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On Generalized Lee Weights for Codes over $\mathbb{Z}_4$ *

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1 Introduction

For a linear code over a finite field, Helleseth, Klove and Mykkeltveit [9] introduced the generalized Hamming weights while studying the weight distribution of irreducible cyclic codes and later Wei ([18]) rediscoveredthe idea of generalized Hamming weights. After that a lot of papers dealing with the weights have been published (cf. [17] etc.). Recently, the generalized Hamming weights for codes over $\mathbb{Z}_4$ have been defined and studied, see [1], [19], [20], [3] and [10] for example.

In this note, we shall define a type of generalized Lee weights for codes over $\mathbb{Z}_4$ and give some fundamental results.

A *linear code* of length n over $\mathbb{Z}_4$ is a $\mathbb{Z}_4$-submodule of $\mathbb{Z}_4^n$. For a linear code C of length n over $\mathbb{Z}_4$, we define the *rank* of C, denoted by rank(C), by the minimum number of generators of C. It is known that a linear code C of length n over $\mathbb{Z}_4$ is permutation-equivalent to a linear code with generator matrix of the form

(1) $$\begin{pmatrix} I_{k_1} & X & Y \\ 0 & 2I_{k_2} & 2Z \end{pmatrix},$$

where X and Z are binary matrices and Y is a $\mathbb{Z}_4$-matrix. In this case, it finds that $|C| = 4^{k_1}2^{k_2}$ and $\mathrm{rank}(C) = k_1 + k_2$. We shall define a code with a generator matrix of the form in 1 as being of type $\{k_1, k_2\}$.

For a vector $\boldsymbol{x} \in \mathbb{Z}_4^n$, we denote the *Hamming weight* and *Lee weight* by $\mathrm{wt}(\boldsymbol{x})$ and L-wt$(\boldsymbol{x})$, respectively.

For a linear code C of length n over $\mathbb{Z}_4$, let $A(C)$ be the $|C| \times n$ array of all codewords in C. It is well-known that each column of $A(C)$ corresponds to the following three cases: (i)

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the column contains only 0 (ii) the column contains 0 and 2 equally often (iii) the column contains all elements of $\mathbb{Z}_4$ equally often (cf. [20]). For the three columns (i), (ii) and (iii), we define the *Lee weights* of these columns by 0, 2 and 1 respectively. Thus we define the *Lee weight* $\mathrm{wt}_L(C)$ of C by the sum of the Lee weights of all columns of $A(C)$. For example, if

$$C = \{(0,0,0),(1,0,1),(2,0,2),(3,0,3),(0,2,2),(1,2,3),(2,2,0),(3,2,1)\},$$

then $\mathrm{wt}_L(C) = 1+2+1 = 4$. We remark that if C is generated by only one vector $\boldsymbol{x}$, then the Lee weight $\mathrm{wt}_L(C)$ corresponds to the original Lee weight L-wt$(\boldsymbol{x})$ of $\boldsymbol{x}$. Then we have the following theorem.

Theorem 1.1 *Let C be a linear code C of length n over $\mathbb{Z}_4$ with type $4^{k_1}2^{k_2}$. Then we have*

$$\begin{aligned}\mathrm{wt}_L(C) &= \frac{1}{4^{k_1-1}2^{k_2}} \sum_{\boldsymbol{x}\in C} (\text{L-wt}(\boldsymbol{x}) - \mathrm{wt}(\boldsymbol{x})) \\ &= \frac{1}{4^{k_1-1}2^{k_2}} \sum_{\boldsymbol{x}\in C} |\{i : x_i = 2\}|.\end{aligned}$$

Now, for $1 \le r \le \mathrm{rank}(C)$, we define the r-th *generalized Lee weight with respect to rank* (GLWR) $d_r^L(C)$ of C as follows:

$$d_r^L(C) := \min\{\mathrm{wt}_L(D) : D \text{ is a } \mathbb{Z}_4\text{-submodule of } C \text{ with } \mathrm{rank}(D) = r\}.$$

We note that $d_1^L(C)$ corresponds to the minimum Lee weight of C.

2 Bounds for GLWR

In this section, we give some bounds for GLWR of linear codes over $\mathbb{Z}_4$.

Lemma 2.1 *If C is a linear code of length n over $\mathbb{Z}_4$ with* $\mathrm{rank}(C) = 2$, *then there exists a codeword* $\mathbf{0} \neq \boldsymbol{v} \in C$ *such that* L-wt$(\boldsymbol{v}) \le \mathrm{wt}_L(C)$.

Using the above lemma, we have the following result.

Theorem 2.2 *Let C be a linear code of length n over $\mathbb{Z}_4$ with* $\mathrm{rank}(C) \ge 2$. *Then we have* $1 \le d_1^L(C) \le d_2^L(C)$.

In [11], the rth generalized Hamming weight with respect to rank (GHWR) of a linear code C is defined by

$$d_r^H(C) := \min\{|\mathrm{Supp}(D)| : D \text{ is a } \mathbb{Z}_4\text{-submodule of } C \text{ with } \mathrm{rank}(D) = r\},$$

where $\mathrm{Supp}(D) := \cup_{\boldsymbol{x}\in D}\mathrm{supp}(\boldsymbol{x})$. We remark that

$$d_r^L(C) \le 2d_r^H(C). \qquad (2)$$

The following lemma is called the *generalized Singleton bound* for linear codes over $\mathbb{Z}_4$ (see

Lemma 2.3 *Let C be a linear code of length n over $\mathbb{Z}_4$. Then, for any r, $1 \leq r \leq \text{rank}(C)$,*

$$d_r^H(C) \leq n - \text{rank}(C) + r.$$

Now, we give a similar type bound for GLWR.

Theorem 2.4 *For a linear code C of length n over $\mathbb{Z}_4$ and any r, $1 \leq r \leq \text{rank}(C)$,*

$$\left\lfloor \frac{d_r^L(C) - 2r + 1}{2} \right\rfloor \leq n - \text{rank}(C).$$

Remark 2.5 In [7] and [15], it is shown that for a linear code C of length n over $\mathbb{Z}_4$ with minimum Lee weight d_L,

$$\left\lfloor \frac{d_L - 1}{2} \right\rfloor \leq n - \text{rank}(C).$$

Since $d_L = d_1^L(C)$, the bound in Theorem 2.4 is a generalization of the above bound.

If a linear code C of length n over $\mathbb{Z}_4$ meets the bound in Theorem 2.4 for r, that is, $\lfloor (d_r^L(C) - 2r + 1)/2 \rfloor = n - \text{rank}(C)$, then we shall call the code C as *r-th maximum Lee distance separable with respect to rank* (r-th MLDR) code. Similarly if a linear code C of length n over $\mathbb{Z}_4$ meets the bound in Lemma 2.3 for r, that is, $d_r^H(C) = n - \text{rank}(C) + r$, then the code C is called *r-th maximum Hamming distance separable with respect to rank* (r-th MHDR) code. Now we shall give a connection between r-th MLDR codes and r-th MHDR codes.

Lemma 2.6 *If C is an r-th MLDR code, then $d_r^L(C) = 2d_r^H(C) - 1$ or $2d_r^H(C)$.*

Theorem 2.7 *Let C be a linear code C of length n over $\mathbb{Z}_4$. If C is an r-th MLDR code, then C is an r-th MHDR code.*

Theorem 2.8 *Let C be an r-th MHDR code of length n over $\mathbb{Z}_4$. C is an r-th MLDR code if and only if $d_r^L(C) = 2d_r^H(C) - 1$ or $2d_r^H(C)$.*

It is known that if C is a linear code of length n over $\mathbb{Z}_4$ with minimum Hamming weight d_H and minimum Lee weight d_L, then

(3) $$d_H \geq \left\lceil \frac{d_L}{2} \right\rceil$$

(cf. [14]). In [16], they have proved the following Griesmer type bound for linear codes over finite quasi-Frobenius rings.

Lemma 2.9 *Let C be a linear code of length n over $\mathbb{Z}_4$ with* $\text{rank}(C) = k$ *and minimum Hamming weight d_H. Then*

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d_H}{2^i} \right\rceil .$$

Using (3) and Lemma 2.9, we have the following Griesmer type bound for minimum Lee weights of linear codes over $\mathbb{Z}_4$.

Proposition 2.10 *Let C be a linear code of length n over $\mathbb{Z}_4$ with* $\text{rank}(C) = k$ *and minimum Lee weight d_L. Then*

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{\lceil d_L/2 \rceil}{2^i} \right\rceil .$$

Now we have a generalized Griesmer type bound for GLWR.

Theorem 2.11 *For a linear code C of length n over $\mathbb{Z}_4$ and any r, $1 \leq r \leq \text{rank}(C)$, we have*

$$d_r^L(C) \geq \sum_{i=0}^{r-1} \left\lceil \frac{\lceil d_1^L(C)/2 \rceil}{2^i} \right\rceil .$$

Let C be a linear code C of length n over $\mathbb{Z}_4$. From the definitions of GLWR and GHWR, we have

$$d_r^H \geq \left\lceil \frac{d_r^L}{2} \right\rceil \tag{4}$$

for any r. We define the *socle* of C as follows:

$$\text{Soc}(C) := \{\boldsymbol{x} \in C \mid 2\boldsymbol{x} = \boldsymbol{0}\}.$$

It is known that if C is a linear code C of length n over $\mathbb{Z}_4$ with $\text{rank}(C) = k$ and minimum hamming weight d_H, then $\text{Soc}(C)$ is isomorphic to a binary $[n, k, d]$ code (cf. [11]).

Lemma 2.12 ([11]) *For any r, $1 \leq r \leq \text{rank}(C)$, we have*

$$d_r^H(C) = d_r^H(\text{Soc}(C)).$$

Using the above lemma and Theorem 3.19 (p. 35 in [5]), the lemma follows:

Lemma 2.13 *Let C be a linear code C of length n over $\mathbb{Z}_4$ with* $\text{rank}(C) = k$. *Then*

$$n \geq d_r^H(C) + \sum_{i=1}^{k-r} \left\lceil \frac{d_r^H(C)}{2^i(2^i - 1)} \right\rceil ,$$

for any r, $1 \leq r \leq k$.

Now we have a generalized Griesmer type bound for GLWR.

Theorem 2.14 *Let C be a linear code C of length n over $\mathbb{Z}_4$ with* $\mathrm{rank}(C) = k$. *Then*

$$n \geq \left\lceil \frac{d_r^L(C)}{2} \right\rceil + \sum_{i=1}^{k-r} \left\lceil \frac{\lceil d_r^L(C)/2 \rceil}{2^i(2^i-1)} \right\rceil ,$$

for any r, $1 \leq r \leq k$.

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