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On the relation between Borel sum and classical solution for Cauchy problem of Airy's and Beam's PDE

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1 Introduction

We consider the following two Cauchy problems for partial differential equations of non-Kowalevski type

$$\begin{array}{llll} (\mathbf{A})_{\mathbb{R}} & \partial_t u(t,x) = \partial_x^3 u(t,x), & u(0,x) = \varphi(x), & (t>0,\ x\in\mathbb{R}) \\ (\mathbf{B})_{\mathbb{R}} & \partial_t u(t,x) = -\partial_x^4 u(t,x), & u(0,x) = \varphi(x), & (t>0,\ x\in\mathbb{R}) \end{array}$$

where the equation $(\mathbf{A})_{\mathbb{R}}$ is called the "Airy equation" and the equation $(\mathbf{B})_{\mathbb{R}}$ is called the "Beam equation", respectively.

The purpose in this note is to give the relationship between the "Classical solution" and the "Borel sum" of each Cauchy problem in complex $\mathbb{C}^2$ plane. Precisely, we shall show that the Classical solution of the Cauchy problem is derived from a deformation of path of integration of the Borel sum in 0 direction under some conditions for the Cauchy data.

We state the contents of the following sections. In Section 2 we shall give the "Classical solution". In Section 3 we shall give the definition of Borel summability, known results and the "Borel sum". In Section 4 our Claim which gives the relationship between the Borel sum and the Classical solution will be stated and their proofs will be given. In Section 5 we shall give the sketch of proof of Proposition 3.4 on the kernel function of the Borel sum. In Section 6 we shall give a generalization of our Claim as a theorem without proof, which will be given in a forthcoming paper.

2 Classical solution

Firstly, we shall give the "Classical solution".

When we consider the Classical solution, we always assume that $t > 0$, $x \in \mathbb{R}$ and for the Cauchy data $\varphi \in \mathcal{S}$, the rapidly decreasing functions in Schwartz' sense, for simplicity. Then it is known that the Cauchy problem $(\mathbf{A})_{\mathbb{R}}$ is uniquely solvable in $\mathcal{S}$ and the solution is given by

$$(2.1) \qquad u_c^A(t,x) = \frac{1}{(3t)^{1/3}} \int_{-\infty}^{\infty} \varphi(x+y) Ai\left(\frac{y}{(3t)^{1/3}}\right) dy, \quad t > 0,\ x \in \mathbb{R}.$$

Here Ai denotes the Airy function which is defined by the following Airy's integral

$$(2.2) \qquad Ai(z) = \frac{1}{2\pi i} \int_{\gamma} \exp\left(zs - \frac{s^3}{3}\right) ds, \quad z \in \mathbb{C},$$

where the path γ is any curve which begins at infinity in the sector $7\pi/6 < \arg z < 3\pi/2$ and ends at infinity in the sector $\pi/2 < \arg z < 5\pi/6$. (see Figure 1 below)

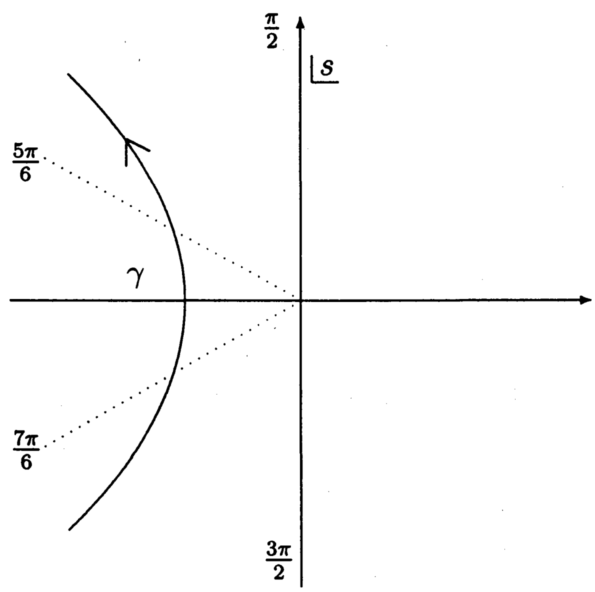

Figure 1: Airy's path γ

In a similar way, the solution of $(\mathbf{B})_{\mathbb{R}}$ in $\mathcal{S}$ is given by

$$(2.3) \qquad u_c^B(t,x) = \frac{1}{(4t)^{1/4}} \int_{-\infty}^{\infty} \varphi(x+y) Be\left(\frac{y}{(4t)^{1/4}}\right) dy,$$

which is well-defined in $\mathrm{Re}\, t > 0$ and $x \in \mathbb{R}$. Here Be is given by

$$(2.4) \qquad Be(z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \exp\left(zs - \frac{s^4}{4}\right) ds, \quad z \in \mathbb{C}.$$

We call these solutions (2.1) and (2.3) the "Classical solutions".

3 Borel sum

Next, we shall give the "Borel sum". Precisely, we shall give the Borel sums of divergent solutions of the Cauchy problems $(\mathbf{A})_{\mathbb{C}}$ and $(\mathbf{B})_{\mathbb{C}}$ which are obtained from $(\mathbf{A})_{\mathbb{R}}$ and $(\mathbf{B})_{\mathbb{R}}$ by changing the real variables into the complex variables.

In order to do so, we consider the following Cauchy problem for partial differential equations, which generalizes the Airy and the Beam equations.

$$(\mathbf{CP})_{\mathbb{C}} \qquad \partial_\tau u(\tau,z) = \alpha \partial_z^q u(\tau,z), \quad u(0,z) = \varphi(z),$$

where $(\tau,z) \in \mathbb{C}^2$, $q \geq 2$, $\alpha \in \mathbb{C}\backslash\{0\}$ and the Cauchy data φ is assumed to be holomorphic in a neighbourhood of the origin.

This Cauchy problem $(\mathbf{CP})_{\mathbb{C}}$ has a unique formal solution

$$\hat{u}(\tau,z) = \sum_{n\geq 0} \alpha^n \varphi^{(qn)}(z) \frac{\tau^n}{n!} \underset{\text{put}}{=} \sum_{n\geq 0} u_n(z)\tau^n. \tag{3.1}$$

By Cauchy's integral formula, we can see that the coefficients $u_n(z)$ have the following estimates: There exist positive constants C and K for a fixed $r > 0$ such that the following estimates hold

$$\max_{|z|\leq r} |u_n(z)| \leq CK^n((q-1)n)!, \quad n = 0, 1, 2, \ldots. \tag{3.2}$$

By the assumption $q \geq 2$, the formal solution $\hat{u}(\tau,z)$ is divergent. Precisely, the formal solution $\hat{u}(\tau,z)$ is called the formal power series of Gevrey order $(q-1)$ in τ variable.

We shall study the Borel summability of the divergent solution and we shall give the Borel sum of the divergent solution.

Before stating the results, let us prepare some notations and definitions (cf. [Bal]).

3.1 Notations and Definitions

1. Sector. For $d \in \mathbb{R}$, $\beta > 0$ and $\rho\,(0 < \rho \leq \infty)$, we define a sector $S(d,\beta,\rho)$ by

$$S(d,\beta,\rho) := \left\{\tau \in \mathbb{C}; |\arg \tau - d| < \frac{\beta}{2}, 0 < |\tau| < \rho\right\}, \tag{3.3}$$

where d, β and ρ are called the direction, the opening angle and the radius of this sector, respectively.

2. Gevrey Formal Power Series. We denote by $\mathcal{O}[[\tau]]$ the ring of formal power series in τ-variable with coefficients in $\mathcal{O}$ (the set of holomorphic functions at $z = 0$).

For $k > 0$, we define $\mathcal{O}[[\tau]]_{1/k}$, the ring of formal power series of Gevrey order $1/k$ in τ-variable, in the following way: $\hat{f}(\tau, z) = \sum_{n=0}^{\infty} f_n(z)\tau^n \in \mathcal{O}[[\tau]]_{1/k}(\subset \mathcal{O}[[\tau]])$ if and only if the coefficients $f_n(z)$ are holomorphic on a common closed disk $B_r := \{z \in \mathbb{C}; |z| \leq r\}$ and there exist positive constants C and K such that for any n, we have

$$\max_{|z|\leq r} |f_n(z)| \leq CK^n \Gamma\left(1 + \frac{n}{k}\right), \tag{3.4}$$

where Γ denotes the gamma function.

By using this terminology, we see that for our formal solution $\hat{u}(\tau, z)$ of $(\mathbf{CP})_{\mathbb{C}}$

$$\hat{u}(\tau, z) \in \mathcal{O}[[\tau]]_{q-1}. \tag{3.5}$$

3. Gevrey Asymptotic Expansion. Let $k > 0$, $\hat{f}(\tau, z) = \sum_{n=0}^{\infty} f_n(z)\tau^n \in \mathcal{O}[[\tau]]_{1/k}$ and $f(\tau, z)$ be an analytic function on $S(d, \beta, \rho) \times B_r$. Then we define that

$$f(\tau, z) \cong_k \hat{f}(\tau, z) \quad \text{in } S = S(d, \beta, \rho), \tag{3.6}$$

if for any relatively compact subsector S' of S, there exist some positive constants C and K such that for any N, we have

$$\max_{|z|\leq r} \left| f(\tau, z) - \sum_{n=0}^{N-1} f_n(z)\tau^n \right| \leq CK^N |\tau|^N \Gamma\left(1 + \frac{N}{k}\right), \quad \tau \in S'. \tag{3.7}$$

4. Borel Summability. For $k > 0$, $d \in \mathbb{R}$ and $\hat{f}(\tau, z) \in \mathcal{O}[[\tau]]_{1/k}$, we define that $\hat{f}(\tau, z)$ is k-summable or Borel summable in d direction if there exist a sector $S = S(d, \beta, \rho)$ with $\beta > \pi/k$ and an analytic function $f(\tau, z)$ on $S \times B_r$ such that $f(\tau, z) \cong_k \hat{f}(\tau, z)$ in S.

Remark 3.1 *Let $\hat{f}(\tau, z) \in \mathcal{O}[[\tau]]_{1/k}$ be given.*

(i) *If $\beta \leq \pi/k$, then for any direction d, there are infinitely many analytic functions $f(\tau, z)$ on $S(d, \beta, \rho) \times B_r$ satisfying $f(\tau, z) \cong_k \hat{f}(\tau, z)$ in $S(d, \beta, \rho)$ by some positive constants ρ and r.*

(ii) *If $\beta > \pi/k$, then there does not exist such a function in general. But if such a function exists, then it is unique. In this sense, such a function $f(\tau, z)$ is called the Borel sum of $\hat{f}(\tau, z)$ in d direction. We write it by $f_B^d(\tau, z)$ and we say that $\hat{f}(\tau, z)$ is Borel summmable in d direction.*

We give some preparations for the special functions.

5. The Generalized Hypergeometric Series. (cf. [Luk, p. 41])
For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p) \in \mathbb{C}^p$ and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_q) \in \mathbb{C}^q$, we define

$$(3.8) \qquad {}_pF_q(\boldsymbol{\alpha}; \boldsymbol{\gamma}; z) = {}_pF_q\left(\begin{array}{c} \boldsymbol{\alpha} \\ \boldsymbol{\gamma} \end{array}; z\right) := \sum_{n=0}^{\infty} \frac{(\boldsymbol{\alpha})_n}{(\boldsymbol{\gamma})_n} \frac{z^n}{n!},$$

where

$$(\boldsymbol{\alpha})_n = \prod_{\ell=1}^{p} (\alpha_\ell)_n, \quad (\boldsymbol{\gamma})_n = \prod_{j=1}^{q} (\gamma_j)_n, \quad (c)_n = \frac{\Gamma(c+n)}{\Gamma(c)} \ (c \in \mathbb{C}).$$

6. The Meijer G-Function. (cf. [MS, p. 2]) For $\boldsymbol{\alpha} \in \mathbb{C}^p$ and $\boldsymbol{\gamma} \in \mathbb{C}^q$ with $\alpha_\ell - \gamma_j \notin \mathbb{N}$ ($\ell = 1, 2, \dots, n; j = 1, 2, \dots, m$), we define

$$(3.9) \qquad G_{p,q}^{m,n}\left(z \,\middle|\, \begin{array}{c} \boldsymbol{\alpha} \\ \boldsymbol{\gamma} \end{array}\right) = \frac{1}{2\pi i} \int_I \frac{\prod_{j=1}^{m} \Gamma(\gamma_j + s) \prod_{\ell=1}^{n} \Gamma(1 - \alpha_\ell - s)}{\prod_{j=m+1}^{q} \Gamma(1 - \gamma_j - s) \prod_{\ell=n+1}^{p} \Gamma(\alpha_\ell + s)} z^{-s} ds,$$

where the path of integration I runs from $\kappa - i\infty$ to $\kappa + i\infty$ for any fixed $\kappa \in \mathbb{R}$ in such a manner that all poles of $\Gamma(\gamma_j + s)$, $\{-\gamma_j - k\,;\, k \geq 0, j = 1, 2, \dots, m\}$, lie to the left of the path and all poles of $\Gamma(1 - \alpha_\ell - s)$, $\{1 - \alpha_\ell + k\,;\, k \geq 0, \ell = 1, 2, \dots, n\}$, lie to the right of the path.

In the following, the integration $\int_0^{\infty(\theta)}$ denotes the integration from 0 to ∞ along the half line of argument θ.

3.2 Known Results

Now, we give a theorem for the Borel summability which is a special case in Miyake's paper [Miy].

Theorem 3.2 (Miyake) *The formal solution $\hat{u}(\tau, z)$ of* $(\mathbf{CP})_{\mathbb{C}}$ *is Borel summable in d direction if and only if there exists a positive constant ε such that*

(i) *the Cauchy data φ can be continued analytically in a domain*

$$(3.10) \qquad \Omega_\varepsilon(d; q, \alpha) := \bigcup_{m=0}^{q-1} S\left(\frac{d + \arg\alpha + 2\pi m}{q}, \varepsilon, \infty\right),$$

(ii) *φ has a growth condition of exponential order at most $q/(q-1)$, that is, there exist positive constants C and γ such that the following growth estimate holds.*

$$(3.11) \qquad |\varphi(z)| \leq C \exp\left(\gamma |z|^{q/(q-1)}\right), \quad z \in \Omega_\varepsilon(d; q, \alpha).$$

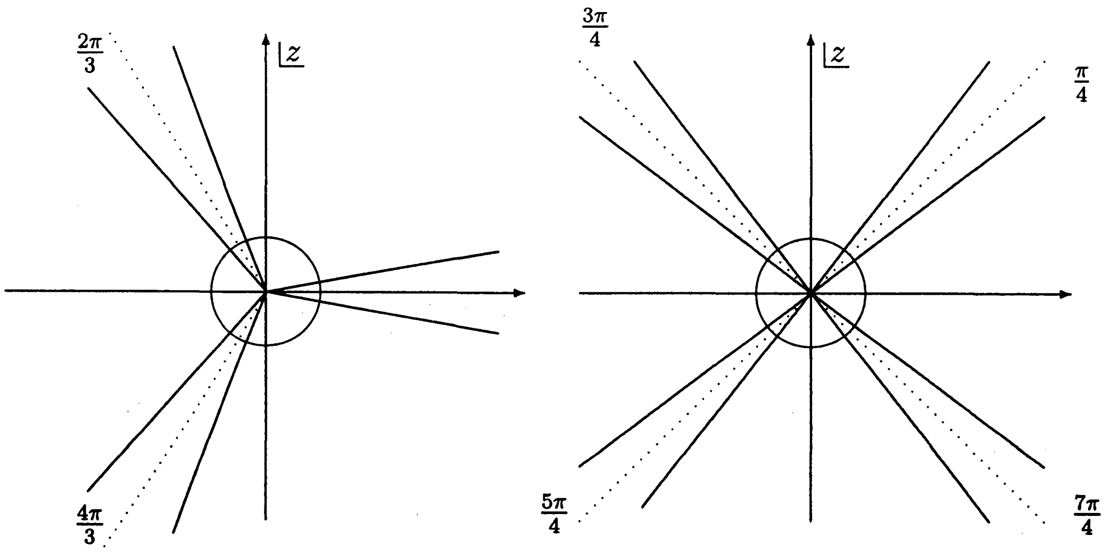

Figure 2: $\Omega_\varepsilon(0,3,1)$ and $\Omega_\varepsilon(0,4,-1)$

Next, we give a theorem for the Borel sum which is a special case in the author's paper [Ich].

Theorem 3.3 (Ichinobe) *Under the above conditions* (i) *and* (ii) *in Theorem* 3.2, *the Borel sum* $u^d_B(\tau,z)$ *is given by the analytic continuation of the following function*

$$u^d_B(\tau,z) = \int_0^{\infty((d+\arg\alpha)/q)} \Phi_q(z,\zeta) k_q(\tau,\zeta;\alpha)\,d\zeta, \tag{3.12}$$

where $(\tau,z) \in S(d,\beta,\rho) \times B_r$ *with* $\beta < (q-1)\pi$,

$$\Phi_q(z,\zeta) = \sum_{m=0}^{q-1} \varphi(z+\omega_q^m \zeta), \quad \omega_q = \exp(2\pi i/q), \tag{3.13}$$

and the kernel function $k_q(\tau,\zeta;\alpha)$ *is given by*

$$k_q(\tau,\zeta;\alpha) = \frac{C_q}{\zeta}\, G^{q-1,0}_{0,q-1}\left(Z_\alpha \,\middle|\, \begin{matrix} \\ 1/q, 2/q, \ldots, (q-1)/q \end{matrix} \right), \tag{3.14}$$

with

$$Z_\alpha = \frac{1}{q^q \alpha}\frac{\zeta^q}{\tau}, \quad C_q = \frac{1}{\prod_{j=1}^{q-1}\Gamma(j/q)}.$$

In special cases the kernel functions are given more explicitly (cf.[LMS], [Ich]).

Proposition 3.4 (i) *When $(q,\alpha)=(2,1)$, that is, the case of the heat equation, the kernel function is given by*

$$k_2(\tau,\zeta;1)=\frac{1}{\sqrt{4\pi\tau}}e^{-\zeta^2/4\tau}. \tag{3.15}$$

(ii) *When $(q,\alpha)=(3,1)$, that is, the case of the Airy equation, the kernel function is given by*

$$k_3(\tau,\zeta;1)=\frac{1}{(3\tau)^{1/3}}Ai\left(\frac{\zeta}{(3\tau)^{1/3}}\right). \tag{3.16}$$

(iii) *When $(q,\alpha)=(4,-1)$, that is, the case of the Beam equation, the kernel function is given by*

$$k_4(\tau,\zeta;-1)=\frac{1}{(4\tau)^{1/4}}\frac{1}{2\pi i}\int_{\gamma_2}\exp\left[\left(\frac{\zeta}{(4\tau)^{1/4}}\right)s-\frac{s^4}{4}\right]ds, \tag{3.17}$$

where the path γ_2 is any curve which begins at infinity in the sector $7\pi/8<\arg s<9\pi/8$ and ends at infinity in the sector $3\pi/8<\arg s<5\pi/8$ (see Figure 3 at Section 4.2).

The statement (i) was given by [LMS] and the statement (ii) was given by [Ich]. The statement (iii) is a new expression which will be proved in Section 5.

4 Main result

As we have shown the Classical solution and the Borel sum are different notion and the integral representations of solutions are also completely different, but we shall show a relationship between these solutions as follows.

Claim *The Classical solutions $u_c(t,x)$ are obtained by deforming the paths of integrations for the Borel sum $u_B^0(\tau,z)$ under some additional conditions for the Cauchy data, where will be specified in each equation in the below.*

4.1 Case of the Airy Equation

In Airy's case, we recall the conditions for the Borel summability in 0 direction for the Cauchy data $\varphi(z)$ which is holomorphic in a neighbourhood of the origin.

The Cauchy data $\varphi(z)$ can be continued analytically in $\Omega_\varepsilon(0;3,1)$ (see Figure 2) with a growth condition of exponential order at most $3/2$ there.

We recall the integral repesentation of the Borel sum $u_B^0(\tau, z)$

$$(4.1)\qquad u_B^0(\tau,z) = \frac{1}{(3\tau)^{1/3}}\left\{\int_0^{+\infty}\varphi(z+\zeta)Ai(X)d\zeta + \int_0^{\infty(2\pi/3)}\varphi(z+\zeta)Ai(X\omega_3^{-1})\omega_3^{-1}d\zeta + \int_0^{\infty(4\pi/3)}\varphi(z+\zeta)Ai(X\omega_3^{-2})\omega_3^{-2}d\zeta\right\},\quad X=\frac{\zeta}{(3\tau)^{1/3}},$$

where $(\tau, z) \in S(0, 2\pi, \rho) \times B_r$.

Now we assume the following additional conditions for the Cauchy data:

(1) φ can be continued analytically in a sector $S(\pi, 2\pi/3, \infty)$ with the same growth condition as in the Borel summability.

(2) There exists a positive constant δ such that in the region $S(\pi, \delta, \infty)$, φ has a decreasing condition of polynomial order, exactly, there exist positive constants C and λ such that

$$|\varphi(z)| \le C|z|^{-3/4-\lambda}, \quad z \in S(\pi, \delta, \infty).$$

Under these assumptions, we can deform the paths of integrations as follows. First, we restrict $\tau > 0$ in the Borel sum $u_B^0(\tau, z)$. Then in the second and the third integrations of the expression (4.1) the paths of integrations can be deformed into the integrations on the negative axis, because the Airy function has the following asymptotic expansion (cf. [Erd, p. 96]).

$$(4.2)\qquad Ai(z) = \frac{1}{2\sqrt{\pi}}z^{-1/4}\exp\left(-\frac{2}{3}z^{3/2}\right)\left[1+O(z^{-3/2})\right],\quad |\arg z| < \pi,\ z \to \infty.$$

Therefore we have

$$(4.3)\qquad u_B^0(\tau,z) = \frac{1}{(3\tau)^{1/3}}\left[\int_0^{+\infty}\varphi(z+\zeta)Ai(X)d\zeta + \int_0^{-\infty}\varphi(z+\zeta)\left\{Ai(X\omega_3^{-1})\omega_3^{-1} + Ai(X\omega_3^{-2})\omega_3^{-2}\right\}d\zeta\right],$$

where $\tau > 0$ and $z \in \mathbb{R}$. Finally, by using the following functional equality of Airy functions

$$(4.4)\qquad w_m(z) + \omega_3 w_{m+1}(z) + \omega_3^2 w_{m+2}(z) = 0, \quad w_m(z) = Ai(\omega_3^m z),$$

(cf. [Erd, p.96]), we have

$$(4.5)\qquad Ai(X\omega_3^{-1})\omega_3^{-1} + Ai(X\omega_3^{-2})\omega_3^{-2} = -Ai(X).$$

Therefore for $\tau > 0$ and $z \in \mathbb{R}$ we obtain

$$(4.6)\qquad u_B^0(\tau,z) = \frac{1}{(3\tau)^{1/3}}\int_{-\infty}^{+\infty}\varphi(z+\zeta)Ai(X)d\zeta, \quad X = \frac{\zeta}{(3\tau)^{1/3}}.$$

4.2 Case of the Beam Equation

Before we give the integral representation of the Borel sum of the Beam equation $(\mathbf{B})_{\mathbb{C}}$, we introduce the following functions.

$$(4.7) \qquad v_j(z) = \frac{1}{2\pi i}\int_{\gamma_j} \exp\left(zs - \frac{s^4}{4}\right) ds, \quad (1 \le j \le 6), \quad z \in \mathbb{C},$$

where γ_j's are given by the following figure.

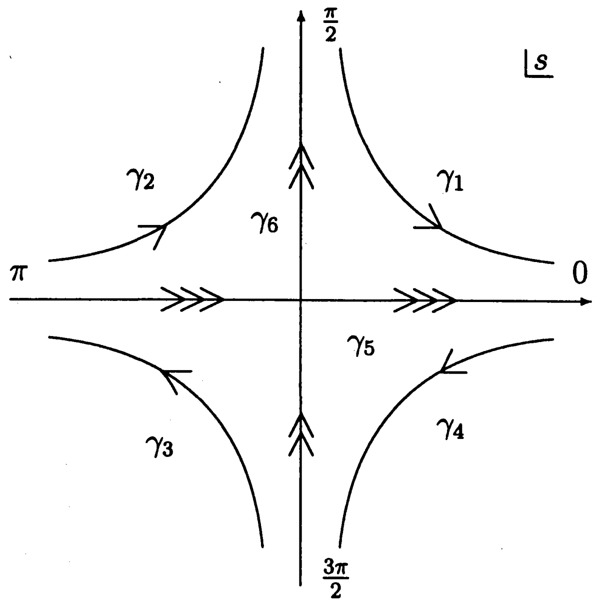

Figure 3: v_j's paths

We remark that the functions v_j's have the following functional equalities.

$$\begin{array}{ll} \textbf{(V1)} & v_2 + v_3 = v_6, \quad v_1 + v_4 = -v_6, \\ \textbf{(V2)} & v_2(z) = v_1(z\omega_4)\omega_4 = v_3(z\omega_4^{-1})\omega_4^{-1} = v_4(z\omega_4^{-2})\omega_4^{-2}, \quad \omega_4 = e^{2\pi i/4}. \end{array}$$

Moreover, there are the following relations between the kernel functions and the function v_j's, that is, the kernel function $Be(z)$ of Classical solution (2.3) is $v_6(z)$

$$(4.8) \qquad Be(z) = v_6(z),$$

and the kernel function (3.17) of the Borel sum is given by v_2

$$(4.9) \qquad k_4(\tau, \zeta; -1) = \frac{C_4}{\zeta}\, G^{3,0}_{0,3}\left(-\frac{\zeta^4}{4^4\tau}\,\middle|\, 1/4, 2/4, 3/4\right) = \frac{1}{(4\tau)^{1/4}}\, v_2\left(\frac{\zeta}{(4\tau)^{1/4}}\right).$$

From (4.8), the Classical solution is rewritten by

$$u_c^B(t,x) = \frac{1}{(4t)^{1/4}} \int_{-\infty}^{+\infty} \varphi(x+y) v_6 \left(\frac{y}{(4t)^{1/4}} \right) dy, \tag{4.10}$$

where $\mathrm{Re}\, t > 0$ and $x \in \mathbb{R}$.

Next, from (4.9) and Theorem 3.3, the Borel sum $u_B^0(\tau, z)$ of $(\mathbf{B})_{\mathrm{C}}$ is given by

$$\begin{aligned} u_B^0(\tau,z) &= \frac{1}{(4\tau)^{1/4}} \left\{ \int_0^{\infty(\pi/4)} \varphi(z+\zeta) v_2(X) d\zeta \right. \\ &+ \int_0^{\infty(3\pi/4)} \varphi(z+\zeta) v_2(X\omega_4^{-1}) \omega_4^{-1} d\zeta + \int_0^{\infty(5\pi/4)} \varphi(z+\zeta) v_2(X\omega_4^{-2}) \omega_4^{-2} d\zeta \\ &+ \left. \int_0^{\infty(7\pi/4)} \varphi(z+\zeta) v_2(X\omega_4^{-3}) \omega_4^{-3} d\zeta \right\}, \quad X = \frac{\zeta}{(4\tau)^{1/4}}, \end{aligned} \tag{4.11}$$

where $(\tau, z) \in S(0, 3\pi, \infty) \times B_r$.

By using the functional equalities $(\mathbf{V2})$, the Borel sum $u_B^0(\tau, z)$ is rewritten in the following form.

$$\begin{aligned} & u_B^0(\tau,z) \\ &= \frac{1}{(4\tau)^{1/4}} \left\{ \int_0^{\infty(\pi/4)} \varphi(z+\zeta) v_2(X) d\zeta + \int_0^{\infty(3\pi/4)} \varphi(z+\zeta) v_1(X) d\zeta \right. \\ &+ \left. \int_0^{\infty(5\pi/4)} \varphi(z+\zeta) v_4(X) d\zeta + \int_0^{\infty(7\pi/4)} \varphi(z+\zeta) v_3(X) d\zeta \right\}. \end{aligned} \tag{4.12}$$

We recall the conditions for the Borel summability in 0 direction for the Cauchy data.

The Cauchy data $\varphi(x)$ can be continued analytically in $\Omega_\varepsilon(0; 4, -1)$ (see Figure 2) with a growth condition of exponential order at most $4/3$ there.

We assume the following additional conditions for the Cauchy data:

- φ can be continued analytically in a sector $S(0, \pi/2, \infty) \cup S(\pi, \pi/2, \infty)$, and has the same growth condition as in the Borel summability.

Then by restricting $\mathrm{Re}\, \tau > 0$, we can deform the paths of integrations in the Borel sum as follows. The paths of integrations of the arguments $\pi/4$ and $7\pi/4$ can be changed into the integrations on the positive real axis, and the paths of integrations of the arguments $3\pi/4$ and $5\pi/4$ can be changed into the integrations on the negative real axis. In fact, it follows from the expression (4.9) and the fact that the G-function has the following asymptotic expansion (cf. [Luk, p. 179]).

$$G_{0,3}^{3,0}\left(z \,\middle|\, \begin{matrix} \\ 1/4, 2/4, 3/4 \end{matrix} \right) = \frac{2\pi}{\sqrt{3}} z^{1/6} \exp\left(-3z^{1/3}\right) \left[1 + O\left(z^{-1/3}\right)\right], \tag{4.13}$$

$$z \to \infty, \ |\arg z| \le 4\pi - \delta, \ \delta > 0.$$

Therefore we have

$$(4.14)\qquad u_B^0(\tau,z) = \frac{1}{(4\tau)^{1/4}}\left[\int_0^{+\infty}\varphi(z+\zeta)\left\{v_2(X)+v_3(X)\right\}d\zeta \right. \\ \left. +\int_0^{-\infty}\varphi(z+\zeta)\left\{v_1(X)+v_4(X)\right\}d\zeta\right],$$

where Re $\tau > 0$ and $z \in \mathbb{R}$. Finally, by using the functional equalities (**V1**), we obtain

$$(4.15)\qquad u_B^0(\tau,z) = \frac{1}{(4\tau)^{1/4}}\int_{-\infty}^{+\infty}\varphi(z+\zeta)v_6(X)d\zeta, \quad X = \frac{\zeta}{(4\tau)^{1/4}}.$$

Therefore $u_c^B(t,x) = u_B^0(t,x)$ by restricting $t>0$ and $x \in \mathbb{R}$ in the above formula.

5 Sketch of Proof of Proposition 3.4, (iii)

We shall prove the statement (iii) of Proposition 3.4 which means

$$(5.1)\qquad \frac{C_4}{\zeta}G_{0,3}^{3,0}\left(Z_{-1}\,\middle|\,\begin{matrix} \\ 1/4,2/4,3/4\end{matrix}\right) = \frac{1}{(4\tau)^{1/4}}\frac{1}{2\pi i}\int_{\gamma_2}\exp\left[\left(\frac{\zeta}{(4\tau)^{1/4}}\right)s-\frac{s^4}{4}\right]ds,$$

where $Z_{-1} = \zeta^4/(4^4 e^{\pi i}\tau)$ (since $\alpha = -1 = e^{\pi i}$) and $C_4 = \prod_{j=1}^{3}\Gamma(j/4)$.

We recall the following formula for the G-function (cf. [Luk, p. 150])

$$(5.2)\qquad z^\sigma G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\gamma}\end{matrix}\right) = G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}\boldsymbol{\alpha}+\sigma\\ \boldsymbol{\gamma}+\sigma\end{matrix}\right),$$

where $\boldsymbol{\alpha}+\sigma = (\alpha_1+\sigma, \alpha_2+\sigma, \dots, \alpha_p+\sigma)$. Then we have

$$\frac{C_4}{\zeta}G_{0,3}^{3,0}\left(Z_{-1}\,\middle|\,\begin{matrix} \\ 1/4,2/4,3/4\end{matrix}\right) = \frac{1}{(4\tau)^{1/4}}\frac{C_4 e^{-\pi i/4}}{4^{3/4}}Z_{-1}^{-1/4}G_{0,3}^{3,0}\left(Z_{-1}\,\middle|\,\begin{matrix} \\ 1/4,2/4,3/4\end{matrix}\right) \\ = \frac{1}{(4\tau)^{1/4}}\frac{C_4 e^{-\pi i/4}}{4^{3/4}}G_{0,3}^{3,0}\left(Z_{-1}\,\middle|\,\begin{matrix} \\ 0,1/4,2/4\end{matrix}\right).$$

Therefore it is enough to prove the following equality

$$(5.3)\qquad \frac{C_4 e^{-\pi i/4}}{4^{3/4}}G_{0,3}^{3,0}\left(Z_{-1}\,\middle|\,\begin{matrix} \\ 0,1/4,2/4\end{matrix}\right) = \frac{1}{2\pi i}\int_{\gamma_2}\exp\left[\left(\frac{\zeta}{(4\tau)^{1/4}}\right)s-\frac{s^4}{4}\right]ds.$$

In order to do so, we shall show that the power series expansions of both sides are the same ones. Precisely, we give the power series expansion at $Z_{-1} = 0$ of the left hand side and at $\zeta/(4\tau)^{1/4} = X = 0$ of the right hand side, respectively. We note the relation between Z_{-1} and X,

$$Z_{-1} = \frac{1}{4^3 e^{\pi i}} X^4 \quad \left(\text{or} \quad X = 4^{3/4} e^{\pi i/4} {Z_{-1}}^{1/4}\right).$$

First, from the integral representation of the G-function on the left hand side of (5.3) we have the following expansion by calculating the residues of the left side of the path of integration $I = \{\mathrm{Re}\ s = \kappa; \kappa > 0\}$.

$$\text{(5.4)} \quad G_{0,3}^{3,0}\left(Z_{-1} \,\middle|\, \begin{matrix} \\ 0, 1/4, 2/4 \end{matrix}\right) = \frac{1}{2\pi i}\int_I \Gamma(s)\Gamma(1/4+s)\Gamma(2/4+s){Z_{-1}}^{-s} ds$$
$$= \sum_{\ell=1}^{3} \prod_{j=1, j\neq\ell}^{3} \Gamma\left(\frac{j-\ell}{4}\right) {Z_{-1}}^{(\ell-1)/4} {}_1F_3\left(\begin{matrix} 1 \\ (1+\ell)/4, (2+\ell)/4, (3+\ell)/4 \end{matrix}; (-1)^3 Z_{-1}\right).$$

Next, on the right hand side of (5.3), by expanding e^{Xs} in the integrand into its power series and by termwise integrating, we have

$$\frac{1}{2\pi i}\int_{\gamma_2} \exp\left[Xs - \frac{s^4}{4}\right] ds = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{X^n}{n!}\int_{\gamma_2} s^n \exp\left(-\frac{s^4}{4}\right) ds.$$

We choose the path of integration γ_2 as the summation of two rays with the arguments $\pi/2$ and π. Then these integrals can be expressed in terms of the gamma functions.

$$\begin{aligned}
&\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{X^n}{n!}\int_{\gamma_2} s^n \exp\left(-\frac{s^4}{4}\right) ds \\
=\;& \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{X^n}{n!}\left\{\int_0^{\infty(\pi/2)} - \int_0^{\infty(\pi)}\right\} s^n \exp\left(-\frac{s^4}{4}\right) ds \\
=\;& \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{X^n}{n!}\left\{e^{\pi i(n+1)/2} - e^{\pi i(n+1)/4}\right\} 4^{(n-3)/4}\Gamma\left(1+\frac{n-3}{4}\right) \\
=\;& -\sum_{n=0}^{\infty}\frac{X^n}{n!}\frac{4^{(n-3)/4}e^{3\pi i(n+1)/4}}{\Gamma(1-(n+1)/4)}.
\end{aligned}$$

The third equality is obtained from the following relations

$$\begin{aligned}
e^{\pi i(n+1)/2} - e^{\pi i(n+1)/4} &= -e^{3\pi i(n+1)/4}(e^{\pi i(n+1)/4} - e^{-\pi i(n+1)/4}) \\
&= -e^{3\pi i(n+1)/4}\, 2i \sin\left(\frac{n+1}{4}\pi\right) \\
&= -\frac{2\pi i\ e^{3\pi i(n+1)/4}}{\Gamma((n+1)/4)\Gamma(1-(n+1)/4)}.
\end{aligned}$$

When $n = 4k+3$ in the above summation, we notice $1/\Gamma(1-(n-1)/4) = 1/\Gamma(-k) = 0$. Therefore by calculating carefully, we have

$$
\begin{aligned}
(5.5)\qquad & -\sum_{n=0}^{\infty} \frac{X^n}{n!} \frac{4^{(n-3)/4} e^{3\pi i(n+1)/4}}{\Gamma(1-(n+1)/4)} \\
&= -\sum_{\ell=0}^{2}\sum_{k=0}^{\infty} \frac{X^{4k+\ell}}{(4k+\ell)!} \frac{4^{k+(\ell-3)/4} e^{3\pi i(\ell+1)/4}(-1)^k}{\Gamma(1-k-(\ell+1)/4)} \\
&= -\sum_{\ell=0}^{2} \frac{e^{3\pi i(\ell+1)/4} 4^{(\ell-3)/4} X^\ell}{\Gamma((3-\ell)/4)\,\ell!} {}_1F_3\left(\begin{matrix} 1 \\ (\ell+2)/4, (\ell+3)/4, (\ell+4)/4 \end{matrix}; \frac{X^4}{4^3}\right) \\
&= -\sum_{\ell=1}^{3} \frac{e^{3\pi i\ell/4} 4^{(\ell-4)/4} X^{\ell-1}}{\Gamma((4-\ell)/4)\,(\ell-1)!} {}_1F_3\left(\begin{matrix} 1 \\ (\ell+1)/4, (\ell+2)/4, (\ell+3)/4 \end{matrix}; \frac{X^4}{4^3}\right).
\end{aligned}
$$

The second equality is obtained from the relations

$$
\begin{aligned}
(4k+\ell)! &= 4^{4k}\left(\frac{\ell+1}{4}\right)_k \left(\frac{\ell+2}{4}\right)_k \left(\frac{\ell+3}{4}\right)_k \left(\frac{\ell+4}{4}\right)_k \ell!, \\
\Gamma\left(1-\frac{\ell+1}{4}-k\right) &= \Gamma\left(\frac{3-\ell}{4}-k\right) = \Gamma\left(\frac{3-\ell}{4}\right)/(-1)^k\left(\frac{\ell+1}{4}\right)_k,
\end{aligned}
$$

and by employing the representation (3.8) of the generalized hypergeometric series.

At the end the proof is comlete by examing the following relations.

$$
(-1)^3 Z_{-1} = \frac{\zeta^4}{4^4 t} = \frac{X^4}{4^3}\ \left(X = \frac{\zeta}{(4\tau)^{1/4}}\right), \quad Z_{-1}^{1/4} = \frac{1}{4^{3/4} e^{\pi i/4}} X,
$$

and

$$
\begin{aligned}
C_4 \times \prod_{j=1, j\neq\ell}^{3} \Gamma\left(\frac{j-\ell}{4}\right) &= \prod_{j=1}^{3} \Gamma\left(\frac{j}{4}\right) \times \prod_{j=1, j\neq\ell}^{3} \Gamma\left(\frac{j-\ell}{4}\right) \\
&= \frac{(-1)^{\ell-1} 4^{\ell-1}}{(\ell-1)!} \frac{1}{\Gamma(1-\ell/4)} = \frac{e^{(\ell-1)\pi i} 4^{\ell-1}}{(\ell-1)!} \frac{1}{\Gamma(1-\ell/4)}
\end{aligned}
$$

which is obtained from the multiplication formula for the gamma function (cf. [Luk, p. 11])

$$
(5.6)\qquad \Gamma(mz) = (2\pi)^{-(m-1)/2} m^{mz-1/2} \prod_{j=0}^{m-1} \Gamma\left(z+\frac{j}{m}\right),
$$

where $z + j/m \notin \mathbb{Z}_{\leq 0} := \{0, -1, -2, \ldots\}$ $(j = 0, 1, \ldots, m-1)$.

6 Generalization of our Claim

In this section, we shall give a generalization of our claim ($q = 3, 4$) as a theorem without proof which will be given in a forthcoming paper.

We consider the following Cauchy problems

$$\begin{array}{llll}
(\mathbf{CP})_{\mathbb{R}} & \partial_t u(t,x) = \alpha \partial_x^q u(t,x), & u(0,x) = \varphi(x), & t > 0,\ x \in \mathbb{R}, \\
(\mathbf{CP})_{\mathbb{C}} & \partial_\tau u(\tau,z) = \alpha \partial_z^q u(\tau,z), & u(0,z) = \varphi(z), & \tau,\ z \in \mathbb{C},
\end{array}$$

where $q \geq 4$, $\alpha = 1$ if $q \notin 4\mathbb{Z}$ and $\alpha = -1$ if $q \in 4\mathbb{Z}$.

Under the above assumptions the Cauchy problem $(\mathbf{CP})_{\mathbb{R}}$ is uniquely solvable in $\mathcal{S}$ and the Classical solution $u_c(t,x)$ is given by

$$u_c(t,x) = \int_{-\infty}^{+\infty} \varphi(x+y) E(t,y) dy, \quad t > 0,\ x \in \mathbb{R}. \tag{6.1}$$

Here the kernel function $E(t,y)$ is given by

$$E(t,y) = \begin{cases} \dfrac{1}{(qt)^{1/q}} \dfrac{1}{2\pi i} \displaystyle\int_\gamma \exp\left(\frac{y}{(qt)^{1/q}} s - \frac{s^q}{q} \right) ds, & \text{if } q \neq 4n+2, \\ \dfrac{1}{(qt)^{1/q}} \dfrac{1}{2\pi i} \displaystyle\int_{-i\infty}^{+i\infty} \exp\left(\frac{y}{(qt)^{1/q}} s + \frac{s^q}{q} \right) ds, & \text{if } q = 4n+2, \end{cases} \tag{6.2}$$

where γ is given as follows:

(I) When $q = 4n - 1$, the path γ is any curve which begins at ∞ in the sector $3\pi/2 - \pi/q < \arg s < 3\pi/2$ and ends at ∞ in the sector $\pi/2 < \arg s < \pi + \pi/q$.

(II) When $q = 4n$, the path γ runs from $-i\infty$ to $+i\infty$.

(III) When $q = 4n + 1$, the path γ is any curve which begins at ∞ in the sector $3\pi/2 < \arg s < 3\pi/2 + \pi/q$ and ends at ∞ in the sector $\pi/2 - \pi/q < \arg s < \pi/2$.

Now, our theorem for the relationship between the Classical solution $u_c(t,x)$ which is given by (6.1) and the Borel sum $u_B^0(\tau,z)$ which is given by (3.12) is stated as follows.

Theorem 6.1 *Under the additional conditions for the Cauchy data which are stated below, the expressions* (6.1) *of the Classical solutions $u_c(t,x)$ are obtained by deforming the paths of integrations* (3.12) *for the Borel sum $u_B^0(\tau,z)$.*

(I) (*Generalization of Airy equation*) *When $q = 4n - 1$, the Cauchy data φ can be continued analytically in a sector $S(0, \pi - 3\pi/q, \infty) \cup S(\pi, \pi - \pi/q, \infty)$ with the* *same growth condition as in the Borel summability in Theorem* 3.3 *and there exists a positive constant* δ *such that in the region* $S(\pi, \delta, \infty)$, φ *has a decreasing condition of polynomial order, exactly, there exist positive constants* C *and* λ *such that*

$$|\varphi(z)| \leq C|z|^{-3/2(q-1)-\lambda}. \tag{6.3}$$

(II) (*Generalization of the heat equation*) *When* $q = 2n$, *the Cauchy data* φ *can be continued analytically in a sector* $S(0, \pi - 2\pi/q, \infty) \cup S(\pi, \pi - 2\pi/q, \infty)$ *with the same growth condition as in the Borel summability in Theorem* 3.3.

(III) *When* $q = 4n + 1$, *the Cauchy data* φ *can be continued analytically in a sector* $S(0, \pi - \pi/q, \infty) \cup S(\pi, \pi - 3\pi/q, \infty)$ *with the same growth condition as in the Borel summability in Theorem* 3.3 *and there exists a positive constant* δ *such that in the region* $S(0, \delta, \infty)$, φ *has the same decreasing condition* (6.3) *as in the case* $q = 4n-1$.

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