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ON DIRECT SUM BANACH SPACES AND UNIFORM NON-SQUARENESS

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Recently the strict convexity and the uniform convexity of the ψ-direct sum $X \oplus_\psi Y$ of Banach spaces X and Y were characterized in [15, 12]. We shall characterize the uniform non-squareness of $X \oplus_\psi Y$.

Let N_a denote the family of all absolute nomalized norms on $\mathbb{C}^2$, that is,

$$\|(z,w)\| = \|(|z|,|w|)\| \quad \text{and} \quad \|(1,0)\| = \|(0,1)\| = 1,$$

and let Ψ denote the family of all continuous convex functions ψ on $[0,1]$ with $\psi(0) = \psi(1) = 1$ satisfying $\max\{1-t,t\} \leq \psi(t) \leq 1 \ \ (0 \leq t \leq 1)$. According to [3], the norms in N_a and the convex functions in Ψ correspond in a one-to-one way under the equation $\psi(t) = \|(1-t,t)\|$. Namely, for every element $\|\cdot\| \in N_a$ the function $\psi(t)$ defined by $\psi(t) = \|(1-t,t)\|$ belongs to Ψ; and conversely for every element $\psi \in \Psi$, define

$$(1)\|(z,w)\|_\psi = \begin{cases} (|z|+|w|)\psi\left(\frac{|w|}{|z|+|w|}\right) & \text{if } (z,w) \neq (0,0), \\ 0 & \text{if } (z,w) = (0,0). \end{cases}$$

Then $\|(\cdot,\cdot)\|_\psi$ is a norm in N_a and satisfies $\psi(t) = \|(1-t,t)\|_\psi$.

In [15], the ψ-direct sum $X \oplus_\psi Y$ of two Banach spaces X and Y was introduced as the direct sum $X \oplus Y$ with the norm $\|(x,y)\|_\psi = \|(\|x\|,\|y\|)\|_\psi (x \in X,\ y \in Y)$. Recently the strict convexity and the uniform convexity of $X \oplus_\psi Y$ were characterized in [15, 12]. In this note we characterize the uniform non-squareness of $X \oplus_\psi Y$. As an application we give an example of Banach spaces which are not uniformly convex but uniformly non-square.

Now recall that a Banach space X is called *uniformly non-square* ([6]; cf. [2, 10]) provided there exists a $\delta\,(0<\delta<1)$ such that, whenever $\|(x-y)/2\| > 1-\delta$, $\|x\| = \|y\| = 1$, one has $\|(x+y)/2\| \leq 1-\delta$. X is called *strictly convex* provided, if $\|x\| = \|y\| = 1$, $x \neq y$, then $\|\frac{x+y}{2}\| < 1$. X is called *uniformly convex* if any $\epsilon > 0$ there is a δ $(0 < \delta < 1)$ such that, whenever $\|x-y\| \geq \epsilon$, $\|x\| \leq 1$, $\|y\| \leq 1$, one has $\|\frac{x+y}{2}\| < 1-\delta$. As is well known, the notion of uniform non-squareness lies between uniform convexity and super-reflexivity. Also, it is well known that there exists a Banach space which is neither uniformly convex nor uniformly non-square but surper-refrexive. (cf. [7], [1].) A function ψ on $[0,1]$ is called *strictly convex* if, for any s, $t \in [0,1]$, $s \neq t$, and for any c $(0 < c < 1)$, one has $\psi((1-c)s+ct) < (1-c)\psi(s)+c\psi(t)$.

THEOREM A ([15, 12]). *Let X and Y be Banach spaces and let $\psi \in \Psi$. Then*

(i) *$X \oplus_\psi Y$ is strictly convex if and only if X and Y are strictly convex, and ψ is strictly convex* ([15, Theorem 1]).

(ii) *$X \oplus_\psi Y$ is uniformly convex if and only if X and Y are uniformly convex, and ψ is strictly convex* ([12, Theorem 1]).

Saito-Kato-Takahashi [13] gave the following characterization of the absolute norms on $\mathbb{C}^2$ which are uniformly non-square.

Proposition 1 ([13]). *Let* $\psi \in \Psi$. *Then the following are equivalent.*

(i) $(\mathbb{C}^2, \|\cdot\|_\psi)$ *is uniformly non-square.*

(ii) $\psi \neq \psi_1$ *and* $\psi \neq \psi_\infty$.

1. Monotonicity Property of Absolute Norms

We discuss the monotonicity property of absolute norms on $\mathbb{C}^2$ for later use. Recall the following fundamental facts. Propostion 2 played an essential role in the proof of Theorem A.

Lemma 1 ([2, p.36, Lemma 2]). *Let* $\|\cdot\| \in N_a$.

(i) *If* $|p| \leq |r|$ *and* $|q| \leq |s|$, *then* $\|(p,q)\| \leq \|(r,s)\|$.

(ii) *If* $|p| < |r|$ *and* $|q| < |s|$, *then* $\|(p,q)\| < \|(r,s)\|$.

Proposition 2 (Takahashi, Kato and Saito [15]). *Let* $\psi \in \Psi$. *Then the following assertions are equivalent:*

(i) *If* $|z| \leq |u|$ *and* $|w| < |v|$, *or* $|z| < |u|$ *and* $|w| \leq |v|$, *then* $\|(z,w)\|_\psi < \|(u,v)\|_\psi$.

(ii) $\psi(t) > \psi_\infty(t)$ *for all* $t \in (0,1)$.

A more precise (component-wise) result is given in [15]. Next we present a condition on (z,w) and (u,v) for which the above assertion (i) is valid (component-wise) for a general $\psi \in \Psi$.

Proposition 3. *Let* $\psi \in \Psi$ *and let* (z,w), $(u,v) \in \mathbb{C}^2$.

(i) *Let* $|z| < |u|$ *and* $|w| = |v|$. *Then* $\|(z,w)\|_\psi = \|(u,v)\|_\psi$ *if and only if* $\|(z,w)\|_\psi = |w|$.

(ii) *Let* $|z| = |u|$ *and* $|w| < |v|$. *Then* $\|(z,w)\|_\psi = \|(z,v)\|_\psi$ *if and only if* $\|(z,w)\|_\psi = |z|$.

Propostion 3 is important in the proof of the uniform non-squareness of $X \oplus_\psi Y$.

3. Uniform Non-squareness of $X \oplus_\psi Y$

We need the following lemma.

Lemma 2. *Let $\{x_n\}$ and $\{y_n\}$ be sequences in a Banach space X whose norms are convergent to non-zero limits.*
(i) $\lim_{n\to\infty} \|x_n + y_n\| = \lim_{n\to\infty}(\|x_n\| + \|y_n\|)$.
(ii) $\lim\limits_{n\to\infty} \left\| \frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|} \right\| = 2.$

By Proposition 3 and Lemma 2, we obtain the following main theorem.

Theorem 1. *Let X and Y be Banach spaces and $\psi \in \Psi$. Then the following are equivalent.*
(i) *$X \oplus_\psi Y$ is uniformly non-square.*
(ii) *X and Y are unformly non-square and $\psi \neq \psi_1, \psi_\infty$.*

Now consider the Lorentz $\ell_{p,q}$-norm $\|\cdot\|_{p,q}$, $1 \leq q \leq p \leq \infty$, $q < \infty$:

$$\|(z_1, z_2)\|_{p,q} = \left\{ z_1^{*q} + 2^{(q/p)-1} z_2^{*q} \right\}^{1/q},$$

where $\{z_1^*, z_2^*\}$ is the non-increasing rearrangement of $\{|z_1|, |z_2|\}$. (Note that in case of $1 \leq p < q \leq \infty$, $\|\cdot\|_{p,q}$ is not a norm but a quasi-norm (cf. [8], [16, p.126]). Clearly $\|\cdot\|_{p,q}$ is an absolute normalized norm and the corresponding convex function $\psi_{p,q}$ is given by

$$(2)\quad psi_{p,q}(t) = \begin{cases} \{(1-t)^q + 2^{q/p-1}t^q\}^{1/q} & \text{if } 0 \leq t \leq 1/2, \\ \{t^q + 2^{q/p-1}(1-t)^q\}^{1/q} & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Then $\psi_{p,q}$ yields the $\ell_{p,q}$-sum $X \oplus_{p,q} Y$:

$$\|(x,y)\|_{p,q} = \left\{\max(\|x\|^q, \|y\|^q) + 2^{(q/p)-1} \min(\|x\|^q, \|y\|^q)\right\}^{1/q} \tag{3}$$

COROLLARY 1. *Let* $1 \leq q \leq p \leq \infty$ *and not* $p = q = 1, \infty$. *Then,* $\ell_{p,q}$*-sum* $X_1 \oplus_{p,q} X_2$ *is uniformly non-sqaure if and only if* X_1 *and* X_2 *are uniformly non-sqaure.*

In particular, ℓ_p*-sum* $X_1 \oplus_p X_2$, $1 < p < \infty$, *is uniformly non-sqaure if and only if* X_1 *and* X_2 *are uniformly non-sqaure.*

Theorem A and Theorem 1 easily gives an example of Banach spaces which are not uniformly convex but uniformly non-square.

EXAMPLE 1 (cf. [12, 13]). Let X and Y be uniformly convex Banch space and let $1/2 < \alpha < 1$. Now we define $\psi_\alpha \in \Psi$ by

$$\psi_\alpha(t) = \begin{cases} \frac{\alpha-1}{\alpha}t + 1 & \text{if } 0 \leq t \leq \alpha, \\ t & \text{if } \alpha \leq t \leq 1. \end{cases} \tag{4}$$

Then the norm of $X \oplus_{\psi_\alpha} Y$ is given by

$$\|(x,y)\|_{\psi_\alpha} = \max\{\|x\| + (2 - \frac{1}{\alpha})\|y\|, \|y\|\}. \tag{5}$$

$X \oplus_{\psi_\alpha} Y$ is an example of uniformly non-square Banach spaces without uniform convexity.

References

[1] J.-B. Baillon and R. Schoneberg, Asymptotic normal structure and fixed points of nonexpansive mappings, Proc. Amer. Math. Soc., **81** (1981), 257–264.

[2] B. Beauzamy, Introduction to Banach Spaces and their Geometry, 2nd ed., North-Holland, 1985.

[3] F. F. Bonsall and J. Duncan, Numerical Ranges II, London Math. Soc. Lecture Note Ser. **10** (1973).

[4] R. Bhatia, Matrix Analysis, Springer, 1997.

[5] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, Cambridge Univ. Press, 1967.

[6] C. James, Uniformly non-square Banach spaces, Ann. of Math. **80** (1964), 542-550.

[7] L. A. Karlovitz, Existence of fixed points of nonexpansive mappings in a space without normal structure, Pacific J. Math., **66** (1976), 153–159.

[8] M. Kato, On Lorentz spaces $\ell_{p,q}\{E\}$, Hiroshima Math. J. **6** (1976), 73-93.

[9] M. Kato, K.-S. Saito and T. Tamura, On the ψ-direct sums of Banach spaces and convexity, submitted.

[10] R. E. Megginson, An Introduction to Banach Space Theory, Springer, 1998.

[11] K. Mitani, K.-S. Saito and T. Suzuki, Smoothness of absolute norms on $\mathbb{C}^n$, to appear in J. Convex Analysis.

[12] K.-S. Saito and M. Kato, Uniform convexity of ψ-direct sums of Banach spaces, to appear in J. Math. Anal. Appl.

[13] K.-S. Saito, M. Kato and Y. Takahashi, Von Neumann-Jordan constant of absolute normalized norms on $\mathbb{C}^2$, J. Math. Anal. Appl. **244** (2000), 515-532.

[14] K.-S. Saito, M. Kato and Y. Takahashi, On absolute norms on $\mathbb{C}^n$, J. Math. Anal. Appl. **252** (2000), 879-905.

[15] Y. Takahashi, M. Kato and K.-S. Saito, Strict convexity of absolute norms on $\mathbb{C}^2$ and direct sums of Banach spaces, J. Inequal. Appl. **7** (2002), 179-186.

[16] H. Triebel, Intepolation Theory, Function spaces, Differential Operators, North-Holland, 1978.