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Summary of GNAVOA, I.

Studies in groups, nonassociative algebras and vertex operator algebras.

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The material in this talk will appear in an article titled GNAVOA, I, due to appear in the proceedings of the Infinite Dimensional Lie Theory Meeting, Fields Institute, 23-27 October, 2000.

Robert L. Griess Jr.[1]
Department of Mathematics, University of Michigan
Ann Arbor, MI 48109

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Abstract

In this talk, we mention a few highlights of the article [GNAVOA, I], which is one in a series which take an exploratory look at some VOAs of CFT type, such as the ones of lattice type, their automorphism groups and the automorphism groups of their degree 2 part.

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1 Summary

Since full details will soon appear in [GNAVOA, I], we indicate only a few highlights.

At this time,we are especially interested in questions about VOAs and their automorphism groups mostly along the following lines:

Q1. What groups occur as $Aut(V)$, for a VOA V?

Q2. Are there reasonable methods for determining $Aut(V)$ in cases of interest?

It seems a good idea to explore interconnections among groups, nonassociative algebras and VOA theory, hence the acronym GNAVOA. Here, we are thinking mainly of finite dimensional commutative nonassociative algebras which occur as some $(V_2, 1^{st})$. It has been known for a long time that the algebra $(V_1, 0^{th})$ is a Lie algebra if $V_n = 0$ for $n < 0$ and $dim(V_0) = 1$. We shall say little about this well-studied role of Lie algebras, and concentrate on degree 2 and higher.

In the seventies decade, the theories of finite simple groups and commutative nonassociative algebras became more closely interconnected. In the mid eighties, VOA theory became established, and developed with ideas from physics, geometry and Lie theory and the algebraic theories involving the monster simple group.

Examples of finite groups acting as automorphisms of finite dimensional algebras were presented, to indicate how certain finite simple groups and actions on nonassociative commutative algebras were discovered.

We take a closer look at how commutative nonassociative algebras come up in the VOA world. Mainly, we are thinking of the cases where $(V_2, 1^{st})$ is commutative. These include classic examples, for instance some Jordan matrix algebras, but also nonfamiliar ones. The algebra $\mathcal{B}_0$ of dimension 196883 associated to construction of the monster has no nontrivial low degree identities [GrMont], so one can not hope for a structure theory like those of Lie and Jordan algebras. Probably classic work with identities is not effective in general for the algebras $(V_2, 1^{st})$. Some questions about the algebras may be answered by dealing with the automorphism group. The advantage of this viewpoint is that both the theories of Lie groups and finite simple groups are well-developed.

We know $Aut(V)$ for only a limited family of V. The ones we are aware of are the lattice VOAs [DN], lattice type VOAs of rank 1 and a few special cases, such as the monster and $O^+(10, 2)$. See the survey in [GrRaleigh].

Since that survey, the following basic result has been obtained [DG2].

Theorem 1.1. *The automorphism group of a finitely generated VOA is an algebraic group.*

The Fischer theory of 3-transposition groups was reviewed. This is a basic theme in finite simple group theory.

An important connection between 3-transposition groups and VOAs was noticed by Miyamoto, whose idea is that to each element ω_i of a Virasoro frame is associated an automorphism $t(\omega_i)$ of order 2 (or order 1, in exceptional situations), based on fusion rules involving $L(\frac{1}{2},0), L(\frac{1}{2},\frac{1}{2})$ and $L(\frac{1}{2},\frac{1}{16})$, the irreducibles for the Virasoro subVOA generated by the ω_i. In case $L(\frac{1}{2},\frac{1}{16})$ does not occur in V, $t(\omega_i)$ belongs to a conjugacy class of 3-transpositions in $Aut(V)$. See [Miy], [DGH], [GrRaleigh].

We think of the 3-transposition concept as a link between the worlds of finite simple groups and basic VOA theory, something worth studying.

Definition 1.2. A VOA V has *CFT type* if V_n is 0 for $n < 0$ and $V_0 = \mathbb{C}\mathbf{1}$ is 1-dimensional.

Definition 1.3. The *OZ property* of a VOA $V = \bigoplus_{n\in\mathbb{Z}} V_n$ means the following set of conditions: $dim(V_n) = 0$ for $n < 0$; $dim(V_0) = 1$; and $dim(V_1) = 0$. (Note that OZ stands for the sequence of dimensions: one, zero). A VOA with the OZ property is called an *OZVOA*, or an *ozzie*, for short.

The OZ property implies the CFT property, but not conversely.

If V has the OZ property, $V_0 = \mathbb{C}\mathbf{1}$ and $(V_2, 1^{st})$ is a commutative nonassociative algebra with an associative, symmetric bilinear form $(x, y) = x_3 y$, $x, y \in V_2$ [FLM].

Definition 1.4. A commutative algebra $(A, *)$ for which there is an OZVOA V such that $(A, *) \cong (V_2, 1^{st})$ is called a *Griess algebra.* We say that such an OZVOA *affords* the algebra $(A, *)$.

The term Griess algebra arose in the VOA literature, due to the role of the 196884-dimensional algebra $\mathcal{B}$ in the construction of the monster and in the theory of $V^\natural$, the moonshine VOA, which has the OZ property. Given a Griess algebra, there seems to be no obvious relation between two VOAs which afford it.

We can create many OZVOAs in the following way.

Definition 1.5. Take a VOA V of CFT type. Let F be a subgroup of $Aut(V)$ which is fixed point free on the degree 1 part. Then the fixed point subVOA V^F is an OZVOA. Call this procedure (of making ozzies from CFTs) *ozzification.*

A given VOA of CFT type may have many ozzifications, depending on choice of F. One can see several rank 1 examples of LTVOA ozzifications in [DG, DGR]. When the lattice is a root lattice, we can use well-developed knowledge of the finite subgroups of Lie groups [GRS][GRQE]. In $E_8(\mathbb{C})$, there are many fixed point free finite subgroups, for example ones isomorphic to $PSL(2,q)$, for at least $q = 5, 9, 16, 31, 32, 41, 49, 61$. A nontoral elementary abelian 2-group of rank 5 in $E_8(\mathbb{C})$ gave the example in [?]. In $E_7(\mathbb{C})$, there is $PSU(3,8)$ and in $E_6(\mathbb{C})$ there is $PSL(2,19)$, for instance. In general, a Lie primitive finite subgroup of a simple Lie group will be fixed point free on the adjoint module (though not conversely). See [GRS], [GRQE] and references therein.

Definition 1.6. Let k be an integer. The *degree-k automorphism group* of a VOA V is $Aut(V,k)$, the restriction of $Aut(V)$ to V_k. It acts as automorphisms of the algebra $(V_k,(k-1)^{th})$, so we have a containment $Aut(V,k) \leq Aut((V_k,(k-1)^{th}))$.

A survey of methods to create VOAs with finite automorphism groups was presented. We shall not give details here.

A new result is that an interesting commutative nonassociative algebra of dimension 27 was created as a Griess algebra. It came with automorphism group containing $3^3{:}GL(3,3)$ and was built inside an E_6 lattice type VOA. Calculations showed that the algebra is not Jordan and has automorphism group exactly $3^3{:}GL(3,3)$. There are groups which contain $3^3{:}SL(3,3)$ as nonnormal subgroup and leave invariant 27 dimensional algebra structures, but our algebra turned out to be not one already known (to the author).

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