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Shadows of Odd Unimodular Lattices

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This is an exposition of our recent joint paper with G. Nebe: G. Nebe and B. Venkov, "Unimodular lattices with long shadow." The paper is submitted to Journal of Number Theory. Full text can be found on the web page of G. Nebe.

After Elkies and Gaulter, we study odd unimodular lattice Λ in $\mathbf{R}^n$, whose all characteristic vectors γ have big norm $(\gamma,\gamma)\geq n-16$, where $n=\dim\Lambda$. We say that such a lattice is an $(n-16)$-lattice. By shadow theory, this condition is equivalent to the fact that the theta series of Λ, which is a polynomial in two standard generators θ_3 and Δ_8, is in fact a sum of three monomials

$$\theta_\Lambda=\theta_3^n+A\theta_3^{n-8}\Delta_8+B\theta_3\Delta_8^2$$

with two constants A and B. If Λ has no elements of norm 1, then $A=-2n$. If moreover $\min\Lambda\geq 3$, i.e. Λ contains no roots, then B is also fixed as a function of n. That fixes θ_Λ and gives, for example, for the number of elements of norm 3 and 4

$$\begin{aligned} n_3 &= \frac{4}{3}n(n^2-69n+1208),\\ n_4 &= 2n(n^3-94n^2+2783n-24425). \end{aligned}$$

Our main result is that such $(n-16)$-lattices without roots can exist only for $n\leq 46$. (Previous bound by Gaulter [Gau] for $(n-16)$-lattices, possibly with roots, was $n\leq 2907$). Our bound $n=46$ is achieved, because the lattice $\Lambda_0=O_{23}\oplus O_{23}$, where O_{23} is the shorter Leech lattice, satisfies our conditions.

We also prove the uniqueness of Λ_0 for $n = 46$, and nonexistence of $(n-16)$-lattice without roots for dimensions $n = 44$ and 45. We also were able to construct examples of $(n-16)$-lattices without roots for $n \leq 35$. It remains unknown, what happens for dimensions $36 \leq n \leq 44$.

To prove these results we consider theta series of Λ with harmonic coefficients. For an odd unimodular lattice Λ and a harmonic polynomial P on $\mathbf{R}^n$ of degree $k \equiv 0 \pmod 2$, the theta series for (Λ, P) is

$$\theta_{\Lambda,P} = \sum_{\lambda \in \Lambda} P(\lambda) q^{(\lambda,\lambda)}$$

where $q = e^{\pi i z}$, $\operatorname{Im} z > 0$. It is a modular form of weight $n/2 + k$ for the theta group with some character which depend on k mod 4. It follows that $\theta_{\Lambda,P}$ is a polynomial in θ_3 and Δ_8 if $k \equiv 0 \pmod 4$ or is of the form $\Phi \cdot$ (polynomial in θ_3, Δ_8), if $k \equiv 2 \pmod 4$. Here

$$\Phi(z) = \theta_2(z)^4 - \theta_3(z)^4.$$

As for usual theta series ($P = 1$), there is a analogue of the shadow theory. If for a modular form ϕ of weight m we define the shadow transformation by

$$S(\phi)(z) = \left(\sqrt{\frac{i}{z}}\right)^{2m} \phi(-\frac{1}{z}+1),$$

then

$$S(\theta_{\Lambda,P}) = (-1)^{k/2} \sum_{\lambda \in S(\Lambda)} P(\lambda) q^{(\lambda,\lambda)},$$

where $S(\Lambda)$ is the shadow of Λ: $S(\Lambda) = \Lambda_0^* - \Lambda$, where Λ_0 is the even part of Λ. If our odd lattice Λ has a long shadow, then $S(\theta_{\Lambda,P})$ starts with a big power of q and that gives extra information on the theta series with harmonic coefficients. For an $(n-16)$-lattice Λ without roots and for $k = 2$ that gives

$$\theta_{\Lambda,P_2} = c\Phi\theta_3^{n-16}\Delta_8^2 = 0$$

so all layers of Λ and $S(\Lambda)$ form spherical 3-designs. From harmonic polynomials of degree $k = 4$ and $k = 6$ we get formulas

$$\sum_{v \in \Lambda_4} (v,\alpha)^4 - 2(n-28) \sum_{u \in \Lambda_3} (u,\alpha)^4 = 24(n-41)(n-46)(\alpha,\alpha)^2,$$

$$\sum_{v \in \Lambda_4} (v,\alpha)^6 - 2(n-40) \sum_{u \in \Lambda_3} (u,\alpha)^6 = 30(\alpha,\alpha) \sum_{u \in \Lambda_3} (u,\alpha)^4 - 240(n-37)(\alpha,\alpha)^3$$

here $\alpha \in \mathbf{R}^n$ is an arbitrary element of $\mathbf{R}^n$. These formulas permit to find an element $v_0 \in \Lambda_4$ of norm 4, such that $|\{u \in \Lambda_3 \mid (u, v_0) = 2\}|$ is big. Considering neighboring lattice to Λ with respect to such a v_0, we get an $(n-1)$-dimensional unimodular lattice with the root system kA_1 with big k. For $n = 46$ this k happens to be 23. Projecting on this 23-dimensional subspace we get an O_{23} and it follows that $\Lambda = O_{23} \oplus O_{23}$. Similarly one proves impossibility of $n = 44$ and 45.

References

[1] N. D. Elkies, Lattices and codes with long shadows, Math. Res. Lett. **2** (1995), no. 5, 643–651.

[Gau] M. Gaulter, Lattices without short characteristic vectors, Math. Res. Lett. **5** (1998), no. 3, 353–362.