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A GALOIS CORRESPONDENCE BETWEEN INTERMEDIATE SUBALGEBRAS AND EQUIVALENCE SUBRELATIONS

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1. Preparation

We assume that all von Neumann algebras in this paper have separable preduals.

Let $(X, \mathcal{B}, \mu)$ be a standard Borel space, we call that $\mathcal{R}$ is a discrete measured equivalence relation on $(X, \mathcal{B}, \mu)$ if $\mathcal{R}$ is an equivalence relation which is a Borel subset of $X \times X$ such that for almost all $x \in X$, the equivalent class

$$\mathcal{R}(x) := \{y \in X \ : \ (x, y) \in \mathcal{R}\}$$

is countable. For each countable group G which acts on (X, μ) as a Borel automorphism, we obtain a discrete measured equivalence relation $\mathcal{R}_G$ which is defined by the following:

$$\mathcal{R}_G := \{(x, gx) \ : \ x \in X, \ g \in G\}.$$

By [4, Theorem 1], any discrete measured equivalence relation $\mathcal{R}$ is equal to $\mathcal{R}_G$ for some countable group G.

Let $\mathcal{R} := \mathcal{R}_G$ be a discrete measured equivalence relation on (X, μ). We say that the measure μ is quasi-invariant for $\mathcal{R}$ if μ is quasi-invariant for G.

In the discussion that follows, we fix a discrete measured equivalence relation $\mathcal{R}_G$ on (X, μ), where μ is quasi-invariant.

We denote the full group of $\mathcal{R}$ by $[\mathcal{R}]$ and the groupoid of $\mathcal{R}$ by $[\mathcal{R}]_*$, i.e.,

$$\begin{aligned}
[\mathcal{R}] := \{\varphi : \varphi & \text{ is a bimeasurable nonsingular transformation on } X \\
& \text{such that } (x, \varphi(x)) \in \mathcal{R} \text{ up to a } \mu\text{-null set}\},\\
[\mathcal{R}]_* := \{\varphi : \varphi & \text{ is a bimeasurable nonsingular map from a measurable} \\
& \text{subset Dom}(\varphi) \text{ of } X \text{ onto a measurable subset Im}(\varphi) \text{ of } X \\
& \text{such that } (x, \varphi(x)) \in \mathcal{R} \text{ up to a } \mu\text{-null set}\}.
\end{aligned}$$

For each $\rho \in [\mathcal{R}]_*$, we write $\Gamma(\rho)$ for the graph of ρ :

$$\Gamma(\rho) := \{(x, \rho(x)) \mid x \in \mathrm{Dom}(\rho)\}.$$

The map π_l into the first coordinate is a projection from $\mathcal{R}$ into X(i.e., $\pi_l(x,y) = x$). The left counting measure μ_l of μ is defined by the following:

$$\mu_l(C) := \int_X |\pi_l^{-1}(x) \cap C| d\mu(x),$$

where $|\cdot|$ stands for the cardinality. We can also define the right counting measure μ_r on $\mathcal{R}$ by the projection into the second coordinate. Since $\mathcal{R}$ is a countable equivalence relation, μ_l and μ_r are equivalence. We write D_μ for the Radon–Nikodym derivative $d\mu_l/d\mu_r$.

For each $n \in \mathbf{N}$, we define a subset $\mathcal{R}^n$ of X^{n+1} by the following:

$$\mathcal{R}^n := \{(x_0, x_1, \ldots, x_n) \in X^{n+1} \ : \ x_i \in \mathcal{R}(x_0) \text{ for all } i\}.$$

By the same manner as μ_l on $\mathcal{R}^1 = \mathcal{R}$, we define a measure μ^{n+1} on $\mathcal{R}^n$.

If a Borel map σ from $\mathcal{R}^2$ to the one-dimensional torus $\mathbf{T}$ satisfies the followings for almost all (x, y, z, w) in $\mathcal{R}^3$, we call σ a normalized 2-cocycle on $\mathcal{R}$:

$$\sigma(x,y,z)\sigma(x,z,w) = \sigma(x,y,w)\sigma(y,z,w),$$
$$\sigma(x,y,z) = 1 \qquad \text{if two of } x, y, z \text{ are equal.}$$

Definition 1. (1) Let f be a Borel function on $\mathcal{R}$. We call f a left finite function if $D_\mu^{1/2} f$ is a finite function and f satisfies the following:

$$\sup_{(x,y)\in\mathcal{R}} \{|\{z : z \sim x, \ f(x,z) \neq 0\}| + |\{z : z \sim y, \ f(z,y) \neq 0\}|\} < \infty.$$

(2) We define a von Neumann algebra $W^*(\mathcal{R}, \sigma)$ which act on $L^2(\mathcal{R}, \mu_l)$ by the following:

$$W^*(\mathcal{R}, \sigma) := \{L^\sigma(f) : f \text{ is a left finite function}\}'',$$

where $L^\sigma(f)$ is defined by

$$\{L^\sigma(f)\xi\}(x,z) := \sum_{(y,x)\in\mathcal{R}} f(x,y)\xi(y,z)\sigma(x,y,z).$$

We regard $L^\infty(X, \mu)$ as functions on the diagonal of $\mathcal{R}$, and define a von Neumann subalgebra $W^*(X)$ of $W^*(\mathcal{R}, \sigma)$ by the following:

$$W^*(X) := \{L(a) : a \in L^\infty(X, \mu)\}'',$$

where $L(a)$ is defined by

$$\{L(a)\xi\}(x,z) := a(x)\xi(x,z).$$

By [5], for each element T in $W^*(\mathcal{R}, \sigma)$, there exists a square integrable function f_T on $\mathcal{R}$ such that

$$(T\xi)(x,z) = \sum_{y\sim x} f_T(x,y)\xi(y,z)\sigma(x,y,z)$$

for any $\xi \in L^2(\mathcal{R}, \mu_l)$. We denote T by $L^\sigma(f_T)$.

For each $L^\sigma(f)$, $L^\sigma(g) \in W^*(\mathcal{R}, \sigma)$, we have $L^\sigma(f)^* = L^\sigma(f^*)$ and $L^\sigma(f)L^\sigma(g) = L^\sigma(f * g)$, where f^* and $f * g$ are square integrable functions on $\mathcal{R}$ which are defined by

$$f^*(x,z) := D_\mu^{-1}(x,z)\overline{f(z,x)},$$
$$(f * g)(x,z) := \sum_{y \sim x} f(x,y)g(y,z)\sigma(x,y,z).$$

(3) Let M be a von Neumann algebra and A be a subalgebra of M. We call A is a Cartan subalgebra of M if A satisfies the following:

(i) A is maximal abelian in M,
(ii) A is regular in M, i.e., the normalizer

$$\mathcal{N}_M(A) := \{u \in M : u \text{ is unitary and } uAu^* = A\}$$

generates M,
(iii) there exists a faithful normal conditional expectation E_A from M onto A.

It is easy to check that $W^*(X)$ is a Cartan subalgebra of $W^*(\mathcal{R}, \sigma)$. Indeed, the conditional expectation E is defined by the restriction $\mathcal{R}$ to the diagonal:

$$E(L^\sigma(f)) := L^\sigma(f|_X).$$

Furthermore, by [5, Proposition 2.9], each element of full group $[\mathcal{R}]$ define a normalizer. So $W^*(X)$ is regular in $W^*(\mathcal{R}, \sigma)$.

Conversely, Feldman and Moore also show that each inclusion of a von Neumann algebra and a Cartan subalgebra arises from an equivalence relation and a 2-cocycle on it.

Theorem 2 ([5, Theorem 1]). *For each inclusion of a von Neumann algebra M and a Cartan subalgebra A of M, there exists a standard Borel space (X, μ) and a discrete measured equivalence relation $\mathcal{R}$ on X with a normalized 2-cocycle σ such that $(A \subseteq M)$ is isomorphic to $(W^*(X) \subseteq W^*(\mathcal{R}, \sigma))$.*

2. Main Theorem

Our main purpose is to characterize intermediate von Neumann subalgebras between an inclusion of a von Neumann algebra and a Cartan subalgebra. For this, we use the following proposition.

Proposition 3 ([6, Remark 2.4]). *Let $A \subseteq M$ be a von Neumann algebra and a Cartan subalgebra. For each $A \subseteq N \subseteq M$, the following assertions are equivalent.*

(1) *There exists a (unique) faithful normal conditional expectation from M onto N.*

(2) *A is also a Cartan subalgebra of N, i.e., there exists a subrelation $\mathcal{S}$ of $\mathcal{R}$. such that $N = W^*(\mathcal{S}, \sigma|_{\mathcal{S}})$.*

Indeed, if E_N^M is a faithful normal conditional expectation, from M onto N, then we have

$$\begin{aligned} N = E_N^M(M) &= E_N^M(\mathcal{N}_M(A)'') \\ &= E_N^M(\mathcal{N}_M(A))'' \qquad \text{(since } E_N^M \text{ is normal)} \\ &\subseteq \mathcal{N}_N(A)'' \subseteq N, \end{aligned}$$

and conclude $\mathcal{N}_N(A)'' = N$.

Conversely, for each subrelation $\mathcal{S}$ of $\mathcal{R}$, a conditional expectation from $W^*(\mathcal{R}, \sigma)$ onto $W^*(\mathcal{S}, \sigma|_{\mathcal{S}})$ is defined by restricting $\mathcal{R}$ to $\mathcal{S}$:

$$E(L^\sigma(f)) := L^\sigma(f|_{\mathcal{S}}).$$

By [2, Theorem 1.5.5], this is the unique conditional expectation.

Our main theorem is the following(cf. [8, Theorem 1.1]).

Theorem 4 ([1, Theorem 1.1]). *Let M be a von Neumann algebra and A be a Cartan subalgebra of M. If N is a von Neumann subalgebra of M such that $A \subseteq N \subseteq M$, then there exists a unique faithful normal conditional expectation from M onto N.*

So we get a "Galois correspondence" for a inclusion of a von Neumann algebra and a Cartan subalgebra.

Corollary 5 (cf. [3, Proposition 6.1]). *Suppose M is a von Neumann algebra with a Cartan subalgebra A of M such that $M = W^*(\mathcal{R}, \sigma)$ and $A = W^*(X)$, where $\mathcal{R}$ is an equivalence relation on (X, μ) with a 2-cocycle σ. Then there exists a bijective correspondence between the set of Borel subrelations $\mathcal{S}$ of $\mathcal{R}$ on (X, μ) and the set of von Neumann subalgebras N of M which contain A:*

$$N \mapsto \mathcal{S}_N \subseteq \mathcal{R}, \qquad \mathcal{S} \mapsto W^*(\mathcal{S}, \sigma|_{\mathcal{S}}) \subseteq M.$$

3. Proof of Main Theorem

This section will be devoted to the proof of the main theorem. In the discussion that follows, we fix a von Neumann algebra M and a Cartan subalgebra A of M with a equivalence relation $\mathcal{R}$ on (X, μ) such that

$$(A \subseteq M) = (W^*(X) \subseteq W^*(\mathcal{R}, \sigma)).$$

To prove our main theorem, we construct an equivalence subrelation of $\mathcal{R}$ for each intermediate subalgebra $A \subseteq N \subseteq M$.

Lemma 6. *There exists a countable subset $\{\rho_n\}_{n \in I}$ of $[\mathcal{R}]_*$ such that $\mathcal{R}$ is a disjoint union of graphs $\{\Gamma(\rho_n)\}_{n \in I}$ up to null sets.*

Proof. By [4, Theorem 1], there exists a countable group G of Borel automorphisms of X such that

$$\mathcal{R} = \mathcal{R}_G := \{(x, gx) : x \in X, g \in G\}.$$

Since G is countable, there exists $l \in \mathbf{N} \cup \{\infty\}$ such that $J := \{n \in \mathbf{Z} : |n| < l\}$ and

$$G = \{g_n : n \in J\}, \quad g_0 = \mathrm{id}, \quad g_{-n} = g_n^{-1} \text{ for each } n \in J.$$

For each $n \in J$, we define a Borel subsets E_n by the following:

$$E_n := \begin{cases} X, & n = 0, \\ \{x \in X : (x, g_n(x)) \notin \bigcup_{j=-n+1}^{n-1} \Gamma(g_j)\}, & n > 0, \\ \{x \in X : (x, g_n(x)) \notin \bigcup_{j=n+1}^{-n-1} \Gamma(g_j)\} = g_{-n}(E_{-n}), & n < 0. \end{cases}$$

Now, we may assume that X is a Borel subset of $[0,1]$. Let us denote by "$<$" the usual order on $[0,1]$. For each $n \in J$, we define a Borel subset F_n of $E_n \cap E_{-n}$ by the following:

$$F_n := \begin{cases} \{x \in E_n \cap E_{-n} : g_n(x) = g_{-n}(x) \text{ and } x < g_n(x)\}, & n \geqq 0, \\ \{x \in E_n \cap E_{-n} : g_n(x) = g_{-n}(x) \text{ and } x > g_n(x)\}, & n < 0. \end{cases}$$

By the definition of $\{F_n \subseteq E_n\}_{n \in J}$, we obtain that $\mathcal{R}$ is a disjoint union of $\{\Gamma(g_n|_{E_n \setminus F_n})\}_{n \in J}$ up to a μ_l-null set. We set $I := \{n \in J : \mu(E_n \setminus F_n) > 0\}$ and $\rho_n := g_n|_{E_n \setminus F_n}$ for each $n \in I$. Since $\rho_n(E_n \setminus F_n) = E_{-n} \setminus F_{-n}$ up to a μ-null set, we have $\rho_{-n} = \rho_n^{-1}$ for each $n \in I$ and $\mathcal{R} = \bigcup_{n \in I} \Gamma(\rho_n)$ up to null sets. By relabeling I, we get the conclusion. □

We denote the normalizing groupoid of A in M by $\mathcal{GN}_M(A)$, i.e.,

$$\mathcal{GN}_M(A) := \{v \text{ is a partial isometry of } M \text{ and satisfies } v^*v,\ vv^* \in A,\ vAv^* = Avv^*\}.$$

For each $n \in I$, we set $v_n := L^\sigma(D_\mu^{-1/2} \chi_{\Gamma(\rho_n)})$. It is easy to check that v_n is in $\mathcal{GN}_M(A)$.

Lemma 7. *For each $T \in \mathcal{N}_M(A)$, $T = \sum_{n \in I} E_A(Tv_n^*)v_n$ in the sense of the strong operator topology.*

Proof. For each $n \in I$, we define $T_n \in M$ by the following:

$$T_n := E_A(Tv_n^*)v_n.$$

Suppose $T = L^\sigma(f)$ and $T_n = L^\sigma(f_n)$. A direct computation shows that

$$f_n(x,y) = (\chi_{\Gamma(\rho_n)} f)(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in \Gamma(\rho_n), \\ 0, & \text{otherwise} \end{cases}$$

for almost all $(x,y) \in \mathcal{R}$. So we have $T_n = L(\chi_{F_n})T_n$, where $F_n := \{x \in \mathrm{Dom}(\rho_n) : f(x, \rho_n(x)) \neq 0\}$. Since T is in $\mathcal{N}_M(A)$, by [5, Proposition 2.9], f comes from the graph of an element of $[\mathcal{R}]$, i.e., $\{F_n\}_{n\in I}$ is a partition of X. So, we have

$$T\xi_0 = \sum_{n\in I} T_n \xi_0,$$

where ξ_0 is a characteristic function of the diagonal. On the other hand, since $\|\sum_{n=-k}^{k} T_n\| \leqq \|T\| = 1$ for each $k \in \mathbf{N}$, $\sum_{n=-k}^{k} T_n$ strongly converges to T . Indeed, suppose $\xi \in L^2(\mathcal{R}, \mu_l)$. Since ξ_0 is cyclic for M', for each $\varepsilon > 0$, there exists $T' \in M'$ such that $\|T'\xi_0 - \xi\| < \varepsilon/3$. By the above argument, there exists $n_0 \in \mathbf{N}$ such that

$$\|T'T\xi_0 - T' \sum_{n=-k}^{k} T_n\xi_0\| = \|TT'\xi_0 - \sum_{n=-k}^{k} T_n T'\xi_0\| < \frac{\varepsilon}{3}$$

for each $k > n_0$. So we have $\|\sum_{n=-k}^{k} T_n\xi - T\xi\| < \varepsilon$, and get the conclusion. □

The following lemma is crucial in our argument.

Lemma 8. *For each $v \in \mathcal{GN}_M(A)$, $E_A(Nv^*)$ is equal to $Avv^* \cap Nv^*$. In particular, $E_A(Sv^*)v$ is in N for each $S \in N$.*

Proof. It suffices to show $E_A(Nv^*) \subseteq Nv^*$. For each $S \in N$, we have

$$\begin{aligned} E_A(Sv^*) &\in \mathrm{conv}\{uSv^*u^* : u \text{ is unitary in } A\}^{-\,\mathrm{stg}} \\ &= \mathrm{conv}\{uSv^*vv^*u^* : u \text{ is unitary in } A\}^{-\,\mathrm{stg}} \\ &= \mathrm{conv}\{uSv^*u^*vv^* : u \text{ is unitary in } A\}^{-\,\mathrm{stg}} \quad (\text{since } vv^* \in A) \\ &\subseteq \mathrm{conv}\{Sv^* : S \text{ is in } N\}^{-\,\mathrm{stg}} \quad (\text{since } v^*u^*v \in N) \\ &= Nv^*. \end{aligned}$$

So we get the conclusion. □

By this lemma, for each $n \in I$, there exists a projection e_n in A such that $e_n \leqq v_n v_n^*$ and $Ae_n := E_A(Nv_n^*)$. For each e_n, we obtain a Borel subset E_n of $\mathrm{Dom}(\rho_n)$ such that $e_n = L(\chi_{E_n})$.

We define a subset $\mathcal{S}_0$ of $\mathcal{R}$ by the following:

$$\mathcal{S}_0 := \bigcup_{n\in I} \Gamma(\rho_n|_{E_n}).$$

Moreover, we define $\mathcal{S}$ as a subset of $\mathcal{R}$ which is constructed by $\Gamma(\rho_n|_{E_n})$'s, i.e.,

$$\mathcal{S} := \langle \mathcal{S}_0 \rangle = \bigcup_{k\geqq 1} \bigcup_{l_1,\dots,l_k\in I} F_{l_1,\dots,l_k},$$

$$F_{l_1,\dots,l_k} := \Gamma(\rho_{l_k}\rho_{l_{k-1}}\cdots\rho_{l_1}|_{E_{l_1}\cap\rho_{l_1}^{-1}(E_{l_2})\cap\cdots\cap\rho_{l_1}^{-1}\cdots\rho_{l_{k-1}}^{-1}(E_{l_k})}).$$

Lemma 9. *The subset $\mathcal{S}$ defined above is a Borel equivalence subrelation of $\mathcal{R}$.*

Proof. Since $\rho_l \in [\mathcal{R}]_*$ and E_l is a Borel subset of X for each $l \in I$, $\mathcal{S}$ is a Borel subset of $\mathcal{R}$. So it suffices to prove that $\mathcal{S}$ is an equivalence relation.

Since $\rho_0 = \mathrm{id}$ and $E_0 = X$ up to a μ-null set, $\mathcal{S}$ contains the diagonal D. If $(x,y) \in \mathcal{S}$, then there exist $l_1,\dots,l_k \in I$ such that $(x,y) \in F_{l_1,\dots,l_k}$. So we conclude that (y,x) is in $F_{-l_k,\dots,-l_1} \subseteq \mathcal{S}$. Finally, if (y,z) is also in $\mathcal{S}$, then $(y,z) \in F_{m_1,\dots,m_j}$ for some $m_1,\dots,m_j \in I$ and we get $(x,z) \in F_{l_1,\dots,l_k,m_1,\dots,m_j} \subseteq \mathcal{S}$. Therefore we complete the proof. □

Lemma 10. *The above subrelation $\mathcal{S}$ coincides with $\mathcal{S}_0$ up to a μ_l-null set, i.e., $\mu_l(\mathcal{S}\setminus\mathcal{S}_0) = 0$.*

Proof. If $\mu_l(\mathcal{S}\setminus\mathcal{S}_0) > 0$, then there exist $l_1,\dots,l_k \in I$ such that

$$\mu_l(F_{l_1,\dots,l_k}\setminus\mathcal{S}_0) > 0.$$

We set $F := F_{l_1,\dots,l_k}\setminus\mathcal{S}_0$ and define measurable functions $\{f_i\}_{i=1}^k$ on $\mathcal{R}$ and $w \in \mathcal{GN}_M(A)$ by the following:

$$f_i := D_\mu^{1/2}\chi_{\Gamma(\rho_{l_i}|_{\rho_{l_{i-1}}\cdots\rho_{l_1}(\pi_l(F))})},$$
$$w := L^\sigma(f_1 * \cdots * f_k).$$

It is easy to see that $\mathrm{supp}(f_1 * \cdots * f_k) = F$ and $E_A(wv_n^*e_n) = 0$ for each $n \in I$.

On the other hand, since $L^\sigma(f_i) \in Ae_{l_i}v_{l_i} \subseteq N$ for each $i = 1,\dots,k$, we get $w \in N$. In particular, by Lemma 8, $E_A(wv_n^*)e_n = E_A(wv_n^*)$ for each $n \in I$. So $E_A(wv_n^*) = 0$ for each $n \in I$. By Lemma 7, we obtain $w = 0$, i.e., $\mu_l(F) = 0$, a contradiction. Thus $\mu_l(\mathcal{S}\setminus\mathcal{S}_0) = 0$. □

Proposition 11. *The von Neumann subalgebra $W^*(\mathcal{S},\sigma|_\mathcal{S})$ of M is equal to N.*

Proof. We set $L := W^*(\mathcal{S},\sigma|_\mathcal{S})$.

$(L \subseteq N)$: It suffices to prove $\mathcal{N}_L(A) \subseteq N$. If $T \in \mathcal{N}_L(A)$, then, by Lemma 7

$$T = \sum_{n\in I} E_A(Tv_n^*)v_n$$

in the sense of the strong operator topology. Since each $E_A(Tv_n^*)v_n$ is in N, T also belongs to N.

$(N \subseteq L)$: If $L^\sigma(f) \in N \setminus L$, then we get $\mu_l(\mathrm{supp}(f) \cap (\mathcal{R} \setminus \mathcal{S})) > 0$ and

$$\begin{aligned} &\mu_l \left(\mathrm{supp}(f) \cap \bigcup_{n \in I} \Gamma(\rho_n|_{\mathrm{Dom}(\rho_n)\setminus E_n}) \right) \\ =&\mu_l \left(\mathrm{supp}(f) \cap (\mathcal{R} \setminus \mathcal{S}) \cap \bigcup_{n \in I} \Gamma(\rho_n) \right) \\ >&0. \end{aligned}$$

So there exists $n \in I$ such that $\mu_l(\mathrm{supp}(f) \cap \Gamma(\rho_n|_{\mathrm{Dom}(\rho_n)\setminus E_n})) > 0$. On the other hand, $E_A(L^\sigma(f)v_n^*)$ is of the form $L(h)$ for some $h \in L^\infty(X)$. A direct computation shows that $\mathrm{supp}(h)$ is equal to $\pi_l(\mathrm{supp}(f) \cap \Gamma(\rho_n))$. Since $\mu(\pi_l(\mathrm{supp}(f) \cap \Gamma(\rho_n|_{\mathrm{Dom}(\rho_n)\setminus E_n}))) > 0$, we obtain $L(h)(1 - e_n) \neq 0$, i.e., $E_A(L^\sigma(f)v_n^*) \notin Ae_n = E_A(Nv_n^*)$. So we get $L^\sigma(f) \notin N$, a contradiction. □

By the above proposition, we construct a subrelation for each intermediate subalgebra. Hence we have proved our main theorem.

We note that our construction of subrelations uses only the subalgebra and the original equivalence relation. It does not use the arguments given in [5, Section 3].

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