start from a differentail field extension $L = \mathbf{C}(t)\langle\lambda\rangle\langle W\rangle$ over $K = \mathbf{C}(T)\langle\lambda\rangle$ with derivation D_t. They are subfields of

$$\mathbf{C}(W,\ T)\langle\lambda\rangle(y_1,y_2,\frac{\partial y_1}{\partial W},\frac{\partial y_2}{\partial W}).$$

Recall that we have

$$D_t(T) = 1, \quad D_t W = \frac{W(W-1(t-\lambda)}{T(T-1)(W-\lambda)}$$

and W is transcentental over the field K.

Let us now review our differential Galois theory of infinite dimension. We start from the differential field extension $L = K(W)/K$ with derivation D_t. We define its Galois group. We consider the universal Taylor morphism $i : L \longrightarrow L^{\natural}[[\tau]]$. Namely we set for an element $a \in L^{\natural}[[\tau]]$

$$i(a) = \sum_{n=0}^{\infty} \frac{1}{n!} D_t^n(a)\tau^n.$$

Here $L^{\natural}$ is the abstract field structure of the differential field L. i is a morphism of rings compatible with derivations D_t and $\partial/\partial\tau$.

Consider now on $L^{\natural}$, the derivation $\partial/\partial W$, which we denote by $(\partial/\partial W)^{\natural}$ to avoide confusions. So we have in the power series ring $L^{\natural}[[\tau]]$ two mutually commutative derivations $\partial/\partial\tau$ and $(\partial/\partial W)^{\natural}$. The latter operates as a drivation of coeficients of a power series.

The quotient field of $L^{\natural}[[\tau]]$ is the field $L^{\natural}[[\tau]][\tau^{-1}]$ of Laurent series that is the differential field with derivations $\partial/\partial\tau$ and $(\partial/\partial W)^{\natural}$. In this differential field $L^{\natural}[[\tau]][\tau^{-1}]$, let $\mathcal{L}$ be the differential sufield generated by $i(L)$ and $L^{\natural}$ and we define $\mathcal{K}$ as the differential subfield generated by $i(K)$ and $L^{\natural}$. Now considering again the Taylor expansion of the coefficients of a Laurent series, we have a differential algebra morphism $L^{\natural}[[\tau]][\tau^{-1}] \longrightarrow L^{\natural\natural}[[\xi]][[\tau]][\tau^{-1}]$, where ξ is the variable appeared when we expanded the coefficients or our Taylor expansion of coefficients is given by

$$L^{\natural} \longrightarrow L^{\natural\natural}[[\xi]] \quad a \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} \left(\left(\frac{\partial}{\partial W} \right)^{\natural} \right)^n (a)\xi^n.$$

So now $\mathcal{L}$ and $\mathcal{K}$ are differential subfields of $L^{\natural\natural}[[\xi]][[\tau]][\tau^{-1}]$ with derivations $\{\partial/\partial\xi, \partial/\partial\tau\}$.

Now we consider the formal deformation functor of $\mathcal{L}/\mathcal{K}$ in

$$L^{\natural\natural}[[\xi]][[\tau]][\tau^{-1}]$$

that is a principal homogeneous space of a formal group $Infgal(L/K)$ of infinite dimension in general). This is the definition of our Galois group.

We want to show $Infgal(L/K) \simeq \widehat{SL}_{2L^{\natural}}$. It follows form Lie's classification of Lie algebras operating on a manifold of dimension 1. We have to show tr.d.$[\mathcal{L} : \mathcal{K}] = 3$. We have to connect L/K with the differential field (3) of §2. Ignoring the technical points, we have to show

Question. *The field of constants of the differential field*

$$\mathbf{C}(T, \lambda, \lambda', W, y_1, \partial y_1/\partial W)$$

with derivation D_t coincides with $\mathbf{C}$?

We can not answer the Question but we can answer an analogue of the Question for the first Painlevé equation.

Theorem 1. *Let us consider a differential field extension $L = \mathbf{C}(T, \lambda, \lambda', W)$ over $K = \mathbf{C}(T, \lambda, \lambda')$ with derivation D_t such that $D_tT = 1$,*

$$D_tW = \frac{1}{2}\frac{1}{W - \lambda(t)}$$

and such that λ satisfies the first Painlevé equation $D_t^2(\lambda) = 6\lambda + t$. We assume that W is transcendental over K. Then

$$Infgal(L/K) \simeq \widehat{SL}_{2L^{\natural}}.$$

The proof of the theorem is as much invilved as the proof of the irreducility theorem.

References

[1] Jule, Drach, *Sur les équations différentielles du premier ordre et du premier degré* C. R. Acad. Sc. Paris, 1914, 926-929.

[2] Masa-Hiko, Saito and Hiroshi, Umemura, *Painlevé equations and deformations of rational surfaces with rational double points* Proc. of Nagoya international workshop on Physics and Combinatorics, World Scientific, Singapore, 2001, 320-365.

[3] Hiroshi, Umemura, *Differential Galois theory of infinite dimension* Nagoya Math. J., 144 (1996) 59-135.