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Relationship among Minimax, Maximin and Cone Saddle Values of Vector-Valued Functions

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Abstract: In this paper, we consider minimax problems for a vector-valued function, which are the following questions: If we give reasonable definitions for minimax and maximin values of a vector-valued function in an ordered vector space, what minimax equation or inequality holds? Also, if we give a suitable definition for saddle points of a vector-valued function, what relationship holds among such minimax, maximin and saddle values? We will give interesting answers to such questions and introduce a recent problem in minimax problems in this paper.

Keywords: Minimax problem, minimax and maximin values, saddle point, vector-valued function.

1 Introduction

Saddle point theorem for a real-valued function is used well in game theory and other wide fields. It says: a real-valued function possesses a saddle point if and only if minimax and maximin values of the function are coincident. This fact is valid based on the total ordering of $\boldsymbol{R}$, but if we consider more general partial orderings on vector spaces, then what kind of results on minimax and maximin values of a vector-valued function are obtained? This kind of researches have been studied from game theoretical aspect and general aspect of saddle point concept; see [1, 5, 6].

Minimax, maximin and saddle values for a vector-valued function are sets under suitable definitions in general. Then, a kind of saddle point theorem for a vector-valued function holds under some conditions. It says: there exists some minimax and maximin values of a vector-valued function such that their values are ordered by a partial ordering and dominated each other whenever the vector-valued function has a saddle point. In this paper, we will give this theorem in more detail.

Accordingly, the organization of the paper is as follows. In Section 2, we give the preliminary terminology used throughout the paper, and then define vector-valued minimax and maximin values and saddle point. In Section 3, we introduce a saddle point theorem for a vector-valued function. In Section 4, we investigate difference between two concepts of minimax and maximin values for a vector-valued function. In Section 5, we shall introduce a recent result in a minimax problem.

2 Preliminary terminology and definitions

We give some settings for mathematics on vector optimization. Throughout this paper, let Z be an ordered vector space with the following partial ordering, for all $x, y \in Z$,

$$x \leq_C y \Longleftrightarrow y - x \in C, \quad x <_C y \Longleftrightarrow y - x \in C\backslash\{\theta\},$$
$$x \not\leq_C y \Longleftrightarrow y - x \notin C, \quad x \not<_C y \Longleftrightarrow y - x \notin C\backslash\{\theta\},$$

where C is a solid (int$C \neq \emptyset$) pointed ($C \cap (-C) = \{\theta\}$) convex cone. By int$C \neq \emptyset$, $C^0 :=$ (int$C$) $\cup \{\theta\}$ is a pointed convex cone and induces another vector ordering $\leq_{C^0}$ weaker than $\leq_C$ in Z. For these orderings, we define minimal and maximal elements of a subset A of Z, i.e., lower efficient points and upper efficient points with respect to C and C^0, respectively.

Definition 1 *$z_0 \in A \subset Z$ is said to be a C-minimal point of A if $z \not<_C z_0$ for all $z \in A$, and a C-maximal point of A if $z_0 \not<_C z$ for all $z \in A$, respectively. We denote the set of such all C-minimal (resp. C-maximal) points of A by* Min$A$ *(resp.* Max$A$*). Also, C^0-minimal and C^0-maximal points of A are defined similarly, and denoted by* Min$_w A$ *and* Max$_w A$*, respectively.*

Under these definitions, we can define (weak) C-saddle point of a vector-valued function as follows, which is an extended notion of usual saddle points.

Definition 2 *Let $f : X \times Y \to Z$ be a vector-valued function, where X and Y are sets. A point (x_0, y_0) is said to be a C-saddle point of f with respect to $X \times Y$ if $f(x_0, y_0) \in \mathrm{Max} f(x_0, Y) \cap \mathrm{Min} f(X, y_0)$, and a point (x_0, y_0) is said to be a weak C-saddle point of f with respect to $X \times Y$ if $f(x_0, y_0) \in \mathrm{Max}_w f(x_0, Y) \cap \mathrm{Min}_w f(X, y_0)$, respectively.*

We denote the set of all C-saddle and weak C-saddle values of f as follows,

$$SV(f) := \{f(x_0, y_0) | (x_0, y_0) \in X \times Y \text{ is a } C\text{-saddle point of } f\} \quad \text{and}$$
$$SV_w(f) := \{f(x_0, y_0) | (x_0, y_0) \in X \times Y \text{ is a weak } C\text{-saddle point of } f\},$$

respectively.

Moreover, by using concepts of efficient points, we can define the following subsets of Z as analogues of minimax and maximin values for real-valued functions.

Definition 3 *Let $f : X \times Y \to Z$ be a vector-valued function, where X and Y are sets. Subsets of Z*

$$\mathrm{Minimax} f := \mathrm{Min} \bigcup_{x \in X} \mathrm{Max} f(x, Y) \quad \textit{and} \quad \mathrm{Maximin} f := \mathrm{Max} \bigcup_{y \in Y} \mathrm{Min} f(X, y)$$

are called the set of all minimax values for f and the set of all maximin values for f, respectively.

Also, we can consider sets of minimax and maximin values for a weak concept in the same way as efficient and C-saddle points, i.e., subsets of Z

$$\text{Minimax}_w f := \text{Min} \bigcup_{x \in X} \text{Max}_w f(x, Y) \quad \text{and} \quad \text{Maximin}_w f := \text{Max} \bigcup_{y \in Y} \text{Min}_w f(X, y)$$

are weaker concepts than those in Definition 3.

Example 1 Let Z and C be a 2-dimensional Euclidean space and its nonnegative orthant of Z, respectively. Also, let X and Y be sets of convex hulls generated by $(1,0)^t$ and $(0,1)^t$ in another 2-dimensional Euclidean space. We consider a matrix type vector-valued function $f(x,y) = (x^t Ay,\ x^t By)^t$ where A and B are 2×2 matrices.

We consider the image of f for specific matrices. The image of f for

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

forms itself into a figure with an envelope; see Figure 1.

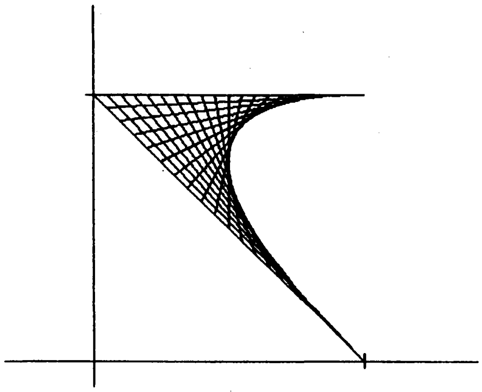

Figure 1: An image set with an envelope (Example 1).

Let

$$\begin{aligned} D_1^{(w)} &= \left\{ (x_0, y_0) \in X \times Y \,\middle|\, f(x_0, y_0) \in \text{Max}_{(w)} f(x_0, Y) \right\} \quad \text{and} \\ D_2^{(w)} &= \left\{ (x_0, y_0) \in X \times Y \,\middle|\, f(x_0, y_0) \in \text{Min}_{(w)} f(X, y_0) \right\}. \end{aligned}$$

$D^{(w)} := D_1^{(w)} \cap D_2^{(w)}$ is the set of all (weak) C-saddle points of f, and $f(D^{(w)}) = SV_{(w)}(f)$. In this example, we get

$$\begin{aligned} D = &\left\{ (x,y) \,\middle|\, x_1 = 0,\ 0 \le y_1 \le \tfrac{1}{2} \right\} \cup \left\{ (x,y) \,\middle|\, 0 \le x_1 < \tfrac{1}{2},\ \tfrac{1}{2} < y_1 \le 1 \right\}, \\ D^w = &\left\{ (x,y) \,\middle|\, x_1 = 0,\ 0 \le y_1 \le \tfrac{1}{2} \right\} \cup \left\{ (x,y) \,\middle|\, 0 \le x_1 \le \tfrac{1}{2},\ \tfrac{1}{2} \le y_1 \le 1 \right\} \cup \\ &\{ (x,y) \mid 0 \le x_1 \le 1,\ y_1 = 0 \} \cup \left\{ (x,y) \,\middle|\, x_1 = 1,\ \tfrac{1}{2} \le y_1 \le 1 \right\} \end{aligned}$$

where $x = (x_1, 1 - x_1)^t$, $y = (y_1, 1 - y_1)^t$. Hence,

$$\begin{aligned} SV(f) &= \{ f(x,y) \mid (x,y) \in D \} \\ &= \left\{ (u,v)^t \,\middle|\, u = y_1,\ v = -y_1 + 1,\ 0 \le y_1 \le \tfrac{1}{2} \right\} \\ &\quad \cup \left\{ (u,v)^t \,\middle|\, \begin{array}{l} u = x_1 + y_1 - 2x_1 y_1,\ v = -y_1 + x_1 y_1 + 1, \\ 0 \le x_1 < \tfrac{1}{2},\ \tfrac{1}{2} < y_1 \le 1 \end{array} \right\}, \\ SV_w(f) &= \{ f(x,y) \mid (x,y) \in D^w \} \\ &= \left\{ (u,v)^t \mid u = x_1,\ v = 1,\ 0 \le x_1 \le 1 \right\} \end{aligned}$$

$$\cup \left\{ (u,v)^t \,\middle|\, u = y_1,\ v = -y_1 + 1,\ 0 \le y_1 \le \tfrac{1}{2} \right\}$$
$$\cup \left\{ (u,v)^t \,\middle|\, \begin{array}{l} u = x_1 + y_1 - 2x_1y_1,\ v = -y_1 + x_1y_1 + 1, \\ 0 \le x_1 \le \frac{1}{2},\ \frac{1}{2} \le y_1 \le 1 \end{array} \right\}.$$

Sets of minimax and maximin values for f in this example are as follows;

$$\begin{aligned} &\mathrm{Minimax} f = \mathrm{Minimax}_w f = \{(u,v)^t \mid u = y_1,\ v = 1 - y_1,\ 0 \le y_1 \le 1\}, \\ &\mathrm{Maximin} f = \left\{ (u,v)^t \,\middle|\, u^2 + 4v^2 - 6u - 8v + 4uv + 5 = 0,\ \tfrac{1}{2} < u \le 1,\ 0 \le v < \tfrac{3}{4} \right\}, \\ &\mathrm{Maximin}_w f = \{(1,1)^t\}. \end{aligned}$$

3 Vector-valued saddle point theorem

A real-valued function possesses a saddle point if and only if minimax and maximin values of the function are coincident and its value is coincident with the saddle value, but its analogy for a vector-valued function can not be expected in general. However, it is well-known that a certain minimax inequality holds under some conditions. If a vector-valued function has weak C-saddle points defined in Section 2, the following saddle point theorem for a vector-valued function is obtained by existence for vector-valued minimax and maximin values.

Theorem 1 *Let X and Y be nonempty compact sets in two topological spaces, respectively. Assume that a vector-valued function $f : X \times Y \to Z$ is continuous and the pointed convex cone C satisfies the condition $\mathrm{cl}C + (C \backslash \{\theta\}) \subset C$. If f has a weak C-saddle point $(x_0, y_0) \in X \times Y$, then there exist*

$$z_1 \in \mathrm{Min} \bigcup_{x \in X} \mathrm{Max}_w f(x, Y), \qquad z_2 \in \mathrm{Max} \bigcup_{y \in Y} \mathrm{Min}_w f(X, y)$$

such that $z_1 \le_C f(x_0, y_0)$ and $f(x_0, y_0) \le_C z_2$.

Refer to [6] about existence of minimax and maximin values and saddle points for a vector-valued function and a proof of the theorem. This theorem can be interpreted in the following way: Minimax and maximin values are lower efficient points and upper efficient points of saddle values, respectively, in the sense of $\le_C$. Moreover, we can get the following vector-valued inequality on the partial ordering from Theorem 1,

$$z_1 \le_C z_2.$$

This inequality is called "minimax inequality". This result means that there exists a maximin value which is greater than a minimax value in the sence of $\le_C$. It seems that this result is similar to the case of a real-valued function.

4 Difference in vector-valued minimax and maximin values for two concepts

In this section, we investigate difference between normal and weak type vector-valued minimax and maximin values defined in Section 2. As to weak type, the corresponding result in Section 3 always holds under some conditions. As to normal type, what kind of thing is said ? An answer for its question is that the vector-valued saddle point theorem does not always hold under the same conditions because sets of minimax and maximin values do not always exist in the normal type. We show the following example.

Example 2 We consider the same settings as in Example 1 in Section 2.

We consider the image of f for the following matrices

$$A = \begin{pmatrix} 3 & -5 \\ -3 & -2 \end{pmatrix}, \ B = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}.$$

The image of f forms itself into a figure with an envelope, which is similar to but more complex than Example 1; see Figure 2.

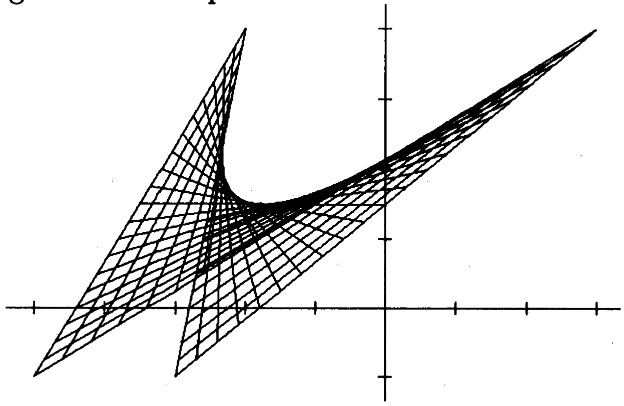

Figure 2: An example of Minimax$f$ $= \emptyset$ (Example 2).

Sets of minimax and maximin values for f in this example are as follows;

$$\begin{aligned}
\mathrm{Minimax}f &= \emptyset, \\
\mathrm{Maximin}f &= \left\{ (u,v)^t \,\middle|\, \begin{array}{l} 100u^2 + 81v^2 + 620u - 768v - 180uv + 1276 = 0, \\ -\frac{7}{3} < u < -\frac{7}{4},\ \frac{3}{2} < v < \frac{58}{27} \end{array} \right\}, \\
\mathrm{Minimax}_w f &= \left\{ (-\tfrac{7}{3}, -\tfrac{4}{9})^t,\ (-\tfrac{7}{2}, \tfrac{3}{2})^t \right\}, \\
\mathrm{Maximin}_w f &= \left\{ (u,v)^t \,\middle|\, \begin{array}{l} 100u^2 + 81v^2 + 620u - 768v - 180uv + 1276 = 0, \\ -\frac{7}{3} < u < -\frac{7}{4},\ \frac{3}{2} < v < \frac{58}{27} \end{array} \right\} \\
&\quad \cup \left\{ (-1, \tfrac{3}{2})^t,\ (-\tfrac{7}{3}, \tfrac{7}{3})^t \right\}.
\end{aligned}$$

5 Recent result

The saddle point theorem for a vector-valued function only guarantees that there exist some minimax and maximin values of the function such that their values are ordered by $\leq_C$ and dominated each other whenever the function has a weak C-saddle point. An interesting recent question in minimax problems is under what kind of conditions minimax and maxmin values are coincident. As to this question, the following theorem holds.

Theorem 2 *Let $f : X \times Y \to Z$ be a vector-valued function and X and Y be convex hulls generated by $(1,0)^t$ and $(0,1)^t$ in 2-dimensional Euclidean space. Assume that f is a bilinear function with respect to $x \in X$ and $y \in Y$, $SV(f) \neq \emptyset$ and* Minimax$f$, Maximin$f$ $\subset SV(f)$*. If either*

$$\forall x \in X,\ d_x \in C \cup (-C) \quad \text{or} \quad \forall y \in Y,\ d_y \in C \cup (-C),$$

$$\text{Minimax}f = \text{Maximin}f$$

where $d_x = f(x,(1,0)^t) - f(x,(0,1)^t)$ *and* $d_y = f((1,0)^t,y) - f((0,1)^t,y)$*, which are called direction vectors.*

We introduce one of the fundamental properties used in the proof of Theorem 2. This property is called "dominance property", which is important in problems on efficient points.

Lemma 1 *(See Lemma 5.2 in [5]) Let* Z *be an ordered vector space with an ordering defined by a solid pointed convex cone* C*, and* A *a subset of* Z*. If the convex cone* C *of* Z *satisfies the condition*

$$\text{cl}C + (C \backslash \{\theta\}) \subset C$$

and if A *is nonempty and compact, then* $\text{Min}A \neq \emptyset$*,* $A \subset \text{Min}A + C$ *and* $\text{Max}A \neq \emptyset$*,* $A \subset \text{Max}A - C$*.*

As to dominance property, more complex one has been proposed, but it is sufficient with this lemma in our setting because Z is the finite-dimensional vector space. We show the proof of Theorem 2 in the following.

Proof of Theorem 2. We assume that $d_x \in C \cup (-C)$ for any $x \in X$. For any $z \in \text{Minimax}f$, there exist $x_0 \in X$ and $y_0 \in Y$ such that $z = f(x_0, y_0)$ and

$$z' \not<_C z \text{ and } z \not<_C f(x_0, y), \quad \forall z' \in \text{Max}f(x, Y),\ x \in X,\ y \in Y.$$

Therefore, we have $z \in \text{Max}f(x_0, Y)$. Since we assume that the set of minimax values is a subset of $SV(f)$,

$$z = f(x_0, y_0) \in \text{Max}f(x_0, Y) \cap \text{Min}f(X, y_0),$$

i.e., $f(x, y_0) \not<_C z$, $\forall x \in X$. Since $d_{x_0} \in C \cup (-C)$, we obtain $\text{Max}f(x_0, Y) = \{z\}$. Moreover, C satisfies the condition in Lemma 1 because C is a closed set, and $f(x, Y)$ is a bounded closed set for each $x \in X$, and then it is a compact set. Hence, $f(x_0, Y) \subset z - C$ by Lemma 1. Here, for given $y \in Y$, let $z_{Min(y)}$ be an element of $\text{Min}f(X, y)$. We suppose that $z_{Min(y)} \in z + C \backslash \{\theta\}$, then

$$f(x_0, y) \leq_C z = f(x_0, y_0) \text{ and } z = f(x_0, y_0) <_C z_{Min(y)}.$$

Hence, we obtain $f(x_0, y) <_C z_{Min(y)}$. This is contradictory to $z_{Min(y)} \in \text{Min}f(X, y)$. Therefore, we have $z_{Min(y)} \notin z + C \backslash \{\theta\}$. Since z is also a saddle value,

$$f(x, y_0) \not<_C z \text{ and } z \not<_C z_{Min(y)},\ \forall z_{Min(y)} \in \text{Min}f(X, y),\ x \in X,\ y \in Y.$$

So, we obtain $z \in \text{Maximin}f$ and hence $\text{Minimax}f \subset \text{Maximin}f$.

On the other hand, for any $z \in \text{Maximin}f$, there exist x_0 and y_0 such that $z = f(x_0, y_0)$ and

$$f(x, y_0) \not<_C z \text{ and } z \not<_C z', \quad \forall z' \in \text{Min}f(X, y),\ x \in X,\ y \in Y.$$

Therefore, we have $z \in \text{Min}f(X, y_0)$. Since we assume that the set of maximin values is a subset of $SV(f)$, we have

$$z = f(x_0, y_0) \in \text{Max}f(x_0, Y) \cap \text{Min}f(X, y_0),$$

i.e., $z \not<_C f(x_0, y)$, $\forall y \in Y$. Here, for given $x \in X$, let $z_{Max(x)}$ be an element of $\text{Max}f(x, Y)$. Then, from $d_x \in C \cup (-C)$, we obtain $\text{Max}f(x, Y) = \{z_{Max(x)}\}$. Moreover, $f(x, Y)$ is a compact set for each $x \in X$ so $f(x, Y) \subset z_{Max(x)} - C$ by Lemma 1. We suppose that $z_{Max(x)} \in z - C\backslash\{\theta\}$, then

$$f(x, y_0) \leq_C z_{Max(x)} \text{ and } z_{Max(x)} <_C z = f(x_0, y_0).$$

Hence, we obtain $f(x, y_0) <_C z = f(x_0, y_0)$. This is contradictory to $z \in \text{Min}f(X, y_0)$. Therefore, we have $z_{Max(x)} \notin z - C\backslash\{\theta\}$. Since z is also a saddle value,

$$z_{Max(x)} \not<_C z \text{ and } z \not<_C f(x_0, y),\ \forall z_{Max(x)} \in \text{Max}f(x, Y),\ x \in X,\ y \in Y.$$

So, we obtain $z \in \text{Minimax}f$ and hence $\text{Minimax}f \supset \text{Maximin}f$.

Consequently, we obtain

$$\text{Minimax}f = \text{Maximin}f.$$

When we also assume that $d_y \in C \cup (-C)$ for any $y \in Y$, we can prove similarly. This completes the proof. □

Note that Theorem 2 does not hold for weak type minimax and maximin values.

Example 3 We consider the same settings as in Example 1 in Section 2.

The image of f for the following matrices

$$A = \begin{pmatrix} 1 & -2 \\ -5 & -2 \end{pmatrix},\ B = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}$$

is shown in Figure 3, whose figure is called 'Fix-Point type' because it has an intersection point of line segments in the image set; see [7].

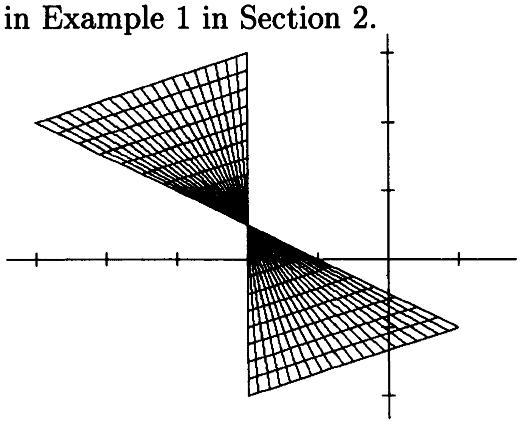

Figure 3: An example of $\text{Minimax}f = \text{Maximin}f$ but $\text{Minimax}_w f \neq \text{Maximin}_w f$ (Example 3).

In this example, $f(x,Y)$ and $f(X,y)$ for $x \in X$ and $y \in Y$ are line-segments which forme the image of f, respectively, because f is a bilinear function with respect to x and y, and vectors d_x and d_y are direction vectors for $f(x,Y)$ and $f(X,y)$, respectively. Moreover, d_x for all $x \in X$ is contained in $C \cup (-C)$. Therefore, from Theorem 2, sets of minimax and maximin values are coincident in this example. In the concrete, we obtain

$$\begin{aligned}
\text{Minimax} f &= \text{Maximin} f \\
&= \left\{ (u,v)^t \;\middle|\; u = 6x_1 - 5,\ v = -3x_1 + 2,\ \tfrac{1}{2} \le x_1 \le 1 \right\}, \\
\text{Minimax}_w f &= \text{Minimax} f, \\
\text{Maximin}_w f &= \left\{ (u,v)^t \;\middle|\; u = 6x_1 - 5,\ v = -3x_1 + 2,\ \tfrac{1}{2} < x_1 \le 1 \right\} \cup \{(-2,3)^t\}.
\end{aligned}$$

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