数理解析研究所講究録 1299 巻 2003 年 85-95

A four-class association scheme derived from a hyperbolic quadric in $PG(3,q)$

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1 Introduction

In the paper [4], Ebert, Egner, Hollmann and Xiang constructed a four-class symmetric association scheme by using the set of secant lines with respect to an elliptic quadric $\mathcal{O}$ of $PG(3,q)$ for $q\geq 4$ a power of 2. We can regard this association scheme as defined on the set of external lines by taking the orthogonal complement with respect to $\mathcal{O}$. In this paper, we consider an analogous construction by hyperbolic quadric. We construct a four-class symmetric association scheme by using the set of external lines with respect to a hyperbolic quadric of $PG(3,q)$. Each relation is invariant under the action of the orthogonal group $O^{+}(4,q)$ but the set of relations is not the set of orbitals on the set of external lines. Indeed, there are more orbitals than relations. Moreover, a quotient of this association scheme forms a strongly regular graph with Latin square type parameters. We also prove that this strongly regular graph is isomorphic to the one constructed from a direct product of a pseudo-cyclic symmetric association scheme defined by the action of $SL(2,q)$ on the right cosets $SL(2,q)/O^{-}(2,q)$, which is a generalization of the construction given by Mathon [6]. This isomorphism is obtained by an isomorphism between $SL(2,q)^{2}$ and $\Omega^{+}(4,q)$.

2 Association schemes, strongly regular graphs and projective spaces

Let X be a finite set and let $\{R_{i}\}_{0\leq i\leq d}$ be relations on X, that is, subsets of $X\times X$. Then $\mathcal{X}=(X,\{R_{i}\}_{0\leq i\leq d})$ is called a *d-class symmetric association scheme* if the following conditions are satisfied.

1. $\{R_{i}\}_{0\leq i\leq d}$ is a partition of $X\times X$.

2. R_{0} is diagonal, that is, $R_{0}=\{(x,x)\mid x\in X\}$.

3. $\{(y,x)\mid(x,y)\in R_{i}\}=R_{i}$ for any i.

4. For any $i,j,k\in\{0,1,\ldots,d\}$, $p_{ij}^{k}:=|\{z\in X\mid(x,z)\in R_{i},(y,z)\in R_{j}\}|$ is independent of the choice of (y,z) in R_{k}.

For $i \in \{0, \ldots, d\}$, let A_i be the adjacency matrix of the relation R_i, that is, A_i is indexed by X and

$$(A_i)_{xy} := \begin{cases} 1 & \text{if} \quad (x,y) \in R_i, \\ 0 & \text{if} \quad (x,y) \notin R_i. \end{cases}$$

Then we have

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$$

for any $i, j \in \{0, \ldots, d\}$. So $A_0, A_1, \cdots, A_d$ form a basis of the commutative algebra generated by $A_0, A_1, \cdots, A_d$ over the complex field (which is called the Bose-Mesner algebra of $\mathcal{X}$). Moreover this algebra has a unique basis $E_0, E_1, \cdots, E_d$ of primitive idempotents. One of the primitive idempotents is $|X|^{-1}J$ where J is the matrix whose entries are all 1. So we may assume $E_0 = |X|^{-1}J$. Let $P = (p_j(i))_{0 \le i,j \le d}$ be the matrix defined by

$$(A_0 \; A_1 \; \cdots \; A_d) = (E_0 \; E_1 \; \cdots \; E_d)P.$$

We call P the first eigenmatrix of $\mathcal{X}$. Remark that $\{p_j(i) \mid 0 \le i \le d\}$ is the set of eigenvalues of A_j. The first eigenmatrix satisfies the orthogonality relation:

$$\sum_{\nu=0}^{d} \frac{1}{k_\nu} p_\nu(i) p_\nu(j) = \frac{|X|}{m_i} \delta_{ij},$$

where $k_i = p_{ii}^0$ and $m_i = \text{rank } E_i$. We say that $\mathcal{X}$ is *pseudo-cyclic* if there exists an integer m such that rank $E_i = m$ for all $i \in \{1, \cdots, d\}$. Remark that in this case, $|X| = dm + 1$ and $k_i = p_{ii}^0 = m$ for all $i \in \{1, \cdots, d\}$ (see [1, p.76]).

Let G be a finite group and K be a subgroup of G. Then G acts naturally on the set $G/K \times G/K$ with orbitals $R_0, R_1, \ldots, R_d$, where we let $R_0 = \{(x,x) \mid x \in X\}$. If all orbitals are self-paired, then $\mathcal{X} = (G/K, \{R_i\}_{0 \le i \le d})$ forms a symmetric association scheme. We denote this association scheme by $\mathcal{X}(G, K)$.

For a strongly regular graph with parameters (n, k, λ, μ), one of the eigenvalues of its adjacency matrix is k, and the others θ_1, θ_2 are the solutions of $x^2 + (\mu - \lambda)x + (\mu - k) = 0$. We can identify the pair of a strongly regular graph and its complement with a two-class symmetric association scheme whose first eigenmatrix is

$$\begin{bmatrix} 1 & k & n-k-1 \\ 1 & \theta_1 & -1-\theta_1 \\ 1 & \theta_2 & -1-\theta_2 \end{bmatrix} \tag{1}$$

In the paper [6], Mathon constructed a strongly regular graph from the pseudo-cyclic symmetric association scheme $\mathcal{X}(SL(2,8), O^-(2,8))$. The next lemma is a generalization of this construction.

Lemma 2.1 *Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ be a pseudo-cyclic symmetric association scheme on $dm+1$ points. Then the graph $\Delta(\mathcal{X})$ whose vertex set is $X \times X$, where two distinct vertices (x, y) and (x', y') are adjacent if and only if $(x, x'), (y, y') \in R_i$ for some $i \neq 0$, is a strongly regular graph with Latin square type parameters $(|X|^2, m(|X|-1), |X| + m(m-3), m(m-1))$.*

Proof) The direct product of $\mathcal{X}$ is $(X \times X, \{R_{ij}\}_{0 \le i,j \le d})$, where

$$R_{ij} := \{((x,y),(x',y')) \mid (x,x') \in R_i,\ (y,y') \in R_j\}.$$

If P is the first eigenmatrix of $\mathcal{X}$, then $P \otimes P$ is the first eigenmatrix of $(X \times X, \{R_{ij}\}_{0 \le i,j \le d})$. The edge set of $\Delta(\mathcal{X})$ is defined to be $\bigcup_{j=1}^{d} R_{jj}$. Then the eigenvalues of the adjacency matrix of $\Delta(\mathcal{X})$ are

$$\left\{ \sum_{j=1}^{d} p_j(i) p_j(i') \,\middle|\, 0 \le i, i' \le d \right\}.$$

Since $\mathcal{X}$ is psuedo-cyclic, $k_0 = m_0 = 1$, $k_j = m_i = m$ for $i, j \neq 0$. Hence the orthogonality relation implies

$$\sum_{j=1}^{d} p_j(i) p_j(i') = \frac{m|X|}{m_i}\delta ii' - m = \begin{cases} m(|X|-1) & \text{if } i = i' = 0, \\ |X| - m & \text{if } i = i' \neq 0, \\ -m & \text{if } i \neq i'. \end{cases}$$

Therefore $\Delta(\mathcal{X})$ has three eigenvalues. This implies that $\Delta(\mathcal{X})$ is strongly regular. The parameters of $\Delta(\mathcal{X})$ can easily be calculated. □

In Lemma 2.1, if $\mathcal{X} = \mathcal{X}(G, K)$ for some finite group G and its subgroup K, then $\Delta(\mathcal{X})$ has the following geometric interpretation.

Lemma 2.2 *Suppose the a finite group G and its subgroup K form a pseudo-cyclic symmetric association scheme $\mathcal{X} = \mathcal{X}(G, K)$. Then the graph $\Delta(\mathcal{X})$ of Lemma 2.1 is isomorphic to the collinearity graph of the coset geometry $(G^2/K^2, G^2/D(G), *)$ where $D(G) := \{(x,x) \mid x \in G\}$ and for $x_1, x_2, y_1, y_2 \in G$, $(x_1, x_2)K^2 * (y_1, y_2)D(G)$ if and only if $(x_1, x_2)K^2 \cap (y_1, y_2)D(G) \neq \emptyset$.*

Proof) Since each relation of $\mathcal{X}(G, K)$ is an orbital of the action of G on $G/K \times G/K$, two pairs $(x_1K, y_1K), (x_2K, y_2K)$ are adjacent in the graph $\Delta(\mathcal{X}(G, K))$ if and only if there exists $w \in G$ such that $y_1K = wx_1K, y_2K = wx_2K$. On the other hand, two pairs $(x_1, y_1)K^2, (x_2, y_2)K^2$ are adjacent in the collinearity graph of $(G^2/K^2, G^2/D(G), *)$ if and only if $(x_1^{-1}x_2, y_1^{-1}y_2)$ is in $K^2D(G)K^2$ (cf. [3, p.15]).

For $x_1, x_2, y_1, y_2 \in G$,

$$\begin{aligned} (x_1^{-1}x_2, y_1^{-1}y_2) \in K^2D(G)K^2 \quad &\Leftrightarrow \quad x_1^{-1}x_2,\ y_1^{-1}y_2 \in KwK \text{ for some } w \in G, \\ &\Leftrightarrow \quad x_1^{-1}x_2 \in Ky_1^{-1}y_2K \\ &\Leftrightarrow \quad y_1kx_1^{-1} = y_2k'x_2^{-1} \text{ for some } k, k' \in K, \\ &\Leftrightarrow \quad y_1 \in wx_1K,\ y_2 \in wx_2K \text{ for some } w \in G, \end{aligned}$$

Hence the mapping $G/K \times G/K \ni (xK, yK) \mapsto (x,y)K^2 \in G^2/K^2$ is an isomorphism between the above two graphs. □

For the rest of this section, we recall some terminologies on finite projective spaces. In this paper, let q be a power of 2 and let $PG(3,q)$ be the three-dimensional projective space over $GF(q)$. For a non-degenerate quadratic form Q on $GF(q)^4$, we say that a point $p = \langle v \rangle$ is *singular* if $Q(v) = 0$, and we say that a line l is *external* (resp. *secant*) if the number of singular points in l is 0 (resp. 2). For a point p, denote by $p^\perp$ the orthogonal complement of p with respect to the symmetric bilinear form obtained from Q. Define for a line l or a plane π, $l^\perp := \bigcap_{p \in l} p^\perp$, $\pi^\perp := \bigcap_{p \in \pi} p^\perp$. We say that a plane π is *tangent* if the point $\pi^\perp$ is singular. Otherwise, we say that π is *oval*.

It is well known that there are two types of non-degenerate quadratic forms on $GF(q)^4$, which are called elliptic type or hyperbolic type. A canonical form of hyperbolic type is

$$Q(x_1, x_2, x_3, x_4) = x_1x_4 + x_2x_3.$$

Denote by $\Omega^+(4,q)$ the commutator group of the orthogonal group defined from the above Q.

3 Main results

For an elliptic type quadratic form, a four-class symmetric association scheme on the set of secant lines was constructed:

Theorem 3.1 ([4]) *Let $q = 2^f \geq 4$. Then the following relations on the set of secant lines of $PG(3,q)$ with respect to an elliptic type quadratic form*

$$\begin{aligned} R_1 &= \{(l,m) \mid l \cap m : \text{a singular point}\} \\ R_2 &= \{(l,m) \mid l \cap m : \text{a nonsingular point}\} \\ R_3 &= \left\{(l,m) \mid l^\perp \cap m \neq \emptyset\right\} \\ R_4 &= \left\{(l,m) \mid l \cap m = \emptyset,\ l^\perp \cap m = \emptyset\right\} \end{aligned}$$

and the diagonal relation R_0 define a four-class symmetric association scheme.

For an elliptic type quadratic form, a line l is secant if and only if $l^\perp$ is external. So we can regard the above association scheme as defined on the set of external lines. The relations R_1, R_2, R_3 and R_4 correspond to the following relations on the set of external lines

$$\begin{aligned} &\{(l,m) \mid \langle l,m \rangle : \text{a tangent plane}\}, \\ &\{(l,m) \mid \langle l,m \rangle : \text{an oval plane}\}, \\ &\left\{(l,m) \mid l^\perp \cap m \neq \emptyset\right\}, \\ &\left\{(l,m) \mid l \cap m = \emptyset,\ l^\perp \cap m = \emptyset\right\}, \end{aligned}$$

respectively.

For a hyperbolic type quadratic form, we can construct a four-class symmetric association scheme similar to the above one. Let $\mathbf{L}$ be the set of external lines with respect to a hyperbolic type quadratic form in $PG(3,q)$.

Theorem 3.2 *Let $q = 2^f \geq 4$. Then the following relations on the set $\mathbf{L}$ of external lines of $PG(3,q)$ with respect to a hyperbolic type quadratic form*

$$\begin{aligned} R_1 &= \{(l,m) \mid l \cap m : \text{a point}\} \\ R_2 &= \left\{(l,m) \mid l : \text{external}, \ m = l^{\perp}\right\} \\ R_3 &= \left\{(l,m) \mid l^{\perp} \cap m : \text{a point}\right\} \\ R_4 &= \left\{(l,m) \mid l \cap m = \emptyset, \ l^{\perp} \cap m = \emptyset\right\} \end{aligned}$$

and the diagonal relation R_0 define a four-class symmetric association scheme.

Moreover we can construct a strongly regular graph from this symmetric association scheme by taking a quotient.

Theorem 3.3 *Let $\Gamma = \Gamma_q$ ($q = 2^f \geq 4$) be the graph with vertex set $\{\{l, l^{\perp}\} \mid l \in \mathbf{L}\}$, where two distinct vertices of Γ, $\{l, l^{\perp}\}$, $\{m, m^{\perp}\}$ are adjacent if and only if $l \cap m \neq \emptyset$ or $l \cap m^{\perp} \neq \emptyset$. Then Γ is a strongly regular graph with Latin square type parameters*

$$v = \frac{1}{4}q^2(q-1)^2, \ k = \frac{1}{2}(q-2)(q+1)^2, \ \lambda = \frac{1}{2}(3q^2 - 3q - 4), \ \mu = q(q+1).$$

Remark that $l \cap m \neq \emptyset$ is equivalent to $l^{\perp} \cap m^{\perp} \neq \emptyset$, and $l \cap m^{\perp} \neq \emptyset$ is equivalent to $l^{\perp} \cap m \neq \emptyset$. So the adjacency in Γ is well-defined.

4 Proof of Theorem 3.2

To prove Theorem 3.2, we recall some facts about $PG(3,q)$ with a hyperbolic type quadratic form from Hirschfeld's book [5, §15]. From now on, put $q = 2^f \geq 4$.

Proposition 4.1 *For a hyperbolic type quadratic form in $PG(3,q)$, the following statements hold.*
(i) *A plane containing an external line is oval.*
(ii) *The number of external lines is $q^2(q-1)^2/2$ and there are $q+1$ oval planes containing a given external line.*
(iii) *The number of oval planes is $q(q^2-1)$ and there are $q(q-1)/2$ external lines in a given oval plane.*
(iv) *For an oval plane π, there is no external line through $\pi^{\perp}$ on π. For a nonsingular point p of π distinct from $\pi^{\perp}$, there are $q/2$ external lines through p on π.*
(v) *There are $q(q-1)/2$ external lines through a given nonsingular point.*

Call $\pi^\perp$ in (iv) the *nucleus* of π. (Remark: when q is an odd prime power, (i),(ii),(iii) and (v) also hold. For an oval plane π, $\pi^\perp$ is not in π.)

First we show that the relations $R_0, \ldots, R_4$ form a partition of $\mathbf{L} \times \mathbf{L}$. It is clear that any pair (l, m) of $\mathbf{L} \times \mathbf{L}$ is in one of $\{R_i\}_{0 \le i \le 4}$. Since any external line l is skew to $l^\perp$, R_1 and R_2 have no intersection. Suppose that $l, m \in \mathbf{L}$ satisfy that l meets m. Then the nucleus of the oval plane $\langle l, m \rangle$ is on $l^\perp$, so m is skew to $l^\perp$ by Proposition 4.1 (iv). Hence R_1 and R_3 have no intersection. Therefore $\{R_i\}_{0 \le i \le 4}$ is a partition of $\mathbf{L} \times \mathbf{L}$.

Next we show that each relation is symmetric. It is clear that R_1, R_2 and R_4 are symmetric. For $(l, m) \in R_3$, the nucleus of the oval plane $\langle l, m^\perp \rangle$ is $l^\perp \cap m$. So $l^\perp$ meets m in a point, hence $(m, l) \in R_3$. Therefore R_3 is also symmetric.

Finally we show that for any $i, j, k \in \{0, \cdots, 4\}$,

$$p_{ij}^k = |\{n \in \mathbf{L} \mid (l, n) \in R_i, (n, m) \in R_j\}|$$

is independent of the choice of $(l, m) \in R_k$. The assertion is clear when $k = 0$. For the moment, we put $p_{ij}(l, m) = |\{n \in \mathbf{L} \mid (l, n) \in R_i, (n, m) \in R_j\}|$. We can easily see that when $(l, m) \in R_k$, $p_{0j}(l, m) = \delta_{jk}$. Since each relation is symmetric, $p_{ji}(l, m) = p_{ij}(m, l)$. Since $R_0, \ldots, R_4$ form a partition of $\mathbf{L} \times \mathbf{L}$, we have

$$\sum_{i=0}^{4} p_{ii}^0 = |\mathbf{L}| = \frac{1}{2} q^2 (q-1)^2,$$

and

$$\sum_{j=0}^{4} p_{ij}(l, m) = p_{ii}^0$$

for any $i \in \{0, \ldots, 4\}$ and for any pair (l, m). Let σ be the permutation $(0,2)(1,3)$ on $\{0, \cdots, 4\}$. Then since $(l, m) \in R_i$ if and only if $(l, m^\perp) \in R_{\sigma(i)}$,

$$p_{ij}(l, m) = p_{i\sigma(j)}(l, m^\perp) = p_{\sigma(i)\sigma(j)}(l, m). \tag{2}$$

Hence we only need to show that p_{11}^k $(1 \le k \le 4)$ are independent of the choice of $(l, m) \in R_k$.

Lemma 4.2 *For $1 \le k \le 4$, p_{11}^k is independent of the choice of $(l, m) \in R_k$ and*

$$p_{11}^0 = \frac{1}{2}(q-2)(q+1)^2,\ p_{11}^1 = q^2 - \frac{3}{2}q - 2,\ p_{11}^2 = 0,\ p_{11}^3 = \frac{1}{2}q^2,\ p_{11}^4 = \frac{1}{2}q(q+1).$$

Proof) Fix $l \in \mathbf{L}$. Any line which meets l in a point is in an oval plane through l, and conversely any line in an oval plane through l meets l in a point. Hence by Proposition 4.1 (ii) and (iii),

$$\begin{aligned} p_{11}^0 &= |\{n \in \mathbf{L} \mid (l, n) \in R_1\}| \\ &= \sum_{\pi \in \Pi_l} |\{n \in \mathbf{L} \mid n \subset \pi,\ n \ne l\}| \\ &= (q+1) \times \left(\frac{1}{2} q(q-1) - 1\right) \\ &= \frac{1}{2}(q-2)(q+1)^2 \end{aligned}$$

where Π_l is the set of oval planes through l.

For $(l,m) \in R_1$, if $n \in \mathbf{L}$ meets both l and m, then n has a point $l \cap m$ or n is in the plane $\langle l,m \rangle$. Hence by Proposition 4.1 (iii)–(v),

$$\begin{aligned} p_{11}^1 &= |\{n \in \mathbf{L} \mid (l,n),(n,m) \in R_1\}| \\ &= |\{n \in \mathbf{L} \mid n \subseteq \langle l,m \rangle,\ n \neq l,m\}| + |\{n \in \mathbf{L} \mid l \cap m \in n \not\subseteq \langle l,m \rangle\}| \\ &= \left(\frac{1}{2}q(q-1)-2\right) + \left(\frac{1}{2}q(q-1)-\frac{1}{2}q\right) \\ &= q^2 - \frac{3}{2}q - 2. \end{aligned}$$

From (2), we have $p_{11}^2 = 0$. For $(l,m) \in R_3 \cup R_4$, we have

$$|\{n \in \mathbf{L} \mid (l,n),(n,m) \in R_1\}| = \sum_{\pi \in \Pi_l} |\{n \in \mathbf{L} \mid m \cap \pi \in n \subseteq \pi\}|.$$

If $(l,m) \in R_3$, then there is just one plane $\pi_0 = \langle l, l^\perp \cap m \rangle \in \Pi_l$ such that $m \cap \pi_0$ is the nucleus of π_0. By Proposition 4.1 (iv), there is no line of $\mathbf{L}$ through $m \cap \pi_0$ and in π_0, and for other plane π, there are $q/2$ lines of $\mathbf{L}$ through $m \cap \pi$ and in π. Hence

$$\begin{aligned} p_{11}^3 &= |\{n \in \mathbf{L} \mid (l,n),(n,m) \in R_1\}| \\ &= \sum_{\pi \in \Pi_l \setminus \{\pi_0\}} |\{n \in \mathbf{L} \mid m \cap \pi \in n \subseteq \pi\}| \\ &= q \times \frac{1}{2}q. \end{aligned}$$

For $(l,m) \in R_4$, any plane π of Π_l has $q/2$ lines of $\mathbf{L}$ through $m \cap \pi$. So

$$p_{11}^4 = |\{n \in \mathbf{L} \mid (l,n),(n,m) \in R_1\}| = (q+1) \times \frac{1}{2}q.$$

□

Therefore $(\mathbf{L}, \{R_i\}_{0 \le i \le 4})$ becomes a symmetric association scheme. For $i \in \{0,\dots,d\}$, let $B_i := (p_{ij}^k)_{0 \le j,k \le 4}$. Then B_0 is the identity matrix,

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ p_{11}^0 & q^2-3/2q-2 & 0 & q^2/2 & q(q+1)/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & q^2/2 & p_{11}^0 & q^2-3/2q-2 & q(q+1)/2 \\ 0 & q^2(q-3)/2 & 0 & q^2(q-3)/2 & (q+1)(q^2-3q-2)/2 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & q^2/2 & p_{11}^0 & q^2-3/2q-2 & q(q+1)/2 \\ 0 & 1 & 0 & 0 & 0 \\ p_{11}^0 & q^2-3/2q-2 & 0 & q^2/2 & q(q+1)/2 \\ 0 & q^2(q-3)/2 & 0 & q^2(q-3)/2 & (q+1)(q^2-3q-2)/2 \end{pmatrix},$$

and

$$B_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & q^2(q-3)/2 & 0 & q^2(q-3)/2 & (q+1)(q^2-3q-2)/2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & q^2(q-3)/2 & 0 & q^2(q-3)/2 & (q+1)(q^2-3q-2)/2 \\ p_{44}^0 & q(q-3)(q^2-3q-2)/2 & p_{44}^0 & q(q-3)(q^2-3q-2)/2 & q(q^3-6q^2+5q+16)/2 \end{pmatrix}$$

where $p_{44}^0 = q(q-2)(q-3)(q+1)/2$. The first eigenmatrix of this association scheme is given by

$$P = \begin{pmatrix} 1 & (q-2)(q+1)^2/2 & 1 & (q-2)(q+1)^2/2 & q(q-2)(q-3)(q+1)/2 \\ 1 & (q-2)(q+1)/2 & -1 & -(q-2)(q+1)/2 & 0 \\ 1 & -(q+1) & -1 & q+1 & 0 \\ 1 & -(q+1) & 1 & -(q+1) & 2q \\ 1 & (q^2-3q-2)/2 & 1 & (q^2-3q-2)/2 & -q(q-3) \end{pmatrix}.$$

5 Proof of Theorem 3.3

In this section, we prove Theorem 3.3 by using Theorem 3.2. The number of vertices of the graph Γ is $|\mathbf{L}|/2 = q^2(q-1)^2/4$. For a pair $\{l, l^\perp\} \in V\Gamma$,

$$\begin{aligned} &\{\{m, m^\perp\} \in V\Gamma \mid \{m, m^\perp\} \text{ is adjacent to } \{l, l^\perp\}\} \\ &\quad = \{\{m, m^\perp\} \in V\Gamma \mid m \text{ meets } l \text{ in a point. }\} \\ &\quad = \{\{m, m^\perp\} \in V\Gamma \mid (l, m) \in R_1\}. \end{aligned}$$

So, the size of this set is $p_{11}^0 = (q-2)(q+1)^2/2$, which is just k in the definition of strongly regular graph. Next choose $\{l, l^\perp\}, \{m, m^\perp\} \in V\Gamma$ which are adjacent in Γ. We may suppose that l meets m in a point. Then

$$\begin{aligned} &\{\{n, n^\perp\} \in V\Gamma \mid \{n, n^\perp\} \text{ is adjacent to both } \{l, l^\perp\} \text{ and } \{m, m^\perp\}\} \\ &\quad = \{\{n, n^\perp\} \in V\Gamma \mid n \text{ meets both } l \text{ and } m \text{ in a point.}\} \\ &\qquad \cup \{\{n, n^\perp\} \in V\Gamma \mid n \text{ meets both } l \text{ and } m^\perp \text{ in a point. }\}, \\ &\quad = \{\{n, n^\perp\} \in V\Gamma \mid (l, n) \in R_1,\ (m, n) \in R_1 \cup R_3\}. \end{aligned}$$

Hence the size of this set is $p_{11}^1 + p_{13}^1 = (3q^2 - 3q - 4)/2$. This is just λ in the definition of strongly regular graph.

Similarly, for $\{l,l^\perp\},\{m,m^\perp\}\in V\Gamma$ which are not adjacent in Γ, since $(l,m)\in R_4$,

$$\left|\left\{\{n,n^\perp\}\in V\Gamma \mid \{n,n^\perp\} \text{ is adjacent to both } \{l,l^\perp\} \text{ and } \{m,m^\perp\}\right\}\right| = p_{11}^4 + p_{13}^4 = q(q+1).$$

This is just μ in the definition of strongly regular graph.

Alternatively, we can prove Theorem 3.3 by using the quotient association scheme (cf. [1, p.139, Thm.9.4]). In the association scheme of Theorem 3.2, $R_0\cup R_2$ is an equivalence relation on $\mathbf{L}$. So we can define a quotient association scheme on the set of equivalence classes $\left\{\{l,l^\perp\} \mid l\in\mathbf{L}\right\}$ whose relations are

$$\left\{(\{l,l^\perp\},\{m,m^\perp\}) \mid (l,m)\in R_1\cup R_3\right\} = \text{the edge set of } \Gamma,$$
$$\left\{(\{l,l^\perp\},\{m,m^\perp\}) \mid (l,m)\in R_4\right\},$$

and the diagonal relation. The first eigenmatrix of this association scheme can be computed from P (cf. [1, p.148]):

$$\begin{pmatrix} 1 & (q-2)(q+1)^2/2 & q(q-2)(q-3)(q+1)/4 \\ 1 & -(q+1) & q \\ 1 & (q^2-3q-2)/2 & -q(q-3)/2 \end{pmatrix}.$$

The first relation forms a strongly regular graph whose parameters are calculated from the second column of the above first eigenmatrix.

6 Another construction of Γ_q

In this section, we will give another construction of the strongly regular graph Γ_q. This construction uses a method which generalizes a construction of Mathon ([6, p.137], see also [2, pp.96–97]).

Let $G=SL(2,q)$, $K=O^-(2,q)$. Then $\mathcal{X}(G,K)$ is a $(q-2)/2$-class pseudo-cyclic symmetric association scheme (cf. [2, p.96]). By Lemma 2.1, we can construct a strongly regular graph $\Delta(\mathcal{X}(G,K))$ with parameters

$$\left(\frac{1}{4}q^2(q-1)^2,\ \frac{1}{2}(q-2)(q+1)^2,\ \frac{1}{2}(3q^2-3q-4),\ q(q+1)\right)$$

which are the same as those of Γ_q. We shall prove that these graphs are isomorphic.

To show this, we use the isomorphism $G^2\simeq\Omega^+(4,q)$ which maps (X,Y) to $X\otimes Y$ (see [7, p.199]). Let l_0 be the external line generated by $v_1={}^t(0,1,1,0), v_2={}^t(1,1,0,\alpha)$, where α is an element of $\mathbf{F}_q$ such that the polynomial $x^2+x+\alpha$ is irreducible over $\mathbf{F}_q$. For an external line l, there are $2(q+1)$ basis (u_1,u_2) of l such that $Q(xu_1+yu_2)=x^2+xy+\alpha y^2$ for any $x,y\in\mathbf{F}_q$. Indeed, by Witt's Theorem, K acts regularly on the set of basis (u_1,u_2) of l with the above condition. It follows that the size of this set is equal to $|K|=2(q+1)$. Let $\mathcal{P}$ be the set of nonsingular points in $PG(3,q)$ and let $\mathcal{L}=\{l\cup l^\perp \mid l\in\mathbf{L}\}$. Then the following lemma holds.

Lemma 6.1 *The group* $\Omega^+(4,q) = \{X \otimes Y \mid X, Y \in G\}$ *is flag-transitive on the incidence structure* $(\mathcal{P}, \mathcal{L}, \in)$. *Under the isomorphism* $G^2 \simeq \Omega^+(4,q)$, *the groups* $D(G)$, K^2 *are the stabilizers of an element of* $\mathcal{P}$, $\mathcal{L}$, *respectively.*

Proof) Let $X = (x_{ij})_{1\le i,j\le 2}$, $Y = (y_{ij})_{1\le i,j\le 2} \in G$. Since

$$X \otimes Y = \begin{pmatrix} x_{11}y_{11} & x_{11}y_{12} & x_{12}y_{11} & x_{12}y_{12} \\ x_{11}y_{21} & x_{11}y_{22} & x_{12}y_{21} & x_{12}y_{22} \\ x_{21}y_{11} & x_{21}y_{12} & x_{22}y_{11} & x_{22}y_{12} \\ x_{21}y_{21} & x_{21}y_{22} & x_{22}y_{21} & x_{22}y_{22} \end{pmatrix},$$

$X \otimes Y$ fixes v_1 if and only if

$$\begin{aligned} x_{11}y_{12} + x_{12}y_{11} = x_{21}y_{22} + x_{22}y_{21} &= 0, \\ x_{11}y_{22} + x_{12}y_{21} = x_{21}y_{12} + x_{22}y_{11} &= 1. \end{aligned}$$

This implies

$$\Omega^+(4,q)_{v_1} = \{X \otimes X \mid X \in G\} \simeq D(G). \tag{3}$$

For $X \in G$, $X \otimes X$ fixes v_2 if and only if

$$\begin{aligned} x_{11}^2 + x_{11}x_{12} + \alpha x_{12}^2 &= 1, \\ x_{11}x_{21} + x_{12}x_{21} + \alpha x_{12}x_{22} &= 0, \\ x_{21}^2 + x_{21}x_{22} + \alpha x_{22}^2 &= \alpha. \end{aligned}$$

From these, we have

$$\Omega^+(4,q)_{v_1,v_2} = \left\{ X \otimes X \;\middle|\; X = \begin{pmatrix} a & b \\ \alpha b & a+b \end{pmatrix} \in G \right\}$$

which is of order $q+1$. Hence

$$\begin{aligned} |\{(Mv_1, Mv_2) \mid M \in \Omega^+(4,q)\}| &= |\Omega^+(4,q)|/(q+1) \\ &= q^2(q-1)^2(q+1). \end{aligned}$$

Since $Q(xv_1 + yv_2) = x^2 + xy + \alpha y^2$ for any $x, y \in \mathbf{F}_q$,

$$\begin{aligned} \left|\left\{(u_1, u_2) \mid Q(xu_1 + yu_2) = x^2 + xy + \alpha y^2 \quad {}^\forall x, y \in \mathbf{F}_q\right\}\right| &= |\mathbf{L}| \times 2(q+1) \\ &= q^2(q-1)^2(q+1). \end{aligned}$$

Hence $\Omega^+(4,q)$ acts transitively on the set of pairs (u_1, u_2) such that $Q(xu_1 + yu_2) = x^2 + xy + \alpha y^2$ for any $x, y \in \mathbf{F}_q$. In particular, $\Omega^+(4,q)$ is flag-transitive on $(\mathcal{P}, \mathcal{L}, \in)$.

The equality (3) means that the stabilizer of $\langle v_1 \rangle \in \mathcal{P}$ is isomorphic to $D(G)$. Let

$$A := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \; B := \begin{pmatrix} a_0 & b_0 \\ \alpha b_0 & a_0 + b_0 \end{pmatrix} \in G$$

such that B is of order $q+1$. Then the group $\langle A, B\rangle$ is isomorphic to K. $A\otimes I, I\otimes A$ interchange l_0 and $l_0^{\perp}$, while $B\otimes I, I\otimes B$ fix l_0 and $l_0^{\perp}$. So $\{X\otimes Y \mid X, Y \in \langle A, B\rangle\}$ is a subgroup of $\Omega^+(4,q)_{l_0\cup l_0^{\perp}}$. Since $\Omega^+(4,q)_{l_0\cup l_0^{\perp}}$ has order $4(q+1)^2 = |K|^2$, we have that $\Omega^+(4,q)_{l_0\cup l_0^{\perp}}$ is isomorphic to K^2. □

Theorem 6.2 *The graph* Γ_q *is isomorphic to* $\Delta(\mathcal{X}(SL(2,q), O^-(2,q)))$.

Proof) The graph Γ_q is isomorphic to the collinearity graph of the dual of the incidence structure $(\mathcal{P}, \mathcal{L}, \in)$. From Lemma 6.1, the dual of $(\mathcal{P}, \mathcal{L}, \in)$ is isomorphic to the coset geometry $(G^2/K^2, G^2/D(G), *)$ defined in Lemma 2.2. From Lemma 2.2, the collinearity graph of $(G^2/K^2, G^2/D(G), *)$ is isomorphic to $\Delta(\mathcal{X}(G,K))$. Therefore Γ_q is isomorphic to $\Delta(\mathcal{X}(G,K))$.
□

References

[1] E. Bannai and T. Ito, "Algebraic Combinatorics I," Benjamin/Cummings, Menlo Park, CA, 1984.

[2] A. E. Brouwer and J. H. van Lint, Strongly regular graphs and partial geometries, in "Enumeration and Design", 85–122, Academic Press, Toronto, ON, 1984.

[3] P. Dembowski, "Finite geometries," Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 44, Springer-Verlag, Berlin-New York, 1968.

[4] G. L. Ebert, S. Egner, H. D. L. Hollmann, and Q. Xiang, On a four-class association scheme, J. Combin. Theory Ser. A 96 (2001), no. 1, 180–191.

[5] J. W. P. Hirschfeld, "Finite Projective Spaces of Three Dimensions," Oxford University Press, 1985.

[6] R. Mathon, 3-class association schemes, Proceedings of the Conference on Algebraic Aspects of Combinatorics, 123–155, Congressus Numerantium XIII, 1975.

[7] D. E. Taylor, "The Geometry of the Classical Groups," Sigma Series in Pure Mathematics, 9, Heldermann Verlag, Berlin, 1992.