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COLLINEATION GROUPS OF TRANSLATION PLANES AND LINEAR GROUPS

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1. Introduction.

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The following description of a finite translation plane is due to André (1954). A finite translation plane is a vector space of dimension $2d$ over a field of q elements of characteristic p, equipped with a spread. This is a set of q^d+1, d-dimensional subspaces such that each non zero vector lies in exactly one of these subspaces. Each one of these subspaces is called a fiber, which is a line incident with the zero vector. (We use the term fiber instead of component because of the term component has special meaning in the finite group theory.)

In this article a translation plane is a finite translation plane. One of the main problem in the Theory of Translation Planes is the following. (See, for example, [8].)

Main Problem. Which non abelian finite simple groups can be collineation groups for a translation plane.

For brevity, we use the term simple group to mean non abelian simple group. In the study of collineation groups of a translation plane, we can apply representation theory to the action of the group on the affine points, and permutation group theory to the action of the group on the points on the line of infinity. The collineation group of a translation plane is a semi-direct product of the translation group and the translation complement. The translation group is a normal elementary subgroup of order q^{2d}. The translation complement is a semi-linear transformation group. This shows that in order to understand a collineation group, one has to study the translation complement. The subgroup of all linear transformations in the translation complement is called the linear complement. Note that perfect subgroups of the translation complement are in the linear complement.

Two types of collineations: affine perspectivities (the set of fixed points is a fiber of the spread) and Baer elements (the set of fixed points is a subplane which is also a d-dimensional subspace) attract most attention. These collineations occur in a simple collineation group of a translation plane in the following way. Being simple, the group is in the linear complement and it does not contain any central homology. Thus perspectivities are affine perspectivities. If the characteristic is

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odd, then involutions are Baer. (See, for example, [5].) If characteristic is even, then an involution is either a Baer involution or an elation with its axis a fiber. Thus in both cases, the dimension of the set of fixed points of any of these collineations is half the dimension of the underlying vector space. This leads to the following project in the study of linear groups: classify all finite groups of linear transformations of a vector space such that the dimension of the set of fixed points of a non identity element is a constant. The following Theorem A of [6] is a result under a weaker condition.

(We use the following notation. For a finite group G, $m_2(G)$ denotes the 2-rank of G, i.e., $2^{m_2(G)}$ is the largest order of an elementary abelian 2-subgroup of G; $O(G)$ denotes the normal subgroup of maximal odd order; and $C_G(i)$ denotes the centralizer of i in G. The two dimensional projective linear group over a field of s elements is denoted by $L_2(s)$; the dihedral group of order $2s$ is denoted by D_{2s}; The cyclic group of order s is denoted by C_s. We use quaternion to mean a quaternion group of order 8 or a general quaternion.)

Theorem A. *Let V be a finite dimensional vector space over a finite field F of characteristic p and $G \leq GL(V)$. Assume $|G|$ is even and for each involution i in G and each $1 \neq x \in C_G(i)$, $dim(C_V(x)) = dim(C_V(i))$. Then one of the following holds:*

(1) *G is the split extension of an elementary abelian 2-group N by a group X of odd order semiregular on N. $F(X)$ and $X/F(X)$ are cyclic.*
(2) *$G \cong L_2(2^a)$ for some $a \geq 2$.*
(3) *p is odd and G is a dihedral group.*
(4) *$G = O(G) < t >$, where t is an involution inverting the abelian group $O(G)$.*
(5) *$m_2(G) = 1$, p is odd, and G is a Frobenius group with Frobenius Kernel $O_p(G)$ and Frobenius complement $C_G(i)$, where i is an involution.*
(6) *p is odd and G is semiregular on $[V, i]$ for i the unique involution in G.*
(7) *$G \cong L_2(t)$ or $PGL_2(t)$, t is a power of the odd prime p, $V = C_V(G) \oplus [V, G]$, and if F is a splitting field for G then each noncentral chief factor for G on V is of dimension 3.*
(8) *p is odd, $G \cong L_2(7)$, $V = C_V(G) \oplus [V, G]$, and $[V, G]$ is the sum of 3-dimensional irreducibles for G.*

Some remarks of Theorem A are in order. In the case in which $m_2(G) \geq 3$, we prove that the centralizer of any involution of G is a 2-subgroup. In an earlier version of [6] we use this fact to apply the famous results of Suzuki on (CIT)-groups. It is interesting to note the following from Suzuki [13, p. 1612]: " We just mentioned that an idea of Thompson [3,7] is used with great advantage and the theory of characters is needed together with an idea similar to the one in ref. [5]." (The references 3, 5, 7 here are respectively 4, 12, 15 in our references.) Note also that Suzuki proves that the incidence structure created is a projective plane of order 4 at the end of the proof of Theorem 4 of [12, Lemma 15 p.467].

The structure of $C_G(i)$ in (5) and (6) can be found, for example, in [11, p.198] for $C_G(i)$ solvable, and [11, p.204] for $C_G(i)$ nonsolvable.

Theorem A seems to hold when we allow $char(F)$ to be zero. A consequence of Theorem A is the following result on non abelian simple collineation groups.

Theorem B [6]. *If G is a non abelian simple collineation group in the translation complement of a finite translation plane V of order n such that each non involutory element in the centralizer of any involution is a perspectivity or a Baer element, then one of the following holds.*

(1) $G \cong L_2(2^a)$ *with* $a \geq 2$.

(2) $G \cong L_2(7)$ *with* $n = m^4$ *prime to 2,3,7,* $m \equiv 1 \bmod 4$, *and* $m^3 \equiv 1 \bmod 7$. *Further $C_V(G)$ is a subplane of order m, elements of order 2 or 3 in G are Baer elements, and $V = C_V(G) \oplus [V, G]$, where $[V, G]$ is a sum of 3-dimensional irreducible modules.*

The following is an application of Theorem A to the collineation groups of a translation plane.

Theorem C [6]. *Let G be a collineation group in the linear complement of a finite translation plane, which is identified with a vector space V over a field F with a spread. Suppose each non identity element in the centralizer of any involution i is an affine perspectivity or a Baer element if i is not the central homology, otherwise the zero vector is the only fixed point. Then one of the conclusions except (7) of Theorem A holds.*

Remark. Note that in a Hall plane of order q^2. There is a collineation group of order $q(q-1)$ which fixes the points of a Baer subplane.

2. Sketch of the proof of Theorem A.

Other notation and terminology in group theory is taken from [1, 3, 9, 14], and in the theory of translation planes, from [2, 10]. All objects considered here are of finite cardinalities.

For a set of non singular linear transformations X on a vector space W, we write $W(X)$ for $C_W(X)$.

First we assume the following Hypothesis.

Hypothesis Hy1. *Let V be a finite dimensional vector space over a finite field F of characteristic p and $G \leq GL(V)$. Let Γ be the set of subgroups H of G such that dim $V(h) = \delta = \delta(H)$ is a constant for all $h \in H^{\#}$.*

2.1. *If $p = 2$ and H is a 2-group, then H is elementary abelian.*

2.2. *If p is odd and H is a 2-group, then H is elementary abelian, cyclic, quaternion or dihedral.*

The next lemma treats the case in which $p = 2$.

2.3. *Assume Hy1. If $p = 2$ and $C_G(i) \in \Gamma$ for each involution i in G, then one of the following holds:*

(1) *G is of odd order.*

(2) *G is the extension of an elementary abelian 2-group N by a group X of odd order acting semiregularly on N with $F(X)$ and $X/F(X)$ cyclic.*
(3) $G \cong L_2(2^a)$ *for some* $a > 1$.
(4) $G = O(G) < t >$, *where t is an involution inverting the abelian group* $O(G)$.

Because of 2.3, we may assume from now on that in addition to Hypothesis Hy1, p is odd. For the next several lemmas we study a subgroup $H \in \Gamma$ such that $Z(H)$ has an involution i.

2.4. *H is a p'-group. If j is an involution in H but not in $< i >$ then either*
(1) $C_H(j)$ *is an elementary abelian Sylow 2-subgroup of H or*
(2) $C_H(j) =< i, j >\cong E_4$ *and H has dihedral Sylow 2-subgroups.*

2.5. *If $m_2(H) = 1$, then $V(i) = V(h)$ for each $h \in H^{\#}$. So H is a Frobenius complement semiregular on $[V, i]$.*

2.6. *If $m_2(H) > 2$, then H is an elementary abelian 2-group.*

2.7. *If $m_2(H) = 2$, then H is a dihedral group.*

2.8. *Assume $H = C_G(i)$, $m_2(G) = 1$, and $i \notin Z(G)$. Then either*
(1) *G is a Frobenius group with Kernel $O_p(G)$ and complement H, or*
(2) $G = O(G) < i >$, *where $O(G)$ is abelian and is inverted by i.*

Because of 2.3, 2.5 and 2.8, which say one of the conclusions (4), (5), (6) of Theorem A holds when $m_2(G) = 1$, we may assume in the rest that the following Hypothesis holds.

Hypothesis Hy2. *In addition to Hypothesis Hy1, we assume that G is of even order, p is odd, $m_2(G) > 1$, and $C_G(j) \in \Gamma$ for each involution $j \in G$.*

2.9. *If Hypothesis Hy2 holds, then one of the following holds:*
(1) *G is a split extension of an elementary abelian 2-group N by a group X of odd order acting semiregularly on N. Further, $F(X)$ and $X/F(X)$ are cyclic.*
(2) $G \cong L_2(2^a)$ *for some* $a \geq 2$.
(3) *G is a dihedral group.*
(4) $G \cong L_2(t)$ *or* $PGL_2(t)$ *with t odd.*

In the rest of the proof, we study the structure of the modules of the groups listed in conclusion (4) of 3.11 and show that conclusions (7) or (8) of Theorem A holds. The proof of Theorem A is then complete.

3. Collineations and proofs of Theorems B and C.

We now consider collineations in the translation complement of a translation plane, which is identified with a vector space V of dimension $2d$ over a field F together with a spread $\mathcal{S}$.

For a subset W of V, we define $\mathcal{S}(W) := \{X \in \mathcal{S} : |X \cap W| > 1\}$, and $\mathcal{S}_W := \{X \cap W : X \in \mathcal{S}(W)\}$.

A collineation which is in a linear transformation is called a linear collineation. The set of fixed points of a collineation carries tremendous information. We generalize some of the results concerning the set of fixed points to the eigenspaces.

3.1. Proposition. *Suppose W is an eigenspace of a linear collineation τ. Then any fiber intersecting W non trivially is τ invariant. An eigenspace is either a subplane or is contained in a fiber.*

3.2. Theorem. *Suppose τ is a linear collineation. Assume $V = U + W$, where U, W are eigenspaces of τ with different eigenvalues. Then either U, W both are fibers, or they are both Baer subplanes and $\mathcal{S}(W) = \mathcal{S}(U)$.*

We use 3.1 and 3.2, to prove the next three lemmas concern translation planes of odd order. These results are then used to proof of Theorems B and C.

3.3. *If σ_1 and σ_2 are two distinct involutions in an elementary abelian group S of order 4 such that each involution is Baer. then the following conclusions hold.*

1. $V(S) = V(\sigma_1) \cap V(\sigma_2)$ *is a Baer subplane of* $V(\sigma_1)$, *and* $n = m^4$, *where* m^4 *is the order of the subplane* $V(S)$.
2. *The subspaces* $V(S)$, $[V(\sigma_1), \sigma_2] = C_{[V,\sigma_2]}(\sigma_1)$, $[V(\sigma_2), \sigma_1]$, $[V(\sigma_1\sigma_2), \sigma_1]$ *are subplanes of order* m, *and* $\mathcal{S}(V(S)) = (X)$ *for any subplane* X *from these four subplanes.*
3. $m \equiv 1$ *mod* 4

3.4. *Suppose V is a translation plane of odd order $n = q^d$, which is identified as a vector space over a field F of characteristic p. Let G be a collineation group in the linear complement, and $G \cong A_4$ or $G \cong S_4$ with $V(s) = V(s^2)$ for an element s of order 4. Let $Q := O_2(G)$. Then the following conclusions hold.*

1. $V = V(Q) \oplus [V, Q]$, *where* $U := [V, Q]$ *is a direct sum of 3-dimensional* Q-*irreducible modules. Involutions in* Q *are Baer. The subspaces* $V(Q)$, $U(\sigma)$ *for* $\sigma \in Q^{\#}$ *are subplanes of order* $n^{\frac{1}{4}}$ *such that* $\mathcal{S}(V(Q)) = \mathcal{S}(U(\sigma))$.
2. *If* $G \cong S_4$, *then* $V(Q) = V(G)$, *every element in* $G^{\#}$ *is Baer,* $p \neq 3$, *and* $[V, G]$ *is the direct sum of the irreducible modules described in 3.12. For* $g \in G^{\#}$, $U(g)$ *is a subplane with same order as* $V(Q)$ *and* $\mathcal{S}(U(g)) = \mathcal{S}(V(Q))$.

3.5. *Assume $G \cong L_2(t)$ with t odd and $t > 5$ is a collineation group of a translation plane V of odd order, then $t = 7$ and $p \neq 3$, 7. Further $n = m^4$ with $m \equiv 1$ mod 4, and $m^3 \equiv 1$ mod 7.*

We now apply Theorem A to prove Theorems B and C. An involution has two possibilities as a collineation of a finite translation plane of order $n = q^d$. It is either a perspectivity or a Baer element. The dimension of the set of fixed points is half the dimension of the underlying vector space V, except in the case in which it is the central homology, i.e., $-I$ on the vector space and n is odd.

The condition (on the linear group) that the dimension of the eigenspace corresponding to the eigenvalue 1 is a constant on the set of non identity elements of a centralizer of an involution becomes the following. If the involution is the central homology, then n is odd and each non identity in its centralizer acts fixed-point-freely on V. In this case G is a Frobenius compliment with $m_2(G) = 1$. If the involution is an affine perspectivity, then each non identity element is either an affine perspectivity or a Baer element.

We now prove Theorem B. In the rest of this proof, simple means non abelian simple. Being simple, G is in the linear complement of the collineation group. As a simple group, G does not contain any central homology. Thus perspectivities in G are affine perspectivities. If $char(F)$ is odd, then involutions in G are Baer by [5]. If $char(F) = 2$, then an involution is either a Baer involution or an elation with its axis a line incident with the zero vector. Thus in both cases the dimension of the fixed point space of an involution in G equals to half the dimension of the underlying vector space. Therefore the hypothesis of Theorem B implies that the dimension of fixed point space of each non identity element of the centralizer of an involution is a constant, namely, half the dimension of the underlying vector space. Theorem B follows from Theorem A and 3.4. Theorem C now follows from Theorems A and B.

4. Further development.

In a simple collineation group of a translation plane, an element h of order 4 acts on the Baer subplane fixed by h^2. The action of h on this subplane could be an involution or trivial. To classify a simple collineation group, thus it is natural to study first the case in which h induces the identity on the Baer subplane fixed by h^2. In a linear group this condition corresponds to the following condition: The dimension of the set of fixed points is a constant on the set of elements of order 2 or 4. We are able to improve the result in 2.1 and 2.2 in [7] to the following theorem.

4.1. Theorem. *Suppose the dimension of the set of fixed points is a constant on the set of elements of order 4 or 2 for a linear group G. Then a Sylow2-subgroup of G is a cyclic, an elementary abelian, a dihedral, a quaternion, or a semidihedral group. If the characteristic is 2, then a Sylow 2-subgroup is an elementary abelian.*

This enables us to classify the simple linear groups. Surprisingly, a semidihedral group cannot occur as a Sylow 2-subgroup of a simple collineation group. During my visit at the Kumamoto University, I was able to eliminate A_7 as a possibility for a linear group. I am working on the M_{11} currently. These and related results will be in the upcoming article [7].

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