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On Characterization of Nash Equilibrium Strategy of Bi-matrix Games with Fuzzy Payoffs

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Abstract. In this paper, we consider fuzzy bi-matrix games, namely, two-person games with fuzzy payoff. Based on fuzzy max order, for such games, we define three kinds of concepts of Nash equilibrium strategies and investigate their properties.
Keywords. Bi-Matrix game; Fuzzy number; Fuzzy max order; Nash equilibrium strategy; Non-dominated Nash equilibrium strategy; Possibility measure; Necessity measures

1 Introduction

Since seminal works by Neumann-Morgenstern([13]) and Nash([11] and [12]), Game theory has played an important role in the fields of decision making theory such as economics, management, and operations research, etc. When we apply the game theory to model some practical problems which we encounter in real situations, we have to know the values of payoffs exactly. However, it is difficult to know the exact values of payoffs and we could only know the values of payoffs approximately, or with some imprecise degree. In such situations, it is useful to model the problems as games with fuzzy payoffs. In this case, since the expected payoffs of the game should be fuzzy-valued, there are no concepts of equilibrium strategies to be accepted widely. So, it is an important task to define the concepts of equilibrium strategies and investigate their properties. Compos([3]) has proposed a methods to solve fuzzy matrix games based on linear programming, but has not defined explicit concepts of equilibrium strategies. For matrix games with fuzzy payoffs, Maeda([9]) has defined minimax equilibrium strategies based on fuzzy max order and investigated their properties. For Bi-matrix games with fuzzy payoffs, Maeda([10]) has defined Nash equilibrium strategies based on possibility and necessity measures and investigated its properties. While, Aubin([2]) has considered fuzzy cooperative games.

In this paper, we consider fuzzy bi-matrix games. For such a game, we shall define three kinds of concepts of Nash equilibrium strategies and investigate their properties.

For that purpose, this paper is organized as follows. In Section 2, we shall give some basic definitions and notations on fuzzy numbers. In Section 3, we shall define fuzzy bi-matrix game with fuzzy payoffs and three kinds of concepts of Nash equilibrium strategies and investigate their properties. In Section 4, we investigate the properties of values of fuzzy matrix games by means of possibility and necessity measures.

2 Preliminary

In this section, we shall give some definitions and notations on fuzzy numbers, which are used throughout the paper.

Let R^n be n-dimensional Euclidean space, and $x \equiv (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in R^n$ be any vector, where $x_i \in R$, $i = 1, 2, \cdots, n$ and T denotes the transpose of the vector. For any two vectors $x, y \in R^n$, we write $x \geqq y$ iff $x_i \geqq y_i$, $i = 1, 2, \cdots, n$, $x \geq y$ iff $x \geqq y$ and $x \neq y$, and $x > y$ iff $x_i > y_i$, $i = 1, 2, \cdots, n$, respectively.

Definition 2.1 *A fuzzy number $\tilde{a}$ is defined as a fuzzy set on the space of real number R, whose membership function $\mu_{\tilde{a}} : R \to [0, 1]$ satisfies the following conditions:*

(i) *there exists a unique real number c, called center of $\tilde{a}$, such that $\mu_{\tilde{a}}(c) = 1$,*

(ii) *$\mu_{\tilde{a}}$ is upper semi-continuous,*

(iii) *$\mu_{\tilde{a}}$ is quasi concave,*

(vi) supp$(\tilde{a})$ *is compact, where* supp$(\tilde{a})$ *denotes the support of $\tilde{a}$.*

We denote the set of all fuzzy numbers by $\mathcal{F}$.

Let $\tilde{a}$, $\tilde{b}$ be any fuzzy numbers and let $\lambda \in R$ be any real number. Then the sum of two fuzzy numbers and scalar product of λ and $\tilde{a}$ are defined by membership functions

$$\mu_{\tilde{a}+\tilde{b}}(t) = \sup_{t=u+v} \min\{\mu_{\tilde{a}}(u), \mu_{\tilde{b}}(v)\}, \qquad \mu_{\lambda\tilde{a}}(t) = \max\{0, \sup_{t=\lambda u} \mu_{\tilde{a}}(u)\}, \tag{1}$$

where we set $\sup\{\emptyset\} = -\infty$.

Definition 2.2 *Let m be any real number and let h be any positive number. A fuzzy number $\tilde{a}$ whose membership function is given by*

$$\mu_{\tilde{a}}(x) \equiv \begin{cases} 1 - \left|\dfrac{x-m}{h}\right| & \text{for} \quad x \in [m-h, m+h] \\ 0 & \qquad\text{otherwise} \end{cases} \tag{2}$$

is called a symmetric triangular fuzzy number, and we denote the set of all symmetric triangular fuzzy numbers by $\mathcal{F}_{\mathrm{T}}$.

Real numbers m and h in (2) are called the center and the deviation parameter of $\tilde{a}$, respectively. Since any symmetric triangular fuzzy number $\tilde{a}$ is characterized by the center m and the deviation parameter h of $\tilde{a}$, we denote the symmetric triangular fuzzy number $\tilde{a}$ by $\tilde{a} \equiv (m, h)_{\mathrm{T}}$.

Let $\tilde{a}$ be any fuzzy number and let $\alpha \in (0, 1]$ be any real number. The set $[\tilde{a}]^\alpha \equiv \{x \in R \mid \mu_{\tilde{a}}(x) \geqq \alpha\}$ is called the α-level set of $\tilde{a}$. For $\alpha = 0$, we set $[\tilde{a}]^0 \equiv \mathrm{cl}\ \{x \in R \mid \mu_{\tilde{a}}(x) > 0\}$, where cl denotes the closure of sets. Since the set $[\tilde{a}]^\alpha$ is a closed interval for each $\alpha \in [0, 1]$, we denote the α-level set of $\tilde{a}$ by $[a_\alpha^L, a_\alpha^R]$, where $a_\alpha^L \equiv \inf[\tilde{a}]^\alpha$ and $a_\alpha^R \equiv \sup[\tilde{a}]^\alpha$.

For any two fuzzy numbers $\tilde{a}, \tilde{b} \in \mathcal{F}_{\mathrm{T}}$, we introduce three kinds of binary relations.

Definition 2.3 *For any symmetric triangular fuzzy numbers* $\tilde{a},\ \tilde{b} \in \mathcal{F}_{\mathrm{T}}$, *we write*

$$\tilde{a} \succeqq \tilde{b} \ \text{ iff } \ (a_\alpha^L, a_\alpha^R)^{\mathrm{T}} \geqq (b_\alpha^L, b_\alpha^R)^{\mathrm{T}}, \quad \forall\alpha \in [0,1], \tag{3}$$

$$\tilde{a} \succeq \tilde{b} \ \text{ iff } \ (a_\alpha^L, a_\alpha^R)^{\mathrm{T}} \geq (b_\alpha^L, b_\alpha^R)^{\mathrm{T}}, \quad \forall\alpha \in [0,1], \tag{4}$$

$$\tilde{a} \succ \tilde{b} \ \text{ iff } \ (a_\alpha^L, a_\alpha^R)^{\mathrm{T}} > (b_\alpha^L, b_\alpha^R)^{\mathrm{T}}, \quad \forall\alpha \in [0,1]. \tag{5}$$

We call binary relations $\succeqq$, $\succeq$ *and* $\succ$ *a fuzzy max order, a strict fuzzy max order and a strong fuzzy max order, respectively.*

From the definition, the fuzzy max order $\succeqq$ defines a partial order on $\mathcal{F}_{\mathrm{T}}$. On the other hand, binary relations $\succeq$ and $\succ$ are not partial orders on $\mathcal{F}_{\mathrm{T}}$.

Theorem 2.1 ([6]) *Let* $\tilde{a} \equiv (a,\alpha)_{\mathrm{T}}$ *and* $\tilde{b} \equiv (b,\beta)_{\mathrm{T}}$ *be any symmetric triangular fuzzy numbers. Then, it holds that*

$$\tilde{a} \succeqq \tilde{b} \ \text{ iff } \ a - b \geqq |\alpha - \beta|, \tag{6}$$

$$\tilde{a} \succ \tilde{b} \ \text{ iff } \ a - b > |\alpha - \beta|. \tag{7}$$

Definition 2.4 *Let* $\tilde{a},\ \tilde{b}$ *be any fuzzy numbers. We define the inequality relations as follows:*

(i) $\mathrm{Pos}\ (\tilde{a} \geq \tilde{b}) \equiv \sup\{\min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)) \mid x \geq y\}$,

(ii) $\mathrm{Nes}\ (\tilde{a} \geq \tilde{b}) \equiv \inf_x \left\{\sup_y\{\max(1-\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)) \mid x \geq y\}\right\}$,

Theorem 2.2 ([15]) *Let* $\tilde{a},\ \tilde{b}$ *be any symmetric triangular fuzzy numbers and let* $\alpha \in (0,\, 1]$ *be any real number. Then we have the following relationships:*

(i) $\mathrm{Pos}(\tilde{a} \geq \tilde{b}) \geqq \alpha$ iff $a_\alpha^R \geqq b_\alpha^L$,

(ii) $\mathrm{Pos}(\tilde{a} \geq \tilde{b}) \leqq \alpha$ iff $a_\alpha^R \leqq b_\alpha^L$,

(iii) $\mathrm{Nes}(\tilde{a} \geq \tilde{b}) \geqq \alpha$ iff $a_{1-\alpha}^L \geqq b_\alpha^L$,

(iv) $\mathrm{Nes}(\tilde{a} \geq \tilde{b}) \leqq \alpha$ iff $a_{1-\alpha}^L \leqq b_\alpha^L$.

3 Bi-matrix Game with Fuzzy Payoffs and Its Equilibrium Strategy

Let $I,\ J$ denote players and let $M \equiv \{1,2,\cdots,m\}$ and $N \equiv \{1,2,\cdots,n\}$ be the sets of all pure strategies available for player I and J, respectively. We denote the sets of all mixed strategies available for players I and J by

$$S_I \equiv \{(x_1,x_2,\cdots,x_m) \in R_+^m \mid x_i \geqq 0,\ i=1,2,\cdots,m,\ \sum_{i=1}^m x_i = 1\},$$

$$S_J \equiv \{(y_1,y_2,\cdots,y_n) \in R_+^n \mid y_j \geqq 0,\ j=1,2,\cdots,n,\ \sum_{j=1}^n y_j = 1\}.$$

By $\tilde{a}_{ij} \equiv (a_{ij}, h_{ij})_{\mathrm{T}}, \tilde{b}_{ij} \equiv (b_{ij}, k_{ij})_{\mathrm{T}} \in \mathcal{F}_{\mathrm{T}}$, we denote the payoffs that player I receives and J receives when player I plays the pure strategy i and player J plays the pure strategy j, respectively. Now we define fuzzy bi-matrix game by

$$\tilde{\Gamma} \equiv \begin{pmatrix} (\tilde{a}_{11}, \tilde{b}_{11}) & (\tilde{a}_{12}, \tilde{b}_{12}) & \cdots & (\tilde{a}_{1n}, \tilde{b}_{1n}) \\ (\tilde{a}_{21}, \tilde{b}_{21}) & (\tilde{a}_{22}, \tilde{b}_{22}) & \cdots & (\tilde{a}_{2n}, \tilde{b}_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\tilde{a}_{m1}, \tilde{b}_{m1}) & (\tilde{a}_{m2}, \tilde{b}_{m2}) & \cdots & (\tilde{a}_{mn}, \tilde{b}_{mn}) \end{pmatrix}.$$

We define two matrix with fuzzy elements by $\tilde{A} = (A, H) = (\tilde{a}_{ij})$ and $\tilde{B} = (B, K) = (\tilde{b}_{ij})$.

Definition 3.1 *A point $(x^*, y^*) \in S_I \times S_J$ is said to be a Nash equilibrium strategy to Game $\tilde{\Gamma}$ if it holds that*

(i) $x^{\mathrm{T}}\tilde{A}y^* \trianglelefteq x^{*\mathrm{T}}\tilde{A}y^*, \quad \forall x \in S_I,$

(ii) $x^{*\mathrm{T}}\tilde{B}y \trianglelefteq x^{*\mathrm{T}}\tilde{B}y^*, \quad \forall y \in S_J.$

Then a point $x^*\tilde{A}y^*$ is said to be the value of Game $\tilde{\Gamma}$

Definition 3.2 *A point $(x^*, y^*) \in S_I \times S_J$ is said to be a non-dominated Nash equilibrium strategy to Game $\tilde{\Gamma}$ if*

(i) *there exist no $x \in S_I$ such that $x^{*\mathrm{T}}\tilde{A}y^* \preceq x^{\mathrm{T}}\tilde{A}y^*$,*

(ii) *there exist no $y \in S_J$ such that $x^{*\mathrm{T}}\tilde{B}y^* \preceq x^{*\mathrm{T}}\tilde{B}y$*

hold.

Definition 3.3 *A point $(x^*, y^*) \in S_I \times S_J$ is said to be a weak non-dominated Nash equilibrium strategy to Game $\tilde{\Gamma}$ if*

(i) *there exist no $x \in S_I$ such that $x^{*\mathrm{T}}\tilde{A}y^* \prec x^{\mathrm{T}}\tilde{A}y^*$,*

(ii) *there exist no $y \in S_J$ such that $x^{*\mathrm{T}}\tilde{B}y^* \prec x^{*\mathrm{T}}\tilde{B}y$*

hold.

By Definition, it is obvious that the following relationship holds among these definitions.

(1) If a strategy $(x^*, y^*) \in S_I \times S_J$ is a Nash equilibrium strategy to Game $\tilde{\Gamma}$, it is a non-dominated Nash strategy.

(2) If a strategy $(x^*, y^*) \in S_I \times S_J$ is a non-dominated Nash equilibrium strategy to Game $\tilde{\Gamma}$, it is a weak non-dominated Nash strategy.

When all elements $\tilde{a}_{ij}s$ are crisp numbers, these definitions coincide with that of bi-matrix games([13]). Therefore, these definitions are natural extensions of Nash equilibrium strategy in bi-matrix to fuzzy bi-matrix game.

From Theorem 2.1, we could derive the following theorems.

Theorem 3.1 *In order that a strategy $(x^*, y^*) \in S_I \times S_J$ be a Nash equilibrium strategy to Game $\tilde{\Gamma}$, it is necessary and sufficient that, for all $x \in S_I,\ y \in S_J$,*

(i) $x^{\mathrm{T}}\mathcal{A}y^* \leqq x^{*\mathrm{T}}\mathcal{A}y^*$,

(ii) $x^{*\mathrm{T}}\mathcal{B}y \leqq x^{*\mathrm{T}}\mathcal{B}y^*$

hold, where $x^{\mathrm{T}}\mathcal{A}y \equiv (x^{\mathrm{T}}A_0^Ly, x^{\mathrm{T}}A_0^Ry)^{\mathrm{T}},\ x^{\mathrm{T}}\mathcal{B}y \equiv (x^{\mathrm{T}}A_0^Ly, x^{\mathrm{T}}A_0^Ry)^{\mathrm{T}}$ hold.

Theorem 3.1 shows that players I, J face a pair of bi-matrix sum games with crisp payoffs

$$\Gamma_1 \equiv \langle\ \{I, J\}, S_I, S_J, A_0^L, B_0^L\ \rangle \text{ and } \Gamma_2 \equiv \langle\ \{I, J\}, S_I, S_J, A_0^R, B_0^R\ \rangle.$$

Next we shall characterize non-dominated and weak non-dominated Nash equilibrium strategies.

Theorem 3.2 *In order that a strategy $(x^*, y^*) \in S_I \times S_J$ be a non-dominated minimax equilibrium strategy to Game $\tilde{\Gamma}$, it is necessary and sufficient that the following conditions hold:*

(i) *there is no $x \in S_I$ such that $x^{*\mathrm{T}}\mathcal{A}y^* \leq x^{\mathrm{T}}\mathcal{A}y^*$ holds,*

(ii) *there is no $y \in S_J$ such that $x^{*\mathrm{T}}\mathcal{A}y \leq x^{*\mathrm{T}}\mathcal{B}y^*$ holds.*

By a similar way, we have the following theorem.

Theorem 3.3 *In order that a strategy $(x^*, y^*) \in S_I \times S_J$ be a weak non-dominated Nash equilibrium strategy to Game $\tilde{\Gamma}$, it is necessary and sufficient that the following conditions hold:*

(i) *there is no $x \in S_I$ such that $x^{*\mathrm{T}}\mathcal{A}y^* < x^{*\mathrm{T}}\mathcal{A}y$ holds,*

(ii) *there is no $y \in S_J$ such that $x^{*\mathrm{T}}\mathcal{B}y\ < x^{*\mathrm{T}}\mathcal{B}y^*$ holds.*

Theorem 3.1, 3.2 and 3.3 show that fuzzy bi-matrix game $\tilde{\Gamma}$ is equivalent to a pair of bi-matrix games with crisp payoffs $\{\Gamma_1, \Gamma_2\}$.

For further discussions, associated with fuzzy bi-matrix game $\tilde{\Gamma}$, we shall define parametric bi-matrix games with crisp payoffs, namely, bi-matrix games whose payoffs are parameterized.

Let $\lambda, \mu \in [0, 1]$ be any real numbers and we set $A(\lambda) \equiv A + (1 - 2\lambda)H,\ B(\mu) \equiv B + (1 - 2\mu)K$. We consider the following bi-matrix game with parameters λ, μ:

$$\Gamma(\lambda, \mu) \equiv \langle \{I, J\},\ S_I,\ S_J,\ A(\lambda),\ B(\mu) \rangle,$$

Noting that

$$\Gamma(\lambda,\mu)=\begin{cases}\langle\{I,J\},\,S_I,\,S_J,\,A^R_{2\lambda},\,B^R_{2\mu}\rangle & \text{if} \quad \lambda,\mu\in(0,1/2],\\ \langle\{I,J\},\,S_I,\,S_J,\,A^R_{2\lambda},\,B^L_{2\mu}\rangle & \text{if} \quad \lambda,\in(0,1/2],\ \mu\in(1/2,1],\\ \langle\{I,J\},\,S_I,\,S_J,\,A^L_{2\lambda},\,B^R_{2\mu}\rangle & \text{if} \quad \lambda\in(1/2,1],\ \mu\in(0,1/2],\\ \langle\{I,J\},\,S_I,\,S_J,\,A^L_{2\lambda},\,B^L_{2\mu}\rangle & \text{if} \quad \lambda,\mu\in(1/2,1]\end{cases}$$

holds.

Definition 3.4 ([12]) *Let $\lambda,\ \mu\in[0,\,1]$ be any real numbers. A strategy $(x^*,y^*)\in S_I\times S_J$ is said to be a Nash equilibrium strategy to Game $\Gamma(\lambda,\ \mu)$ if it holds that*

$$x^{\mathrm{T}}A(\lambda)y^*\leqq x^{*\mathrm{T}}A(\lambda)y^*,\qquad \forall x\in S_I \tag{8}$$

$$x^{*\mathrm{T}}B(\mu)y\leqq x^{*\mathrm{T}}B(\mu)y^*,\qquad \forall y\in S_J. \tag{9}$$

The following theorems give relationships between Game $\tilde{\Gamma}$ and Game $\Gamma(\lambda,\mu)$.

Theorem 3.4 *In order that a strategy $(x^*,y^*)\in S_I\times S_J$ be a non-dominated Nash strategy to Game $\tilde{\Gamma}$, it is necessary and sufficient that there exist positive real numbers $\lambda,\ \mu\in(0,1)$ such that (x^*,y^*) be a Nash equilibrium strategy to bi-matrix Game $\Gamma(\lambda,\mu)$.*

By a similar way, we have the following theorem.

Theorem 3.5 *In order that a strategy $(x^*,y^*)\in S_I\times S_J$ be a weak non-dominated Nash equilibrium strategy to Game $\tilde{\Gamma}$, it is necessary and sufficient that there exist positive real numbers $\lambda,\ \mu\in[0,1]$ such that (x^*,y^*) be a Nash equilibrium strategy to bi-matrix Game $\Gamma(\lambda,\mu)$.*

From Theorem 3.4 and 3.5, in order to find non-dominated or weak non-dominated Nash equilibrium strategy to Game $\tilde{\Gamma}$, it suffices to find Nash equilibrium strategy to Game $\Gamma(\lambda,\mu)$. In this sense, Game $\tilde{\Gamma}$ is equivalent to a family of bi-matrix games $\{\Gamma(\lambda,\mu)\}_{\lambda,\mu}$.

For any real numbers $\lambda,\ \mu\in[0,1]$, it is well known that there exists at least one Nash equilibrium strategy to Game $\Gamma(\lambda,\ \mu)$([1]). Therefore, from Theorem 3.4 and 3.5, we have the following theorem.

Theorem 3.6 *In Game $\tilde{\Gamma}$, the following holds:*

(i) *There exists at least one non-dominated Nash equilibrium strategy.*

(ii) *There exists at least one weak non-dominated Nash equilibrium strategy.*

4 Properties of Values of Fuzzy Matrix Games

In the previous section, we have shown that a fuzzy bi-matrix game is equivalent to a family of parametric bi-matrix games. However, this implies that there are infinite number of non-dominated Nash equilibrium strategies. In this section, we investigate the properties of the value of Game $\tilde{\Gamma}$.

Let $(x^*, y^*) \in S_I \times S_J$ be any non-dominated Nash equilibrium strategy to Game $\tilde{\Gamma}$. Then from Theorem 3.4, there exist real numbers $\lambda, \mu \in (0,1)$ such that

$$x^{*\mathrm{T}}(A + (1-2\lambda)H)y^* \geqq x^{\mathrm{T}}(A + (1-2\lambda)H)y^*, \quad \forall x \in S_I, \tag{10}$$

$$x^{*\mathrm{T}}(B + (1-2\mu)K)y^* \geqq x^{*\mathrm{T}}(B + (1-2\mu)K)y, \quad \forall y \in S_J. \tag{11}$$

Now we set $v^* \equiv x^{*\mathrm{T}}(A + (1-2\lambda)H)y^*$ and $w^* \equiv x^{*\mathrm{T}}(B + (1-2\mu)K)y^*$. In case that $\lambda, \mu \in (0, 1/2]$, from Theorem 2.2, (10) and (11) imply that

$$2\lambda = \mathrm{Pos}(x^{*\mathrm{T}}\tilde{A}y^* \geqq v^*) \geqq \mathrm{Pos}(x^{\mathrm{T}}\tilde{A}y^* \geqq v^*), \quad \forall x \in S_I, \tag{12}$$

$$2\mu = \mathrm{Pos}(x^{*\mathrm{T}}\tilde{B}y^* \geqq w^*) \geqq \mathrm{Pos}(x^{*\mathrm{T}}\tilde{B}y \geqq w^*), \quad \forall y \in S_J. \tag{13}$$

On the other hand, in case that $\lambda, \mu \in (1/2, 1)$, we have

$$2\lambda - 1 = \mathrm{Nes}(x^{*\mathrm{T}}\tilde{A}y^* \geqq v^*) \geqq \mathrm{Nes}(x^{\mathrm{T}}\tilde{A}y^* \geqq v^*), \quad \forall x \in S_I, \tag{14}$$

$$2\mu - 1 = \mathrm{Nes}(x^{*\mathrm{T}}\tilde{B}y^* \geqq w^*) \geqq \mathrm{Nes}(x^{*\mathrm{T}}\tilde{B}y \geqq w^*), \quad \forall y \in S_J. \tag{15}$$

Namely, the strategy x^* maximizes the possibility(or necessity) that fuzzy expected payoff $x^T\tilde{A}y^*$ is greater than or equal to v^*, given player J's strategy y^* and maximum value of the possibility(or necessity) is 2λ (or $2\lambda - 1$). On the other hand, the strategy y^* maximizes the possibility(or necessity) that fuzzy expected payoff $x^{*T}\tilde{B}y$ is greater than or equal to w^*, given player I's strategy y^* and maximum value of the possibility(or necessity) is 2μ (or $2\mu - 1$). These facts induce us to define another types of games.

Let $v \in R$ be any real numbers and we define real-valued functions $P_v{}^A : S_I \times S_J \to [0,1], N_v{}^A : S_I \times S_J \to [0,1], P_v{}^B : S_I \times S_J \to [0,1]$ and $N_v{}^B : S_I \times S_J \to [0,1]$ by $P_v{}^A(x,y) \equiv \mathrm{Pos}^A(x^{\mathrm{T}}\tilde{A}y \geqq v), N_v^A(x,y) \equiv \mathrm{Nes}(x^{\mathrm{T}}\tilde{A}y \geqq v), P_v{}^B(x,y) \equiv \mathrm{Pos}^B(x^{\mathrm{T}}\tilde{B}y \geqq v)$, and $N_v^B(x,y) \equiv \mathrm{Nes}(x^{\mathrm{T}}\tilde{B}y \geqq v)$, respectively. Then we consider the following four kinds of two-person games:

$$\Gamma^{\mathrm{PP}}(v,w) \equiv \langle \{I,J\}, S_I, S_J, P_v{}^A(\cdot,\cdot),\ P_w{}^B(\cdot,\cdot)\rangle,$$
$$\Gamma^{\mathrm{PN}}(v,w) \equiv \langle \{I,J\}, S_I, S_J, P_v{}^A(\cdot,\cdot),\ N_w{}^B(\cdot,\cdot)\rangle,$$
$$\Gamma^{\mathrm{NP}}(v,w) \equiv \langle \{I,J\}, S_I, S_J, N_v{}^A(\cdot,\cdot),\ P_w{}^B(\cdot,\cdot)\rangle,$$
$$\Gamma^{\mathrm{NN}}(v,w) \equiv \langle \{I,J\}, S_I, S_J, N_v{}^A(\cdot,\cdot),\ N_w{}^B(\cdot,\cdot)\rangle,$$

In each Game, player I chooses a strategy that maximizes possibility or necessity which the fuzzy expected payoff $x^T\tilde{A}y^*$ is greater than or equal to v, which is a inspiration level of expected payoff player I claims to get, given player J's strategy. While player J chooses a strategy that maximizes possibility or necessity which the fuzzy expected payoff $x^{*T}\tilde{B}y$ is greater than or equal to w, which is a inspiration level of expected value player J accepts to lose, given player I's strategy.

From the above discussions, we have the following theorem.

Theorem 4.1 *Let a strategy* $(x^*, y^*) \in S_I \times S_J$ *be any non-dominated Nash equilibrium strategy to Game* $\tilde{\Gamma}$. *Then there exist real numbers* $v^*, w^* \in R$ *such that* (x^*, y^*) *is a Nash equilibrium strategy to one of Game* $\Gamma^{\mathrm{PP}}(v^*, w^*), \Gamma^{\mathrm{PN}}(v^*, w^*)$, $\Gamma^{\mathrm{NP}}(v^*, w^*)$, *and* $\Gamma^{\mathrm{NN}}(v^*, w^*)$.

Theorem 4.1 shows that each player I, J faces one of the games $\Gamma^{\mathrm{PP}}(v, w)$, $\Gamma^{\mathrm{PN}}(v, w)$, $\Gamma^{\mathrm{NP}}(v, w)$, and $\Gamma^{\mathrm{NN}}(v, w)$.

Next we shall show that converse relationships holds among them. First we investigate the relationships between $\Gamma^{\mathrm{PP}}(v, w)$ and $\tilde{\Gamma}$.

Theorem 4.2 *Let* $v, w \in R$ *be any real numbers and let a strategy* $(x^*, y^*) \in S_I \times S_J$ *be any Nash equilibrium strategy to Game* $\Gamma^{\mathrm{PP}}(v, w)$. *If* ${P_v}^A(x^*, y^*), {P_w}^B(x^*, y^*) \in (0, 1)$, *then* (x^*, y^*) *is a non-dominated Nash equilibrium strategy to Game* $\tilde{\Gamma}$.

In Theorem 4.2, conditions $P_v(x^*, y^*), P_w(x^*, y^*) \in (0, 1)$ are important. In fact, if parameters v, w are sufficiently small or sufficiently large, all strategies will be Nash equilibrium strategies to Game $\Gamma^{\mathrm{PP}}(v, w)$. In order to exclude such a case, we need these conditions.

Next we consider the relationships between Game $\Gamma^{\mathrm{NN}}(v, w)$ and $\tilde{\Gamma}$.

Theorem 4.3 *Let* $v, w \in R$ *be any real numbers and let a strategy* $(x^*, y^*) \in S_I \times S_J$ *be any Nash equilibrium strategy to Game* $\Gamma^{\mathrm{NN}}(v, w)$. *If* ${N_v}^A(x^*, y^*), {N_w}^B(x^*, y^*) \in (0, 1)$, *then* (x^*, y^*) *is a non-dominated Nash equilibrium strategy to Game* $\tilde{\Gamma}$.

By a similar way, we could show that the following theorem hold.

Theorem 4.4 *Let* $v, w \in R$ *be any real numbers and let a strategy* $(x^*, y^*) \in S_I \times S_J$ *be any Nash equilibrium strategy to Game* $\Gamma^{\mathrm{PN}}(v, w)$. *If* ${P_v}^A(x^*, y^*)$, ${N_w}^B(x^*, y^*) \in (0, 1)$, *then* $(x^*,\ y^*)$ *is a non-dominated Nash equilibrium strategy to Game* $\tilde{\Gamma}$.

Theorem 4.5 *Let* $v, w \in R$ *be any real numbers and let a strategy* $(x^*, y^*) \in S_I \times S_J$ *be any Nash equilibrium strategy to Game* $\Gamma^{\mathrm{NP}}(v, w)$. *If* ${N_v}^A(x^*, y^*)$, ${P_w}^B(x^*, y^*) \in (0, 1)$, *then* $(x^*,\ y^*)$ *is a non-dominated Nash equilibrium strategy to Game* $\tilde{\Gamma}$.

5 Conclusion

In this paper, we considered fuzzy bi-matrix games and defined three kinds of concepts of Nash equilibrium strategies to fuzzy bi-matrix games based on the concepts of fuzzy max order and investigated their properties. Especially, we have shown that the sets of all these Nash equilibrium strategies coincide with sets of Nash equilibrium strategies of a family of parametric bi-matrix games with crisp payoffs. In addition, we have investigated the properties of values of the fuzzy bi-matrix games based on possibility or necessity measures.

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