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Proximity Theorems of Discrete Convex Functions

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Abstract

A proximity theorem is a statement that, given an optimization problem and its relaxation, an optimal solution to the original problem exists in a certain neighborhood of a solution to the relaxation. Proximity theorems have been used successfully, for example, in designing efficient algorithms for discrete resource allocation problems. After reviewing the recent results for L-convex and M-convex functions, this paper establishes proximity theorems for larger classes of discrete convex functions, L_2-convex functions and M_2-convex functions, that are relevant to the polymatroid intersection problem and the submodular flow problem.

1 Introduction

In the area of discrete optimization, nonlinear optimization problems have been investigated as well as linear optimization problems. Submodular (set) functions and separable convex functions are well-known examples of tractable nonlinear functions, in that the submodular function minimization problem can be solved in polynomial time (see [13, 14, 24]), and separable convex functions have been treated successfully in many different discrete optimization problems (see [11]).

Recently, certain classes of "discrete convex functions" were proposed: $\{\mathrm{L},\mathrm{M},\mathrm{L}_2,\mathrm{M}_2\}$-convex functions of Murota [18, 19]. L-convex functions contain the class of submodular set functions. M-convex functions possess structures of matroids and polymatroids. Separable discrete convex functions can be characterized as functions with both L-convexity and M-convexity (in their variants). L_2-convex functions and M_2-convex functions constitute larger classes of discrete convex functions that are relevant to the polymatroid intersection problem, where an L_2-convex function is, by definition, the infimal convolution of two L-convex functions and an M_2-convex function is the sum of two M-convex functions. The M_2-convex function minimization problem is equivalent to the M-convex submodular flow problem [20] which is an extension of the submodular flow problem [3].

Those classes C of discrete convex functions f possess the following features in com-

Discreteness: f is defined on an integral lattice $\mathbf{Z}^n$, i.e., $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$, where $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of integers and reals, respectively.

Convex Extendibility: There exists a continuous convex function $\overline{f}$ such that $\overline{f}(x) = f(x)$ for all $x \in \mathbf{Z}^n$.

Optimality Criterion: There exists a neighborhood $N_C(x^*) \subset \mathbf{Z}^n$ with center x^* such that

$$f(x^*) \leq f(x) \ (\forall x \in \mathbf{Z}^n) \ \Leftrightarrow \ f(x^*) \leq f(x) \ (\forall x \in N_C(x^*)).$$

Optimality criterion says that global minimality is implied by local minimality defined in terms of the neighborhood $N_C(x^*)$. This is a significant feature inherited from continuous convex functions.

Moreover, L-/M-convex functions have a "proximity property" described as

Proximity Property: Given a positive integer α and a point $x^\alpha \in \mathbf{Z}^n$, there exists a function $d_C(n, \alpha)$ such that

$$f(x^\alpha) \leq f(x) \ (\forall x \in N_C^\alpha(x^\alpha)) \ \Rightarrow \ \exists x^* \in \arg\min f : \ ||x^* - x^\alpha||_\infty \leq d_C(n, \alpha),$$

where $N_C^\alpha(x^\alpha) = \{x^\alpha + \alpha(x - x^\alpha) \mid x \in N_C(x^\alpha)\}$ and $\arg\min f$ denotes the set of all minimizers of f, i.e.,

$$\arg\min f = \{x \in \mathbf{Z}^n \mid f(x) \leq f(y) \ (\forall y \in \mathbf{Z}^n)\}.$$

The proximity property says that a locally minimal solution x^α of a "scaled" function

$$f^\alpha(x) = f(x^\alpha + \alpha x) \qquad (x \in \mathbf{Z}^V)$$

is close to a minimizer x^* of f in terms of $d_C(n, \alpha)$. For L-/M-convex functions, $d_C(n, \alpha) = (n-1)(\alpha-1)$ is a valid choice ([15] and [16], respectively). The proximity property can be exploited in developing an efficient scaling algorithm for minimizing f. In fact, the L-convex function minimization problem can be solved in polynomial-time by combining submodular set function minimization algorithms and the proximity property [12] (see also [21]). For the M-convex function minimization, polynomial-time scaling algorithms based on the proximity property and its generalization are known [25, 26]. Proximity theorems for separable discrete convex functions are found in [8, 9, 17] in developing efficient algorithms for resource allocation problems. Different types of theorems on proximity have also been investigated: proximity between integral and real optimal solutions in [1, 2, 7, 9, 10] and proximity for a number of resource allocation problems with min-max type objective functions in [5].

This paper addresses proximity properties of L_2-/M_2-convex functions. Our main results say:

- for an essentially bounded L_2-convex function f and a positive integer α, if $x^\alpha \in \mathrm{dom}\, f$ satisfies
$$f(x^\alpha) \leq f(x^\alpha + \alpha\chi_S)$$
for all $S \subseteq V$, then there exists $x^* \in \arg\min f$ such that
$$||x^* - x^\alpha||_\infty \leq 2(n-1)(\alpha-1),$$

- for an M_2-convex function f represented as the sum of two M-convex functions f_1 and f_2, and a positive integer α, if $x^\alpha \in \mathrm{dom}\, f$ satisfies
$$\sum_{i=1}^{k}(f_1(x^\alpha-\alpha\chi_{u_i}+\alpha\chi_{w_i}) - f_1(x^\alpha)) + \sum_{i=1}^{k}(f_2(x^\alpha-\alpha\chi_{u_{i+1}}+\alpha\chi_{w_i}) - f_2(x^\alpha)) \geq 0$$
for any ordered sets $U=\{u_1,\ldots,u_k\}, W=\{w_1,\ldots,w_k\} \subset V$ with $U \cap W = \emptyset$ where $u_{k+1} = u_1$, then there exists $x^* \in \arg\min f$ such that
$$||x^* - x^\alpha||_\infty \leq \frac{n^2}{2}(\alpha-1).$$

Section 2 states definitions, optimality criteria and proximity properties for several classes of discrete convex functions.

2 Definitions, Optimality Criteria and Proximity Theorems

In this section, we introduce four classes of discrete convex functions, namely, {L, M, L_2, M_2}-convex functions with respect to definitions, optimality criteria and proximity theorems. While other variants of these classes, e.g., $L^\natural$-/$L_2^\natural$-convex functions due to [6] and $M^\natural$-/$M_2^\natural$-convex functions due to [22], are known, we concentrate on the above four classes because the results can be easily extended to the variants.

Subsections 2.3 and 2.4 present new results, an optimality criterion (Theorem 2.8) and a proximity property (Theorem 2.9) for L_2-convex functions, and proximity properties (Theorems 2.12 and 2.13) for M_2-convex functions. Subsection 2.2 also gives a new proximity property (Theorem 2.6) for M-convex functions in terms of ℓ_1-norm. Subsections 2.1 and 2.2 explain known results, optimality criteria and proximity theorems for L-convexity and M-convexity, respectively. Subsection 2.4 introduce optimality criteria for M_2-convexity, which are direct consequences of results for the M-convex submodular flow problem.

We first introduce notations. Let V be a nonempty finite set and put $n = |V|$. We denote by $\mathbf{Z}^V$ the set of all integral vectors $x = (x(v) : v \in V)$ indexed by V, and by $\mathbf{Z}_{++}$ the set of all positive integers. Given a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{\pm\infty\}$, the *effective domain* of f is defined by
$$\mathrm{dom}\, f = \{x \in \mathbf{Z}^V \mid f(x) \neq \pm\infty\}.$$

For each $S \subseteq V$, we denote by χ_S the *characteristic vector* of S defined by

$$\chi_S(v) = \begin{cases} 1 & (v \in S) \\ 0 & (v \notin S) \end{cases} \qquad (v \in V)$$

and write simply χ_u instead of $\chi_{\{u\}}$ for each $u \in V$. We also denote by $\mathbf{0}$ and $\mathbf{1}$ the vectors of all zeros and ones, respectively. For two vectors $x, y \in \mathbf{Z}^V$ with $x \leq y$, $[x, y]_{\mathbf{Z}}$ denotes the set $\{z \in \mathbf{Z}^V \mid x \leq z \leq y\}$.

2.1 L-convex Functions

For any $x, y \in \mathbf{Z}^V$, the vectors $x \wedge y$ and $x \vee y$ in $\mathbf{Z}^V$ are such that

$$(x \wedge y)(v) = \min\{x(v), y(v)\}, \quad (x \vee y)(v) = \max\{x(v), y(v)\} \quad (v \in V).$$

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be *L-convex* if $\mathrm{dom}\, f \neq \emptyset$ and it satisfies the following two conditions:

(SBF) f is *submodular*, i.e.,

$$f(x) + f(y) \geq f(x \wedge y) + f(x \vee y) \quad (\forall x, y \in \mathbf{Z}^V),$$

(TRF) $\exists r \in \mathbf{R}$ such that $f(x + \mathbf{1}) = f(x) + r \quad (\forall x \in \mathbf{Z}^V)$.

Global optimality of an L-convex function is characterized by local optimality.

Theorem 2.1 (L-optimality criterion, [21])
For an L-convex function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and $x^* \in \mathrm{dom}\, f$, we have

$$f(x^*) \leq f(x) \quad (\forall x \in \mathbf{Z}^V) \quad \Longleftrightarrow \quad \begin{cases} f(x^*) \leq f(x^* + \chi_S) \quad (\forall S \subseteq V), \\ f(x^* + \mathbf{1}) = f(x^*). \end{cases}$$

The above local optimality criterion can be checked in polynomial time because the first condition can be verified by using submodular function minimization algorithms and the second condition is easy.

We next introduce a proximity theorem of L-convex functions.

Theorem 2.2 (L-proximity theorem, [15])
Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an L-convex function with $f(x + \mathbf{1}) = f(x)$ $(\forall x \in \mathbf{Z}^V)$ and let $\alpha \in \mathbf{Z}_{++}$. If $x^\alpha \in \mathrm{dom}\, f$ satisfies

$$f(x^\alpha) \leq f(x^\alpha + \alpha\chi_S) \quad (\forall S \subseteq V),$$

then $\arg\min f \neq \emptyset$ and there exists $x^* \in \arg\min f$ with

$$x^\alpha \leq x^* \leq x^\alpha + (n-1)(\alpha - 1)\mathbf{1}.$$

Remark 2.3 Theorems 2.1 and 2.2 are extended to a more general class of "quasi" L-convex functions [23].

2.2 M-convex Functions

We define the *positive support* and *negative support* of a vector $x = (x(v) : v \in V) \in \mathbf{Z}^V$ by

$$\mathrm{supp}^+(x) = \{v \in V \mid x(v) > 0\} \quad \text{and} \quad \mathrm{supp}^-(x) = \{v \in V \mid x(v) < 0\}.$$

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called *M-convex* if $\mathrm{dom}\, f \neq \emptyset$ and it satisfies

(M-EXC) for $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x - y)$, there exists $v \in \mathrm{supp}^-(x - y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

We note that (M-EXC) is also represented as: for $x, y \in \mathrm{dom}\, f$,

$$f(x) + f(y) \geq \max_{u \in \mathrm{supp}^+(x-y)} \min_{v \in \mathrm{supp}^-(x-y)} [\, f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v)\,],$$

where the maximum and the minimum over an empty set are $-\infty$ and $+\infty$, respectively. From (M-EXC), the effective domain $\mathrm{dom}\, f$ lies on a hyperplane $\{x \in \mathbf{R}^V \mid x(V) = \text{constant}\}$, where $x(V) = \sum_{v \in V} x(v)$. It is also known that $\mathrm{dom}\, f$ is the set of integer points of the base polyhedron of an integral submodular system (see [4] for submodular systems).

The minimizers of an M-convex function have a nice characterization which can be checked efficiently.

Theorem 2.4 (M-optimality criterion, [18, 19])
For an M-convex function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and $x^* \in \mathrm{dom}\, f$, we have

$$f(x^*) \leq f(x) \;\; (\forall x \in \mathbf{Z}^V) \quad \Longleftrightarrow \quad f(x^*) \leq f(x^* - \chi_u + \chi_v) \quad (\forall u, v \in V).$$

We next introduce a proximity theorem of M-convex functions.

Theorem 2.5 (M-proximity theorem, [16])
Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an M-convex function and let $\alpha \in \mathbf{Z}_{++}$. If $x^\alpha \in \mathrm{dom}\, f$ satisfies

$$f(x^\alpha) \leq f(x^\alpha - \alpha\chi_u + \alpha\chi_v) \quad (\forall u, v \subseteq V),$$

then $\arg\min f \neq \emptyset$ *and there exists* $x^* \in \arg\min f$ *with*

$$|x^\alpha(v) - x^*(v)| \leq (n-1)(\alpha - 1) \quad (\forall v \in V).$$

By slightly modifying the proof of [16], we also obtain the following proximity theorem in terms of ℓ_1-norm.

Theorem 2.6 *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *be an M-convex function and let* $\alpha \in \mathbf{Z}_{++}$. *If* $x^\alpha \in \operatorname{dom} f$ *satisfies*

$$f(x^\alpha) \leq f(x^\alpha - \alpha\chi_u + \alpha\chi_v) \quad (\forall u, v \subseteq V), \tag{1}$$

then $\arg\min f \neq \emptyset$ *and there exists* $x^* \in \arg\min f$ *with*

$$||x^* - x^\alpha||_1 \leq \frac{n^2}{2}(\alpha - 1). \tag{2}$$

Remark 2.7 Theorems 2.4 and 2.5 are extended to a more general class of "quasi" M-convex functions [23].

2.3 L$_2$-convex Functions

For any functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, the *infimal convolution* of f_1 and f_2, denoted by $f_1 \Box f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{\pm\infty\}$, is defined by

$$(f_1 \Box f_2)(x) = \inf\{f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x,\ x_1, x_2 \in \mathbf{Z}^V\} \quad (x \in \mathbf{Z}^V).$$

It is easy to show that if $f_1 \Box f_2 > -\infty$ then the effective domain of $f_1 \Box f_2$ coincides with the Minkowski sum of the effective domains of f_1 and f_2, that is,

$$\operatorname{dom}(f_1 \Box f_2) = (\operatorname{dom} f_1) + (\operatorname{dom} f_2) \equiv \{x_1 + x_2 \mid x_1 \in \operatorname{dom} f_1,\ x_2 \in \operatorname{dom} f_2\}.$$

It is known that the infimal convolution of two M-convex functions is also M-convex, but the infimal convolution of two L-convex functions may not be L-convex [18]. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be *L$_2$-convex* if $\operatorname{dom} f \neq \emptyset$ and $f = f_1 \Box f_2$ for some L-convex functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. We say that an L-/L$_2$-convex function f is *essentially bounded* if $\operatorname{dom} f \cap \{x \in \mathbf{Z}^V \mid x(v) = 0\}$ is bounded for some $v \in V$. If an L$_2$-convex function $f = f_1 \Box f_2$ is essentially bounded, then f_1 and f_2 are also essentially bounded, because $\operatorname{dom} f = (\operatorname{dom} f_1) + (\operatorname{dom} f_2)$ holds for L$_2$-convex function f.

The following optimality criterion and the proximity theorem for L$_2$-convex functions are new results. We emphasize that the optimality criterion is the same as that for L-convex functions stated in Theorem 2.1 and that the proximity theorem is almost the same as that stated in Theorem 2.2.

Theorem 2.8 (L$_2$-optimality criterion)
For an L$_2$-convex function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *and* $x^* \in \operatorname{dom} f$, *we have*

$$f(x^*) \leq f(x) \quad (\forall x \in \mathbf{Z}^V) \quad \Longleftrightarrow \quad \begin{cases} f(x^*) \leq f(x^* + \chi_S) & (\forall S \subseteq V), \\ f(x^* + \mathbf{1}) = f(x^*). \end{cases}$$

Theorem 2.9 (L_2-proximity theorem)

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an essentially bounded L_2-convex function with $f(x+\mathbf{1}) = f(x)$ $(\forall x \in \mathbf{Z}^V)$ and let $\alpha \in \mathbf{Z}_{++}$. If $x^\alpha \in \mathrm{dom}\, f$ satisfies

$$f(x^\alpha) \leq f(x^\alpha + \alpha\chi_S) \quad (\forall S \subseteq V),$$

then $\arg\min f \neq \emptyset$ and there exists $x^* \in \arg\min f$ with

$$x^\alpha \leq x^* \leq x^\alpha + 2(n-1)(\alpha-1)\mathbf{1}.$$

2.4 M_2-convex Functions

It is known that the sum of two M-convex functions is not necessarily M-convex. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be *M_2-convex* if $\mathrm{dom}\, f \neq \emptyset$ and $f = f_1 + f_2$ for some M-convex functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. It is easy to show that $\mathrm{dom}\, f = (\mathrm{dom}\, f_1) \cap (\mathrm{dom}\, f_2)$. Obviously, if $\mathrm{dom}\, f_1 = \mathrm{dom}\, f_2$ and f_2 is identically zero, then $f = f_1$ is M-convex, and hence, the class of M_2-convex functions includes that of M-convex functions. The M_2-convex function minimization problem contains the polymatroid intersection problem as a special case. Thus, optimality criteria for M_2-convexity below are extensions of known results for the matroid intersection problem and the polymatroid intersection problem.

For a vector $p \in \mathbf{R}^V$, let us define functions $\langle p, x\rangle$ and $f[p](x)$ by

$$\langle p, x\rangle = \sum_{v \in V} p(v)x(v) \quad \text{and} \quad f[p](x) = f(x) + \langle p, x\rangle \qquad (x \in \mathbf{Z}^V).$$

If f is M-convex, then $f[p]$ is also M-convex.

Several results on optimality of M_2-convexity are known.

Theorem 2.10 (M-convex intersection theorem, [18])

For M-convex functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and a point $x^* \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$, we have

$$f_1(x^*) + f_2(x^*) \leq f_1(x) + f_2(x) \quad (\forall x \in \mathbf{Z}^V)$$

if and only if there exists $p^* \in \mathbf{R}^V$ such that

$$\begin{aligned} f_1[-p^*](x^*) &\leq f_1[-p^*](x) \quad (\forall x \in \mathbf{Z}^V), \\ f_2[+p^*](x^*) &\leq f_1[+p^*](x) \quad (\forall x \in \mathbf{Z}^V), \end{aligned}$$

and furthermore, we have

$$\arg\min(f_1 + f_2) = \arg\min(f_1[-p^*]) \cap \arg\min(f_2[+p^*])$$

for such p^*.

Optimality criteria of M$_2$-convex functions can be transformed from those of the M-convex submodular flow problem in [19], because the M$_2$-convex function minimization and the M-convex submodular flow problem are equivalent to each other. The following theorem is a direct consequence of the results in [19].

Theorem 2.11 (M$_2$-optimality criteria, see [19])
For M-convex functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and a point $x^* \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$, three conditions below are equivalent:

(a) $x^* \in \arg\min(f_1 + f_2)$.

(b) *For any ordered sets $U=\{u_1, \ldots, u_k\}, W=\{w_1, \ldots, w_k\} \subset V$ with $U \cap W = \emptyset$,*

$$\sum_{i=1}^{k}(f_1(x^*-\chi_{u_i}+\chi_{w_i}) - f_1(x^*)) + \sum_{i=1}^{k}(f_2(x^*-\chi_{u_{i+1}}+\chi_{w_i}) - f_2(x^*)) \geq 0,$$

where $u_{k+1} = u_1$.

(c) $(f_1 + f_2)(x^*) \leq (f_1 + f_2)(x^* - \chi_U + \chi_W) \quad (\forall U, W \subset V,\ |U| = |W|)$.

The optimality for M$_2$-convexity can be checked in polynomial time by transforming (b) of Theorem 2.11 to a network problem (see Remark 2.14), although checking condition (c) of Theorem 2.11 seems to be a hard problem. In view of polynomial time verifiability, we relax (b) of Theorem 2.11 to formulate a proximity theorem of M$_2$-convex functions. This is the main result of this paper.

Theorem 2.12 (M$_2$-proximity theorem)
Let $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be M-convex functions and let $\alpha \in \mathbf{Z}_{++}$. If $x^\alpha \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ satisfies

$$\sum_{i=1}^{k}(f_1(x^\alpha-\alpha\chi_{u_i}+\alpha\chi_{w_i}) - f_1(x^\alpha)) + \sum_{i=1}^{k}(f_2(x^\alpha-\alpha\chi_{u_{i+1}}+\alpha\chi_{w_i}) - f_2(x^\alpha)) \geq 0$$

for any ordered sets $U=\{u_1, \ldots, u_k\}, W=\{w_1, \ldots, w_k\} \subset V$ with $U \cap W = \emptyset$ where $u_{k+1} = u_1$, then $\arg\min(f_1 + f_2) \neq \emptyset$ and there exists $x^* \in \arg\min(f_1 + f_2)$ with

$$||x^* - x^\alpha||_\infty \leq \frac{n^2}{2}\,(\alpha - 1). \tag{3}$$

The proof of Theorem 2.12 relies heavily on the following result.

Theorem 2.13 *Let $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be M-convex functions with $\arg\min(f_1 + f_2) \neq \emptyset$. For a given point $x \in \mathbf{Z}^V$ with $x(V) = y(V)$ for any $y \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$, and for $d \in \mathbf{Z}$, if there exist $x^1 \in \arg\min f_1$ and $x^2 \in \arg\min f_2$ such that*

$$||x^1 - x||_1 \leq d, \qquad ||x^2 - x||_1 \leq d, \tag{4}$$

then there exists $x^* \in \arg\min(f_1 + f_2)$ with

$$||x^* - x||_\infty \leq d.$$

Remark 2.14 Condition (b) of Theorem 2.11 can be checked in polynomial time. Given two M-convex functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, a point $x \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and a positive integer $\alpha \in \mathbf{Z}_{++}$, we construct a directed graph $G_x^\alpha = (V_1 \cup V_2, A)$ and an arc length $\ell_x^\alpha \in \mathbf{R}^A$ as follows. Let V_1 and V_2 be copies of V, i.e.,

$$V_1 = \{v_1 \mid v \in V\}, \quad V_2 = \{v_2 \mid v \in V\},$$

where v_1 and v_2 are the copies of $v \in V$. Arc set A consists of three disjoint parts:

$$\begin{aligned} A_b &= \{(v_1, v_2) \mid v \in V\} \cup \{(v_2, v_1) \mid v \in V\}, \\ A_1 &= \{(u_1, v_1) \mid u, v \in V,\ u \neq v,\ x - \alpha\chi_u + \alpha\chi_v \in \mathrm{dom}\, f_1\}, \\ A_2 &= \{(v_2, u_2) \mid u, v \in V,\ u \neq v,\ x - \alpha\chi_u + \alpha\chi_v \in \mathrm{dom}\, f_2\}. \end{aligned} \tag{5}$$

We define $\ell_x^\alpha \in \mathbf{R}^A$ by

$$\ell_x^\alpha(a) = \begin{cases} 0 & (a \in A_b) \\ f_1(x - \alpha\chi_u + \alpha\chi_v) - f_1(x) & (a = (u_1, v_1) \in A_1) \\ f_2(x - \alpha\chi_u + \alpha\chi_v) - f_2(x) & (a = (v_2, u_2) \in A_2). \end{cases} \tag{6}$$

Lemma 2.15 below guarantees that (b) of Theorem 2.11 can be checked in polynomial time by applying shortest path algorithms.

Lemma 2.15 *For two M-convex functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, a point $x \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and $\alpha \in \mathbf{Z}_{++}$, two conditions below are equivalent:*

(a) *There exists no negative cycle in G_x^α with length ℓ_x^α.*

(b) *For any ordered sets $U=\{u_1, \ldots, u_k\}, W=\{w_1, \ldots, w_k\} \subset V$ with $U \cap W = \emptyset$,*

$$\sum_{i=1}^{k}(f_1(x-\alpha\chi_{u_i}+\alpha\chi_{w_i}) - f_1(x)) + \sum_{i=1}^{k}(f_2(x-\alpha\chi_{u_{i+1}}+\alpha\chi_{w_i}) - f_2(x)) \geq 0, \tag{7}$$

where $u_{k+1} = u_1$.

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