数理解析研究所講究録 1301 巻 2003 年 139-156

Intuitionistic Linear Logics with Communication Principle

Norihiro Kamide (上出 哲広 : 東京工業大学、研究生)
Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology,
Ookayama, Megro, Tokyo 152-8552, JAPAN.
kamide2@is.titech.ac.jp

September 10, 2002

Abstract

We prove the cut-elimination theorems, the completeness theorems (with respect to Kripke models), the relevance principle, the interpolation property, and a new property named the "communication principle" for some variants of intuitionistic linear logic, and propose computational and process-algebraic interpretations for these logics and properties.

1 Introduction

It is well-known that Girard's linear logic is useful for formalizing concurrent computation [9, 2, 3, 4, 17, 5]. In this paper, we consider a number of extensions (or modifications) obtained from modal intuitionistic linear logic by adding communication-merge rules and give computational interpretations for these logics.

The *communication-merge rules* [1] are as follows:

$$\frac{\Gamma \Rightarrow \quad \Delta \Rightarrow \gamma}{\Gamma, \Delta \Rightarrow \gamma}\ (\text{mix}),$$

$$\frac{\Gamma_1, \Sigma \Rightarrow \gamma \quad \Gamma_2, \Sigma \Rightarrow \gamma}{\Gamma_1, \Gamma_2, \Sigma \Rightarrow \gamma}\ (\text{mingle}), \qquad \frac{!\Gamma_1, \Sigma \Rightarrow \gamma \quad !\Gamma_2, \Sigma \Rightarrow \gamma}{!\Gamma_1, !\Gamma_2, \Sigma \Rightarrow \gamma}\ (!\text{mingle})$$

where $!\Gamma$ denotes the multiset $\{!\gamma \mid \gamma \in \Gamma\}$. The rules mix and !mingle are new rules introduced in this paper. The mix rule is regarded as an intuitionistic version of Girard's MIX in classical linear logic [9]. The mingle rule was originally introduced by Ohnishi and Matumoto in 1964 [16]. In the original form, Σ in (mingle) above is empty. The mingle of the present form was proposed by Hori et al. in 1994 [10]. Sequent calculi and the cut-elimination theorems for propositional substructural logics with mingle were also discussed by Avron in the 1980s, Hori et al. in 1994, Girard in 1987, and Kamide [12, 13, 15]. For a historical overview of mingle, see [12]. The cut-elimination theorem does not hold for the propositional intuitionistic linear logic with the original mingle, but holds for that with the mingle by Hori et al. [10]. The communication-merge rules proposed here are strictly weaker than the weakening rule in the sequent calculus LJ for the intuitionistic logic.

The motivation for introducing the communication-merge rules is that the rules can be used to formalize the "communication process". For example, (mix) intuitively means that Γ and Δ communicate [9]. A process algebraic interpretation of (mix) and (mingle) is "communication merge" (introduced by Milner) in concurrent computation. (mix) and (mingle) correspond to the

[1] This name is borrowed from process algebra with communication merge. See the next paragraph.

following two transition rules in an operational semantics of process algebra:

$$\frac{x \xrightarrow{v} \surd \quad y \xrightarrow{w} y'}{x||y \xrightarrow{\gamma(v,w)} y'} \qquad \frac{x \xrightarrow{v} \surd \quad y \xrightarrow{w} \surd}{x||y \xrightarrow{\gamma(v,w)} \surd}$$

respectively, where (1) $||$ is the merge operator and represents the parallel execution of two process terms x and y, (2) γ is a communication function for each pair of atomic actions v and w, (3) $x \xrightarrow{v} \surd$ expresses the ability of the process term x to terminate successfully by the execution of action v, and (4) $y \xrightarrow{w} y'$ expresses the ability of the process term y to evolve into process term y' by the execution of action w. For the details of the transition rules, see [8] and section 6 in the present paper. In section 6, we will discuss the correspondence between the transition rules and the communication-merge rules.

We introduce eight variants, having (mix), (mingle) or (!mingle), of the ?-free fragment ILL! of modal intuitionistic linear logic. These logics give us a good formalization of the "communication process" in concurrent computation. This is verified using a new characteristic property termed the communication principle and the relevance principle (or variable sharing property) for these logics. The communication principle and the relevance principle do not hold for ILL!.

Computational interpretations of linear logics have been proposed by many logicians and computer scientists [9, 2, 3, 4, 17, 5]. For example, Bellin and Scott [4] propose an interpretation for the π-calculus introduced by Milner et al., and a process-algebraic interpretation of positive linear logic was proposed by Dam [5]. Abramsky's "proofs-as-processes" paradigm [2, 3] produces some good interpretations. An example of these interpretations is Okada's interpretation [17] of concurrent process calculus based on ILL!. We consider, in this paper, a modified version of Okada's calculus, with the addition of the communication-merge rules.

This paper is organized as follows. In section 2, we introduce eight variants MILL (= ILL! − !weakening + !mingle), MILL_x (= MILL + mix), MILL_m (= MILL + mingle), MILL_c (= MILL + contraction), MILL_xm (= MILL + mix + mingle), MILL_xc, MILL_mc and MILL_xmc, and prove the cut-elimination theorems for these logics. Using these theorems, we can derive the fact that the eight logics are completely separable. In section 3, using the cut-elimination theorems, we prove the relevance principle for these eight logics, and the interpolation property for MILL_xmc and MILL_mc. In section 4, we introduce the communication principle, which expresses the property of communication in concurrent computation, and prove that the principle holds for the eight logics. The proof is based on the cut-elimination theorems. We also give a counter example of the principle for ILL!. In section 5, we give a computational interpretation for a subsystem CC of MILL_xm. In section 6, we consider the correspondence between a process algebra with communication merge and a subsystem of MILL_xm. In section 7, we introduce Hilbert-style axiomatizations for the eight logics and present Kriple-completeness of some logics.

In addition, we mention the related work [15] which is a classical analogue of the present work. The paper [15] introduces five variants of classical linear logic. These variants have the new structural rule:

$$\frac{!\Gamma_1, \Pi \Rightarrow \Sigma, ?\Delta_1 \quad !\Gamma_2, \Pi \Rightarrow \Sigma, ?\Delta_2}{!\Gamma_1, !\Gamma_2, \Pi \Rightarrow \Sigma, ?\Delta_1, ?\Delta_2} \ (!?\text{mingle})$$

and some logics presented have the classical versions of the mix and mingle rules:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \ (\text{mix}) \qquad \frac{\Gamma_1, \Pi \Rightarrow \Sigma, \Delta_1 \quad \Gamma_2, \Pi \Rightarrow \Sigma, \Delta_2}{\Gamma_1, \Gamma_2, \Pi \Rightarrow \Sigma, \Delta_1, \Delta_2} \ (\text{mingle}).$$

The cut-elimination theorems, the completeness theorems (with respect to Girard's phase models) and a characteristic property named the "mix-separation principle" are proved for these logics.

Prior to the detailed discussion, we introduce the language used in this paper. *Formulas* are constructed from propositional variables, the constants $\mathbf{1}, \mathbf{0}, \top$, and $\bot$, $\to$ (implication), $\wedge$ (conjunction), $*$ (fusion), $\vee$ (disjunction) and ! (modal operator called "of course"). We will follow the notation for the constants $\mathbf{1}, \mathbf{0}, \top$, and $\perp$ in [20], which differs from that in [9]. Furthermore, $\wedge, \vee$, and $*$ correspond to $\&, \oplus$, and $\otimes$ in [9]. Lower case letters $p, q,\ldots$ are used for propositional variables, lower case Greek letters $\alpha, \beta, \ldots$ are used for formulas, and Greek capital letters $\Gamma, \Delta, \ldots$ are used for finite (possibly empty) multisets of formulas. $!\Gamma$ denotes the multiset $\{!\gamma \mid \gamma \in \Gamma\}$. A *sequent* is an expression of the form $\Gamma \Rightarrow \Pi$, where Π consists at most of one formula. The symbol $\equiv$ is used to denote equality as sequences (or multisets) of symbols. We adopt the convention of association to the right for omitting parentheses. For example $(\alpha\to\beta\to\gamma)\to\beta\to\alpha\to\gamma \equiv (\alpha\to(\beta\to\gamma))\to(\beta\to(\alpha\to\gamma))$.

As all logics discussed in this paper are formulated as sequent calculi, we will occasionally identify a sequent calculus with the logic determined by it.

2 Variants of intuitionistic linear logic

We give a precise definition of the basic system MILL which is an extended propositional intuitionistic linear logic with !mingle. In the following definitions, γ in expression $\Gamma \Rightarrow \gamma$ for any Γ means empty or single formula.

Initial sequents of MILL are of the forms:

$$\alpha \Rightarrow \alpha, \qquad \Rightarrow \mathbf{1}, \qquad \mathbf{0} \Rightarrow, \qquad \Gamma \Rightarrow \top, \qquad \perp, \Delta \Rightarrow \alpha.$$

The rules of inferences of MILL are as follows:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \gamma}{\Gamma, \Sigma \Rightarrow \gamma}\ (\text{cut}),$$

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Sigma \Rightarrow \gamma}{\alpha\to\beta, \Gamma, \Sigma \Rightarrow \gamma}\ (\to\text{left}), \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\to\beta}\ (\to\text{right}),$$

$$\frac{\alpha, \beta, \Delta \Rightarrow \gamma}{\alpha * \beta, \Delta \Rightarrow \gamma}\ (*\text{left}), \qquad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha * \beta}\ (*\text{right}),$$

$$\frac{\alpha, \Delta \Rightarrow \gamma}{\alpha \wedge \beta, \Delta \Rightarrow \gamma}\ (\wedge\text{left1}), \qquad \frac{\beta, \Delta \Rightarrow \gamma}{\alpha \wedge \beta, \Delta \Rightarrow \gamma}\ (\wedge\text{left2}), \qquad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge\text{right}),$$

$$\frac{\alpha, \Delta \Rightarrow \gamma \quad \beta, \Delta \Rightarrow \gamma}{\alpha \vee \beta, \Delta \Rightarrow \gamma}\ (\vee\text{left}), \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee\text{right1}), \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee\text{right2}),$$

$$\frac{\Gamma \Rightarrow \gamma}{\mathbf{1}, \Gamma \Rightarrow \gamma}\ (\mathbf{1}\text{we}), \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \mathbf{0}}\ (\mathbf{0}\text{we}),$$

$$\frac{\alpha, \Delta \Rightarrow \gamma}{!\alpha, \Delta \Rightarrow \gamma}\ (!\text{left}), \qquad \frac{!\Gamma \Rightarrow \alpha}{!\Gamma \Rightarrow !\alpha}\ (!\text{right}),$$

$$\frac{!\alpha, !\alpha, \Gamma \Rightarrow \gamma}{!\alpha, \Gamma \Rightarrow \gamma}\ (!\text{co}), \qquad \frac{!\Gamma_1, \Sigma \Rightarrow \gamma \quad !\Gamma_2, \Sigma \Rightarrow \gamma}{!\Gamma_1, !\Gamma_2, \Sigma \Rightarrow \gamma}\ (!\text{mingle}).$$

Remark that exchange rule is omitted since we have agreed that the antecedents of the sequents in this system are multisets. Also remark that the ?-free fragment ILL! of the modal intuitionistic linear logic is obtained from MILL by adding the inference rule:

$$\frac{\Gamma \Rightarrow \gamma}{!\alpha, \Gamma \Rightarrow \gamma}\ (!\text{we}).$$

We consider the following structural rules:

$$\frac{\Gamma \Rightarrow \quad \Delta \Rightarrow \gamma}{\Gamma, \Delta \Rightarrow \gamma}\ (\text{mix}), \qquad \frac{\Gamma_1, \Sigma \Rightarrow \gamma \quad \Gamma_2, \Sigma \Rightarrow \gamma}{\Gamma_1, \Gamma_2, \Sigma \Rightarrow \gamma}\ (\text{mingle}), \qquad \frac{\alpha, \alpha, \Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma}\ (\text{co}).$$

We define the following systems:

$\mathrm{MILL_m}$ = MILL + (mingle),
$\mathrm{MILL_x}$ = MILL + (mix),
$\mathrm{MILL_c}$ = MILL + (co),
$\mathrm{MILL_{mc}}$ = MILL + (mingle) + (co),
$\mathrm{MILL_{xm}}$ = MILL + (mix) + (mingle),
$\mathrm{MILL_{xc}}$ = MILL + (mix) + (co),
$\mathrm{MILL_{xmc}}$ = MILL + (mix) + (mingle) + (co).

(We use any combination of suffixes x, m and c to denote the systems obtained from MILL by adding structural rules corresponding to these suffixes.) They shed a new light upon the study on linear logics in computer science.

Next we prove the cut-elimination theorems for the logics defined above. Before the proof, we introduce some notions. A rule R is said to be *admissible* in a system L if the following condition is satisfied: for any instance

$$\frac{S_1 \quad S_2}{S}$$

of R, if S_i $(i \in \{1,2\})$ is provable in L, then S is provable in L. Moreover, R is said to be *derivable* in L if there is a derivation from S_1 and S_2 to S in L. Note that derivability implies admissibility.

Theorem 2.1 (Cut-Elimination Theorem) *Let L be* MILL, $\mathrm{MILL_m}$, $\mathrm{MILL_x}$, $\mathrm{MILL_c}$, $\mathrm{MILL_{mc}}$, $\mathrm{MILL_{xm}}$, $\mathrm{MILL_{xc}}$ *or* $\mathrm{MILL_{xmc}}$. *The rule* (cut) *is admissible in cut-free L.*

Proof First, we consider the theorems for the logics without (co) (i.e., L is MILL, $\mathrm{MILL_x}$, $\mathrm{MILL_m}$ or $\mathrm{MILL_{xm}}$).

Let P be a proof

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Gamma \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\, R\\ \alpha, \Sigma \Rightarrow \gamma\end{array}}{\Gamma, \Sigma \Rightarrow \gamma}\ (\text{cut})$$

where Q and R are cut-free proofs in L. The *grade* g of the proof P is the length of the cut formula α. The *rank* r of the cut is the length of the proof P. We prove, by double induction on the grade and rank of this cut (i.e., by transfinite induction on $\omega \cdot g + r$), that there is a cut-free proof of $\Gamma, \Sigma \Rightarrow \gamma$ in L.

We distinguish cases according to the forms of the cut formula α. (Case 1): the cut formula α is not of the form $!\beta$, and (Case 2): the cut formula α is of the form $!\beta$.

Now we show (Case 1). We distinguish cases according to the forms of Q and R in the proof P. It is enough to show the following cases: the last inferences of R and Q are (mix), (mingle) and (!mingle). The other cases are similar to the cases in the propositional intuitionistic linear logic ILL. Here we show only the case for (mix). The case for (mingle) or (!mingle) is the same as that in [10].

(Case (mix)(1)): The last inference of Q is (mix). In this case, P is of the form

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, Q_1\\ \Gamma_1 \Rightarrow\end{array} \quad \begin{array}{c}\vdots\, Q_2\\ \Gamma_2 \Rightarrow \alpha\end{array}}{\Gamma_1, \Gamma_2 \Rightarrow \alpha}\ (\text{mix}) \quad \begin{array}{c}\vdots\, R\\ \alpha, \Sigma \Rightarrow \gamma\end{array}}{\Gamma_1, \Gamma_2, \Sigma \Rightarrow \gamma}\ (\text{cut}).$$

Then the required proof is obtained from the proof

$$\dfrac{\begin{array}{c}\vdots\, Q_1\\ \Gamma_1 \Rightarrow\end{array} \quad \dfrac{\begin{array}{c}\vdots\, Q_2\\ \Gamma_2 \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\, R\\ \alpha, \Sigma \Rightarrow \gamma\end{array}}{\Gamma_2, \Sigma \Rightarrow \gamma}\ (\text{cut})}{\Gamma_1, \Gamma_2, \Sigma \Rightarrow \gamma}\ (\text{mix})$$

by the induction hypothesis.

(Case (mix)(2-1)): The last inference of R is (mix), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Gamma \Rightarrow \alpha\end{array} \quad \dfrac{\begin{array}{c}\vdots\, R_1\\ \alpha, \Delta_1 \Rightarrow\end{array} \quad \begin{array}{c}\vdots\, R_2\\ \Delta_2 \Rightarrow \gamma\end{array}}{\alpha, \Delta_1, \Delta_2, \Rightarrow \gamma}\,(\text{mix})}{\Gamma, \Delta_1, \Delta_2 \Rightarrow \gamma}\,(\text{cut}).$$

Then the required proof is obtained from the proof

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, Q\\ \Gamma \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\, R_1\\ \alpha, \Delta_1 \Rightarrow\end{array}}{\Gamma, \Delta_1 \Rightarrow}\,(\text{cut}) \quad \begin{array}{c}\vdots\, R_2\\ \Delta_2 \Rightarrow \gamma\end{array}}{\Gamma, \Delta_1, \Delta_2 \Rightarrow \gamma}\,(\text{mix})$$

by the induction hypothesis.

(Case (mix)(2-2)): The last inference of R is (mix), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Gamma \Rightarrow \alpha\end{array} \quad \dfrac{\begin{array}{c}\vdots\, R_1\\ \Delta_1 \Rightarrow\end{array} \quad \begin{array}{c}\vdots\, R_2\\ \alpha, \Delta_2 \Rightarrow \gamma\end{array}}{\Delta_1, \alpha, \Delta_2, \Rightarrow \gamma}\,(\text{mix})}{\Delta_1, \Gamma, \Delta_2 \Rightarrow \gamma}\,(\text{cut}).$$

Then the required proof is obtained from the proof

$$\dfrac{\begin{array}{c}\vdots\, R_1\\ \Delta_1 \Rightarrow\end{array} \quad \dfrac{\begin{array}{c}\vdots\, Q\\ \Gamma \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\, R_2\\ \alpha, \Delta_2 \Rightarrow \gamma\end{array}}{\Gamma, \Delta_2, \Rightarrow \gamma}\,(\text{cut})}{\Delta_1, \Gamma, \Delta_2 \Rightarrow \gamma}\,(\text{mix})$$

by the induction hypothesis.

Next we consider (Case 2): the cut formula α is of the form $!\beta$. To consider the case, we introduce the following rule:

$$\dfrac{\Gamma \Rightarrow !\beta \quad \Delta \Rightarrow \gamma}{\Gamma, \Delta^{\star} \Rightarrow \gamma}\,(!\text{cut})$$

where Δ has at least one occurrence of $!\beta$, and $\Delta^{\star}$ is a multiset of formulas obtained from Δ by deleting at least one occurrence of $!\beta$. We also call this (!cut) cut. Then we consider the following proof P:

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Gamma \Rightarrow !\beta\end{array} \quad \begin{array}{c}\vdots\, R\\ \Delta \Rightarrow \gamma\end{array}}{\Gamma, \Delta^{\star} \Rightarrow \gamma}\,(!\text{cut})$$

where Q and R are cut-free proofs in L. The grade and the rank of this (!cut) are same definition of (cut). We distinguish cases according to the forms of Q and R. We consider the case: Q is the initial sequent $!\beta \Rightarrow !\beta$ or $\bot, \Delta \Rightarrow !\beta$. This case is obvious. We consider the cases: the last inferences of Q are (mix), (mingle), (co), (!mingle), (!co), (!left), ($\rightarrow$left), ($*$left), ($\wedge$left1), ($\wedge$left2), ($\vee$left) and (1we). These cases are straightforward. Here we show only the case for (mix).

(Case (mix)): The last inference of Q is (mix). In this case, P is of the form

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, Q_1\\ \Gamma_1 \Rightarrow\end{array} \quad \begin{array}{c}\vdots\, Q_2\\ \Gamma_2 \Rightarrow !\beta\end{array}}{\Gamma_1, \Gamma_2 \Rightarrow !\beta}\,(\text{mix}) \quad \begin{array}{c}\vdots\, R\\ \Delta \Rightarrow \gamma\end{array}}{\Gamma_1, \Gamma_2, \Delta^{\star} \Rightarrow \gamma}\,(!\text{cut})$$

where Δ has at least one occurrence of of $!\beta$, and $\Delta^\star$ is a multiset of formulas obtained from Δ by deleting at least one occurrence of $!\beta$. Then the required proof is obtained from the proof

$$\dfrac{\begin{array}{c}\vdots\; Q_1\\ \Gamma_1 \Rightarrow\end{array} \quad \dfrac{\begin{array}{c}\vdots\; Q_2\\ \Gamma_2 \Rightarrow !\beta\end{array} \quad \begin{array}{c}\vdots\; R\\ \Delta \Rightarrow \gamma\end{array}}{\Gamma_2, \Delta^\star \Rightarrow \gamma}\,(!\mathrm{cut})}{\Gamma_1, \Gamma_2, \Delta^\star \Rightarrow \gamma}\,(\mathrm{mix})$$

by the induction hypothesis.

The critical case is that the last inference of Q is (!right). Thus we consider the case for (!right) in the following. We consider the case that R is initial sequent $!\beta \Rightarrow !\beta$, $\Gamma \Rightarrow \top$ or $\bot, \Gamma \Rightarrow \gamma$ where Γ has at least one occurrence of $!\beta$. This case is obvious. We consider the cases that the last inferences of R are (mix), (mingle), (co), (!co), ($\rightarrow$left), ($\rightarrow$right), ($*$left), ($*$right), ($\wedge$left1), ($\wedge$left2), ($\wedge$right), ($\vee$right1), ($\vee$right2), ($\vee$left), (**1**we), (**0**we), (!left) and (!right). These cases are straightforward.

We show only (Case (mix)): the last inference of Q is (!right) and the last inference of R is (mix).

(Case (mix)(1)): P is of the form

$$\dfrac{\dfrac{\begin{array}{c}\vdots\; Q\\ !\Gamma \Rightarrow \beta\end{array}}{!\Gamma \Rightarrow !\beta}\,(!\mathrm{right}) \quad \dfrac{\begin{array}{c}\vdots\; R_1\\ \Delta_1 \Rightarrow\end{array} \quad \begin{array}{c}\vdots\; R_2\\ \Delta_2 \Rightarrow \gamma\end{array}}{\Delta_1, \Delta_2 \Rightarrow \gamma}\,(\mathrm{mix})}{!\Gamma, \Delta_1^\star, \Delta_2^\star \Rightarrow \gamma}\,(!\mathrm{cut})$$

where Δ_1 (and Δ_2) has at least one occurrence of $!\beta$, and $\Delta_1^\star$ (and $\Delta_2^\star$) is a multiset of formulas obtained from Δ_1 (and Δ_2 respectively) by deleting at least one occurrence of $!\beta$. Then the required proof is obtained from the proof

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\; Q\\ !\Gamma \Rightarrow \beta\end{array}}{!\Gamma \Rightarrow !\beta}\,(!\mathrm{right}) \quad \begin{array}{c}\vdots\; R_1\\ \Delta_1 \Rightarrow\end{array}}{!\Gamma, \Delta_1^\star \Rightarrow}\,(!\mathrm{cut}) \quad \dfrac{\dfrac{\begin{array}{c}\vdots\; Q\\ !\Gamma \Rightarrow \beta\end{array}}{!\Gamma \Rightarrow !\beta}\,(!\mathrm{right}) \quad \begin{array}{c}\vdots\; R_2\\ \Delta_2 \Rightarrow \gamma\end{array}}{!\Gamma, \Delta_2^\star \Rightarrow \gamma}\,(!\mathrm{cut})}{!\Gamma, \Delta_1^\star, !\Gamma, \Delta_2^\star \Rightarrow \gamma}\,(\mathrm{mix})\\ \vdots\;(!\mathrm{co})\\ !\Gamma, \Delta_1^\star, \Delta_2^\star \Rightarrow \gamma\end{array}$$

by the induction hypothesis.

(Case (mix)(2)): The lower sequent of the !cut is $!\Gamma, \Delta_1^\star, \Delta_2 \Rightarrow \gamma$. This case is straightforward.

(Case (mix)(3)): The lower sequent of the !cut is $!\Gamma, \Delta_1, \Delta_2^\star \Rightarrow \gamma$. This case is also straightforward.

Next we consider the theorems for the logics with (co) (i.e., L is $\mathrm{MILL_c}$, $\mathrm{MILL_{mc}}$, $\mathrm{MILL_{xc}}$ or $\mathrm{MILL_{xcm}}$). To consider the case, we introduce the following:

$$\dfrac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \gamma}{\Gamma, \Delta^\star \Rightarrow \gamma}\,(\text{multi-cut})$$

where Δ has at least one occurrence of α, and $\Delta^\star$ is a multiset of formulas obtained from Δ by deleting at least one occurrence of α. We can prove the theorems by using (multi-cut). ∎

By using Theorem 2.1, we can show the following.

Theorem 2.2 (Separation) *The systems* MILL, $\mathrm{MILL_m}$, $\mathrm{MILL_x}$, $\mathrm{MILL_c}$, $\mathrm{MILL_{mc}}$, $\mathrm{MILL_{xm}}$, $\mathrm{MILL_{xc}}$ *and* $\mathrm{MILL_{xmc}}$ *are completely separable. That is, we have the following:* MILL $\subset$ $\mathrm{MILL_x}$ $\subset$ $\mathrm{MILL_{xm}}$ $\subset$ $\mathrm{MILL_{xmc}}$, MILL $\subset$ $\mathrm{MILL_m}$ $\subset$ $\mathrm{MILL_{mc}}$ $\subset$ $\mathrm{MILL_{xmc}}$, MILL $\subset$ $\mathrm{MILL_c}$ $\subset$ $\mathrm{MILL_{xc}}$ $\subset$ $\mathrm{MILL_{xmc}}$, $\mathrm{MILL_m}$ $\subset$ $\mathrm{MILL_{xm}}$, $\mathrm{MILL_c}$ $\subset$ $\mathrm{MILL_{mc}}$, $\mathrm{MILL_x}$ $\subset$ $\mathrm{MILL_{xc}}$ *where* $\subset$ *denotes the proper inclusion between the sets of provable sequents.*

For example, let $S_1 \equiv \mathbf{0} \Rightarrow \mathbf{1}$, $S_2 \equiv p \Rightarrow p{\rightarrow}p$ and $S_3 \equiv p{\rightarrow}p{\rightarrow}q \Rightarrow p{\rightarrow}q$ where p and q are distinct propositional variables. Then S_1 is provable in $\mathrm{MILL_x}$ but not provable in $\mathrm{MILL_{mc}}$, S_2 is provable in $\mathrm{MILL_m}$ but not provable in $\mathrm{MILL_{xc}}$ and S_2 is provable in $\mathrm{MILL_c}$ but not provable in $\mathrm{MILL_{xm}}$. Remark that ILL! is not an extension of $\mathrm{MILL_x}$ or $\mathrm{MILL_m}$ since S_1 and S_2 are not provable in ILL!, but ILL! is an extension of MILL since (!mingle) is strictly weaker than (!we).

We also have the following using Theorem 2.1.

Theorem 2.3 (Conservativity) MILL *is a conservative extension of the (modal-free) propositional fragment* ILL *of the intuitionistic linear logic.*

3 Relevance principle and interpolation property

In the area of relevant logic (in philosophical motivation), the relevance principle (or variable sharing property) is important in formalizing "relevant implication" in pure human reasoning (see [1, 13]). (In section 5, we will present a computational meaning of the principle.) Here we prove that the relevance principle holds for the eight logics introduced in the previous section.

Before the precise discussion, we introduce the following notation. For a sequence Δ of formulas, $\mathrm{V}(\Delta)$ means the set of all propositional variables which occur in some formula in Δ. We say that a logic L has the *variable sharing property*, when for any sequent $\alpha \Rightarrow \beta$ without propositional constants, if $\alpha \Rightarrow \beta$ is provable in L then $\mathrm{V}(\alpha) \cap \mathrm{V}(\beta)$ is nonempty. It is clear by the definition that if a logic L has the variable sharing property and L' is weaker than L (i.e., every formula provable in L' is also provable in L), then L' has also the property.

Theorem 3.1 (Relevance Principle) *Let* L *be* MILL, $\mathrm{MILL_x}$, $\mathrm{MILL_c}$, $\mathrm{MILL_{mc}}$, $\mathrm{MILL_{xm}}$, $\mathrm{MILL_{xc}}$ *or* $\mathrm{MILL_{xmc}}$. L *has the variable sharing property.*

The theorem does not hold for ILL!. A counter example is $!p \Rightarrow q{\rightarrow}q$ where p and q are distinct propositional variables. We remark that the theorem does not hold for logics with (!we) or weakening in genaral.

To show the theorem, it is enough to prove the following.

Lemma 3.2 *For any sequent* $\Gamma \Rightarrow \alpha$ *with nonempty* Γ *and without propositional constants, if* $\Gamma \Rightarrow \alpha$ *is provable in* $\mathrm{MILL_{xmc}}$ *then* $\mathrm{V}(\Gamma) \cap \mathrm{V}(\alpha)$ *is nonempty.*

Proof Suppose that $\Gamma \Rightarrow \alpha$ satisfies the conditions in the theorem and is provable in $\mathrm{MILL_{xmc}}$. Then, it has a cut-free proof P, in which neither initial sequents for propositional constants nor rules for them appear. Thus, for any sequent $\Sigma \Rightarrow \Lambda$ in P, Λ is nonempty. This can be shown by checking that for any rule of inference in cut-free $\mathrm{MILL_{xmc}}$ without constants, if the upper sequent(s) have no empty conclusion, then the lower sequent has no empty conclusion. Further we remark that P has no application of the rule (mix), because P has no occurrence of the sequent which is of the form $\Sigma \Rightarrow$. We will show that there exists the *subtree* (of the proof-tree) in P starting from initial sequent(s) and ending at the sequent $\Gamma \Rightarrow \alpha$ such that every sequent $\Pi \Rightarrow \pi$ in the subtree contains at least two formulas (in other words, such subtree is obtained from the proof-tree of P by deleting every sequent of the form $\Rightarrow \delta$). The existence of the subtree can be shown by checking that for any rule of inference of cut-free $\mathrm{MILL_{xmc}}$ (without mix and the rules for the propositional constants), if the lower sequent contains at least two formulas then (at least one of) the upper sequent(s) contains at least two formulas. (Of course, the above does not hold in general when we have (!we) or weakening.) Now, let T be such subtree. For any sequent $\Pi \Rightarrow \pi$ in T containing at least two formulas, we say that $\langle \Pi_1; \Pi_2, \pi \rangle$ is a *partition* of $\Pi \Rightarrow \pi$ if the multiset union of Π_1 and Π_2 is equal to Π, and Π_1 is nonempty. By our assumption on $\Pi \Rightarrow \pi$, there exists at least one partition of $\Pi \Rightarrow \pi$. Moreover, we say that a partition $\langle \Pi_1; \Pi_2, \pi \rangle$ shares variables if $\mathrm{V}(\Pi_1) \cap \mathrm{V}(\Pi_2, \pi)$ is nonempty. Then, we can show the following.

For any sequent $\Pi \Rightarrow \pi$ in T, any partition of $\Pi \Rightarrow \pi$ shares variables. (In particular, taking $\Gamma \Rightarrow \alpha$ for $\Pi \Rightarrow \pi$, we have that the partition $\langle \Gamma; \alpha \rangle$ shares variables.)

We prove this by induction on T. In the following, we show this only when the last rule of inference in T is (mingle).

(Case (mingle)): Suppose that Π is (Ψ, Δ, Σ) and that T is of the form:

$$\dfrac{\begin{array}{cc} \vdots & \vdots \\ \Psi, \Sigma \Rightarrow \pi & \Delta, \Sigma \Rightarrow \pi \end{array}}{\Psi, \Delta, \Sigma \Rightarrow \pi} \ (\text{mingle}).$$

Let $\langle \Psi_1, \Delta_1, \Sigma_1 \ ; \Psi_2, \Delta_2, \Sigma_2, \pi \rangle$ be any partition of $\Psi, \Delta, \Sigma \Rightarrow \pi$, where the multiset union of Ψ_1 (Δ_1 or Σ_1) and Ψ_2 (Δ_2 or Σ_2) is equal to Ψ (Δ or Σ respectively). Then, we consider the following cases.

(1) Ψ_1 is nonempty (in this case, the left upper sequent of the mingle is in T),

(2) Δ_1 is nonempty (in this case, the right upper sequent of the mingle is in T),

(3) Σ_1 is nonempty (in this case, both the right upper sequent and the left upper sequent of the mingle are in T).

The case (1) can be treated symmetrically in the case (2). Also the case (3) can be treated similarly in the case (2). Thus, we consider only the case (2). By the hypothesis of induction, the following holds.

(*) the partition is $\langle \Delta_1, \Sigma_1; \Delta_2, \Sigma_2, \pi \rangle$, and $\mathrm{V}(\Delta_1, \Sigma_1) \cap \mathrm{V}(\Delta_2, \Sigma_2, \pi)$ is nonempty.

By (*), we can show that $\mathrm{V}(\Psi_1, \Delta_1, \Sigma_1) \cap \mathrm{V}(\Psi_2, \Delta_2, \Sigma_2, \pi)$ is nonempty. ∎

Next we show the interpolation theorem for $\mathrm{MILL_{xmc}}$ and $\mathrm{MILL_{mc}}$. The theorem for the other systems is unknown whether holds or not.

We introduce the following notation. For any given finite multiset Γ of formulas, we call a pair $\langle \Gamma_1; \Gamma_2 \rangle$ of (possibly empty) multisets of formulas Γ_1 and Γ_2, a partition of Γ, if the multiset union of Γ_1 and Γ_2 is equal to Γ.

Theorem 3.3 (Interpolation for $\mathbf{MILL_{xmc}}$ and $\mathbf{MILL_{mc}}$) *Let L be $\mathrm{MILL_{xmc}}$ or $\mathrm{MILL_{mc}}$. If a sequent $\alpha \Rightarrow \beta$ is provable in L then there exists a formula δ such that both $\alpha \Rightarrow \delta$ and $\delta \Rightarrow \beta$ are provable in L, and moreover that $\mathrm{V}(\delta) \subseteq \mathrm{V}(\alpha) \cap \mathrm{V}(\beta)$.*

To show the theorem, it is enough to prove the following.

Lemma 3.4 *Let L be $\mathrm{MILL_{xmc}}$ or $\mathrm{MILL_{mc}}$. Suppose that a sequent $\Gamma \Rightarrow \gamma$ is provable in L and that $\langle \Gamma_1; \Gamma_2 \rangle$ is any partition of Γ. Then there exists a formula δ such that both $\Gamma_1 \Rightarrow \delta$ and $\delta, \Gamma_2 \Rightarrow \gamma$ are provable in L, and moreover that $\mathrm{V}(\delta) \subseteq \mathrm{V}(\Gamma_1) \cap \mathrm{V}(\Gamma_2, \gamma)$.*

Proof Since the cut elimination theorem holds for L, we can take a cut-free proof P of $\Gamma \Rightarrow \gamma$. We prove this lemma by induction on P. We distinguish cases according to the last inference in P. We show only the following case.

(Case (mingle)): Suppose that Γ is $(\Lambda, \Delta, \Sigma)$ and that P is of the form:

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ \Lambda, \Sigma \Rightarrow \gamma & \Delta, \Sigma \Rightarrow \gamma\end{array}}{\Lambda, \Delta, \Sigma \Rightarrow \gamma}\ (\text{mingle}).$$

Let $\langle \Lambda_1, \Delta_1, \Sigma_1; \Lambda_2, \Delta_2, \Sigma_2 \rangle$ be any partition of Γ, where the multiset union of Λ_1 (Δ_1 or Σ_1) and Λ_2 (Δ_2 or Σ_2) is equal to Λ (Δ or Σ respectively). By the hypothesis of induction, there exist formulas ψ and ϕ such that

(1a) both $\Lambda_1, \Sigma_1 \Rightarrow \psi$ and $\psi, \Lambda_2, \Sigma_2 \Rightarrow \gamma$ are provable,

(1b) $\mathrm{V}(\psi) \subseteq \mathrm{V}(\Lambda_1, \Sigma_1) \cap \mathrm{V}(\Lambda_2, \Sigma_2, \gamma)$,

(2a) both $\Delta_1, \Sigma_1 \Rightarrow \phi$ and $\phi, \Delta_2, \Sigma_2 \Rightarrow \gamma$ are provable and

(2b) $\mathrm{V}(\phi) \subseteq \mathrm{V}(\Delta_1, \Sigma_1) \cap \mathrm{V}(\Delta_2, \Sigma_2, \gamma)$.

Then, by (1a) and (2a) we have that both $\Lambda_1, \Delta_1, \Sigma_1 \Rightarrow \psi * \phi$ and $\psi * \phi, \Lambda_2, \Delta_2, \Sigma_2 \Rightarrow \gamma$ are provable. The proofs are shown by the following.

$$\begin{array}{c}\dfrac{\Lambda_1, \Sigma_1 \Rightarrow \psi \quad \Delta_1, \Sigma_1 \Rightarrow \phi}{\Lambda_1, \Sigma_1, \Delta_1, \Sigma_1 \Rightarrow \psi * \phi}\ (*\text{right}) \\ \vdots\ (\text{co}) \\ \Lambda_1, \Delta_1, \Sigma_1 \Rightarrow \psi * \phi.\end{array}$$

$$\dfrac{\dfrac{\psi, \Lambda_2, \Sigma_2 \Rightarrow \gamma \quad \phi, \Delta_2, \Sigma_2 \Rightarrow \gamma}{\psi, \Lambda_2, \phi, \Delta_2, \Sigma_2 \Rightarrow \gamma}\ (\text{mingle})}{\psi * \phi, \Lambda_2, \Delta_2, \Sigma_2 \Rightarrow \gamma}\ (*\text{left}).$$

Moreover, by (1b) and (2b), we can show $\mathrm{V}(\psi * \phi) \subseteq \mathrm{V}(\Lambda_1, \Delta_1, \Sigma_1) \cap \mathrm{V}(\Lambda_2, \Delta_2, \Sigma_2, \gamma)$. Thus, $\psi * \phi$ is an interpolant. ∎

4 Communication principle

For any multiset Γ, we say that $\langle \Gamma_1; \Gamma_2 \rangle$ is an *nonempty partition* of Γ if the multiset union of Γ_1 and Γ_2 is equal to Γ and both Γ_1 and Γ_2 are nonempty.

Theorem 4.1 (Communication Principle) *Let L be* MILL, MILL$_{\mathrm{m}}$, MILL$_{\mathrm{x}}$, MILL$_{\mathrm{c}}$, MILL$_{\mathrm{mc}}$, MILL$_{\mathrm{xm}}$, MILL$_{\mathrm{xc}}$ *or* MILL$_{\mathrm{xmc}}$. *For any $\{constants, \rightarrow, \vee\}$-free formulas α, β and any $\{constants, \rightarrow, \vee, *\}$-free formula γ, if $\alpha * \beta \Rightarrow \gamma$ is provable in L then both $\alpha \Rightarrow \gamma$ and $\beta \Rightarrow \gamma$ are also provable in L.*

An intuitive meaning of the principle is that a trivial process communication is separable (or executable in parallel), where the trivial process corresponds to $\{constants, \rightarrow, \vee\}$-free formula, which is not a theorem. In other words, an important (or a context sensitive) process communication is difficult to separate (or to execute in parallel). Another intuitive interpretation of the principle will be discussed in section 5.

This theorem does not hold for ILL!. A counter example is $!p * q \Rightarrow q$ where p and q are distinct propositional variables. We remark that the theorem does not hold for logics with (!we) or weakening in general. Hence the principle is a characteristic property of logics with (mix), (mingle) or (!mingle).

To prove this theorem, it is enough to prove the following generalized form.

Theorem 4.2 (Generalized Communication Principle) *Let L be* MILL, MILL$_{\mathrm{m}}$, MILL$_{\mathrm{x}}$, MILL$_{\mathrm{c}}$, MILL$_{\mathrm{mc}}$, MILL$_{\mathrm{xm}}$, MILL$_{\mathrm{xc}}$ *or* MILL$_{\mathrm{xmc}}$. *Let Γ be a multiset of $\{constants, \rightarrow, \vee\}$-free formulas and γ be a $\{constants, \rightarrow, \vee, *\}$-free formula. Suppose that $\Gamma \Rightarrow \gamma$ is provable in L and $\langle \Gamma_1; \Gamma_2 \rangle$ is any nonempty partition of Γ. Then both $\Gamma_1 \Rightarrow \gamma$ and $\Gamma_2 \Rightarrow \gamma$ are provable in L.*

Proof Since the cut-elimination theorem holds for L, we can take a cut-free proof P of $\Gamma \Rightarrow \gamma$. We prove this theorem by induction on P. By the cut-elimination theorem, the subformula property holds for L. Thus we consider only when the last rule of inference in P is (mingle), (!mingle), (co), (!co), (!left), (!right), ($*$left), ($\wedge$left1), ($\wedge$left2) or ($\wedge$right). Of course we don't have to consider the case "$\Gamma \Rightarrow \gamma$ is $\gamma \Rightarrow \gamma$" because in the considering any nonempty partition $\langle \Gamma_1; \Gamma_2 \rangle$ of Γ, Γ_1 and Γ_2 are nonempty. Also we don't have to consider the case that the last rule of inference in P is (mix). Because in the considering any sequent $\Gamma' \Rightarrow \gamma'$ in P, the succedent γ' is nonempty since the propositional constants are not occuring in $\Gamma' \Rightarrow \gamma'$ (by the subformula property).

In the following, we show only the following case.

(Case (!mingle)): The last inference of P is (!mingle). Suppose that Γ is $(!\Lambda, !\Delta, \Sigma)$ and that P is of the form:

$$\dfrac{\begin{array}{c}\vdots\\ !\Lambda, \Sigma \Rightarrow \gamma\end{array} \quad \begin{array}{c}\vdots\\ !\Delta, \Sigma \Rightarrow \gamma\end{array}}{!\Lambda, !\Delta, \Sigma \Rightarrow \gamma.}\ (!\mathrm{mingle})$$

Let $\langle !\Lambda_1, !\Delta_1, \Sigma_1; !\Lambda_2, !\Delta_2, \Sigma_2 \rangle$ be any nonempty partition of Γ (both $!\Lambda_1, !\Delta_1, \Sigma_1$ and $!\Lambda_2, !\Delta_2, \Sigma_2$ are nonempty), where the multiset union of $!\Lambda_1$ ($!\Delta_1$ or Σ_1) and $!\Lambda_2$ ($!\Delta_2$ or Σ_2) is equal to $!\Lambda$ ($!\Delta$ or Σ respectively).

We show that both $!\Lambda_1, !\Delta_1, \Sigma_1 \Rightarrow \gamma$ and $!\Lambda_2, !\Delta_2, \Sigma_2 \Rightarrow \gamma$ are provable in L. To show this, we must consider the following cases.

(1) $(!\Lambda_1, \Sigma_1) \neq \emptyset$, $(!\Lambda_2, \Sigma_2) \neq \emptyset$, $(!\Delta_1, \Sigma_1) \neq \emptyset$, $(!\Delta_2, \Sigma_2) \neq \emptyset$,

(2) $(!\Lambda_1, \Sigma_1) \neq \emptyset$, $(!\Lambda_2, \Sigma_2) \neq \emptyset$, $(!\Delta_1, \Sigma_1) \neq \emptyset$, $(!\Delta_2, \Sigma_2) \equiv \emptyset$,

(3) $(!\Lambda_1, \Sigma_1) \neq \emptyset$, $(!\Lambda_2, \Sigma_2) \neq \emptyset$, $(!\Delta_1, \Sigma_1) \equiv \emptyset$, $(!\Delta_2, \Sigma_2) \neq \emptyset$,

(4) $(!\Lambda_1, \Sigma_1) \neq \emptyset$, $(!\Lambda_2, \Sigma_2) \equiv \emptyset$, $(!\Delta_1, \Sigma_1) \neq \emptyset$, $(!\Delta_2, \Sigma_2) \neq \emptyset$,

(5) $(!\Lambda_1, \Sigma_1) \equiv \emptyset$, $(!\Lambda_2, \Sigma_2) \neq \emptyset$, $(!\Delta_1, \Sigma_1) \neq \emptyset$, $(!\Delta_2, \Sigma_2) \neq \emptyset$,

(6) $(!\Lambda_1, \Sigma_1) \neq \emptyset$, $(!\Lambda_2, \Sigma_2) \neq \emptyset$, $(!\Delta_1, \Sigma_1) \equiv \emptyset$, $(!\Delta_2, \Sigma_2) \equiv \emptyset$,

(7) $(!\Lambda_1, \Sigma_1) \neq \emptyset$, $(!\Lambda_2, \Sigma_2) \equiv \emptyset$, $(!\Delta_1, \Sigma_1) \equiv \emptyset$, $(!\Delta_2, \Sigma_2) \neq \emptyset$,

(8) $(!\Lambda_1, \Sigma_1) \equiv \emptyset$, $(!\Lambda_2, \Sigma_2) \neq \emptyset$, $(!\Delta_1, \Sigma_1) \neq \emptyset$, $(!\Delta_2, \Sigma_2) \equiv \emptyset$,

(9) $(!\Lambda_1, \Sigma_1) \equiv \emptyset$, $(!\Lambda_2, \Sigma_2) \equiv \emptyset$, $(!\Delta_1, \Sigma_1) \neq \emptyset$, $(!\Delta_2, \Sigma_2) \neq \emptyset$.

It is enough to prove the cases (1), (2), (3), (6) and (7), because (2) ((3), (6) or (7)) is symmetric to (5) ((4), (9) or (8) respectively).

We consider (1). By the hypothesis of induction, we get the following.

(1a) both $!\Lambda_1, \Sigma_1 \Rightarrow \gamma$ and $!\Lambda_2, \Sigma_2 \Rightarrow \gamma$ are provable in L,

(1b) both $!\Delta_1, \Sigma_1 \Rightarrow \gamma$ and $!\Delta_2, \Sigma_2 \Rightarrow \gamma$ are provable in L.

Then, by using (!mingle) we can show that both $!\Lambda_1, !\Delta_1, \Sigma_1 \Rightarrow \gamma$ and $!\Lambda_2, !\Delta_2, \Sigma_2 \Rightarrow \gamma$ are provable in L.

We consider (2). In this case, P is of the form:

$$\dfrac{\begin{array}{c}\vdots\\ !\Lambda_1, !\Lambda_2, \Sigma_1 \Rightarrow \gamma\end{array} \quad \begin{array}{c}\vdots\\ !\Delta_1, \Sigma_1 \Rightarrow \gamma\end{array}}{!\Lambda_1, !\Lambda_2, !\Delta_1, \Sigma_1 \Rightarrow \gamma}\ (!\mathrm{mingle}).$$

By the hypothesis of induction, we get the following.

(2a) both $!\Lambda_1, \Sigma_1 \Rightarrow \gamma$ and $!\Lambda_2 \Rightarrow \gamma$ are provable in L.

Then we can show that both $!\Lambda_1, !\Delta_1, \Sigma_1 \Rightarrow \gamma$ and $!\Lambda_2 \Rightarrow \gamma$ are provable in L. By (2a), $!\Lambda_2 \Rightarrow \gamma$ is provable in L. By (2a) and the hypothesis that $!\Delta_1, \Sigma_1 \Rightarrow \gamma$ is provable in L, we get

$$\dfrac{\begin{array}{c}\vdots\\ !\Lambda_1, \Sigma_1 \Rightarrow \gamma\end{array} \quad \begin{array}{c}\vdots\\ !\Delta_1, \Sigma_1 \Rightarrow \gamma\end{array}}{!\Lambda_1, !\Delta_1, \Sigma_1 \Rightarrow \gamma}\ (!\text{mingle}).$$

We consider (3). In this case, P is of the form:

$$\dfrac{\begin{array}{c}\vdots\\ !\Lambda_1, !\Lambda_2, \Sigma_2 \Rightarrow \gamma\end{array} \quad \begin{array}{c}\vdots\\ !\Delta_2, \Sigma_2 \Rightarrow \gamma\end{array}}{!\Lambda_1, !\Lambda_2, !\Delta_2, \Sigma_2 \Rightarrow \gamma}\ (!\text{mingle}).$$

By the hypothesis of induction, we get the following.

(3a) both $!\Lambda_1 \Rightarrow \gamma$ and $!\Lambda_2, \Sigma_2 \Rightarrow \gamma$ are provable in L.

Then we can show that both $!\Lambda_1 \Rightarrow \gamma$ and $!\Lambda_2, !\Delta_2, \Sigma_2 \Rightarrow \gamma$ are provable in L. By (3a), $!\Lambda_1 \Rightarrow \gamma$ is provable in L. By (3a) and the hypothesis that $!\Delta_2, \Sigma_2 \Rightarrow \gamma$ is provable in L, we get

$$\dfrac{\begin{array}{c}\vdots\\ !\Lambda_2, \Sigma_2 \Rightarrow \gamma\end{array} \quad \begin{array}{c}\vdots\\ !\Delta_2, \Sigma_2 \Rightarrow \gamma\end{array}}{!\Lambda_2, !\Delta_2, \Sigma_2 \Rightarrow \gamma}\ (!\text{mingle}).$$

We consider (6). In this case, P is of the form:

$$\dfrac{\begin{array}{c}\vdots\\ !\Lambda_1, !\Lambda_2 \Rightarrow \gamma\end{array} \quad \begin{array}{c}\vdots\\ \Rightarrow \gamma\end{array}}{!\Lambda_1, !\Lambda_2 \Rightarrow \gamma}\ (!\text{mingle}).$$

By the hypothesis of induction, we have that both $!\Lambda_1 \Rightarrow \gamma$ and $!\Lambda_2 \Rightarrow \gamma$ are provable in L.

We consider (7). In this case, P is of the form:

$$\dfrac{\begin{array}{c}\vdots\\ !\Lambda_1 \Rightarrow \gamma\end{array} \quad \begin{array}{c}\vdots\\ !\Delta_2 \Rightarrow \gamma\end{array}}{!\Lambda_1, !\Delta_2 \Rightarrow \gamma}\ (!\text{mingle}).$$

By the hypothesis, we have that both $!\Lambda_1 \Rightarrow \gamma$ and $!\Delta_2 \Rightarrow \gamma$ are provable in L. ∎

5 Computational interpretation

In this section, we present an intuitive interpretation in concurrent computation for a subsystem of a linear logic with communication-merge rules. This interpretation is a modified version of Okada's paradigm [17].

We consider the following informal correspondence between logical notions and notions from concurrency theory. A logical connective is an action name. A logical (complex) formula is a process or an action (or specification). A propositional variable is a token or a message. A finite multiset of formulas is a specification of a process schedule. An inference rule (on a logical connective) represents state transition (by the corresponding action).

We consider the following interpretation. (1) $\alpha * \beta$ (parallel action): processes α and β are started in parallel. (2) $p * \beta$ (sending action): token (or message) p is sent, and process β is started at the same time. (3) $p \rightarrow \beta$ (receiving action): token p is received, and process β is started. (4) $\alpha \wedge \beta$ (choice action): process α (or β) is chosen, and the chosen process is started. (5) $!\alpha$ (duplication action): as many copies of α as needed are produced, and a copy α is started.

Next we introduce a calculus named *communication calculus* CC.

Initial sequents of CC are of the form: $p \Rightarrow p$.

The rules of inferences of CC are as follows:

$$\frac{\alpha,\beta,\Delta \Rightarrow \gamma}{\alpha * \beta,\Delta \Rightarrow \gamma}\ \text{(parallel)},\quad \frac{p,\beta,\Delta \Rightarrow \gamma}{p * \beta,\Delta \Rightarrow \gamma}\ \text{(sending)},\quad \frac{\beta,\Gamma \Rightarrow \gamma}{p, p \rightarrow \beta,\Gamma \Rightarrow \gamma}\ \text{(receiving)},$$

$$\frac{\alpha,\Delta \Rightarrow \gamma}{\alpha \wedge \beta,\Delta \Rightarrow \gamma}\ \text{(choice 1)},\quad \frac{\beta,\Delta \Rightarrow \gamma}{\alpha \wedge \beta,\Delta \Rightarrow \gamma}\ \text{(choice 2)},\quad \frac{\alpha, !\alpha,\Delta \Rightarrow \gamma}{!\alpha,\Delta \Rightarrow \gamma}\ \text{(duplication)},$$

$$\frac{\Gamma_1 \Rightarrow \quad \Gamma_2 \Rightarrow \gamma}{\Gamma_1,\Gamma_2 \Rightarrow \gamma}\ \text{(communication 1)},\quad \frac{\Gamma_1,\Sigma \Rightarrow \gamma \quad \Gamma_2,\Sigma \Rightarrow \gamma}{\Gamma_1,\Gamma_2,\Sigma \Rightarrow \gamma}\ \text{(communication 2)}.$$

An interpretation of the rules defined above is given as follows. Each action corresponds to a logical inference, by reading each logical inference rule bottom-up. (parallel): parallel-action $\alpha * \beta$ invokes two processes α and β in parallel. Here, Δ represents a finite multiset of processes in the environment. (sending): sending-action $p*\beta$ sends a token p and invokes subprocess β. (receieving): receiving-action $p \rightarrow \beta$ receives a token p and invokes β (when p exists in the environment). (choice 1, 2): choice-action $\alpha \wedge \beta$ chooses either α or β, and invokes it. (duplication): $!\alpha$ produces a copy α and invokes it. The above interpretation of the rules is from Okada [17]. (communication 1, 2): parallel specification (or environment) (Γ_1, Γ_2) can be separated, and Γ_1 and Γ_2 are executed in parallel. These two communication rules represent communication environment (or context) splitting.

Some examples of figures generated by CC are given below. The following two different figures in CC represent two different process schedules.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{r \Rightarrow r \quad r \Rightarrow r}{r, r \Rightarrow r}\ \mathrm{S}}{r, q \wedge r \Rightarrow r}\ \mathrm{R}}{r, p \wedge q \wedge r \Rightarrow r}\ \mathrm{R}}{q \wedge r, p \wedge q \wedge r \Rightarrow r}\ \mathrm{R}}{p \wedge q \wedge r, p \wedge q \wedge r \Rightarrow r}\ \mathrm{R}}{(p \wedge q \wedge r) * (p \wedge q \wedge r) \Rightarrow r}\ \mathrm{Q}}{p, p \rightarrow (q \wedge q \wedge r) * (p \wedge q \wedge r) \Rightarrow r}\ \mathrm{P} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{r \Rightarrow r}{q \wedge r \Rightarrow r}\ \mathrm{R}}{p \wedge q \wedge r \Rightarrow r}\ \mathrm{R} \quad \dfrac{\dfrac{r \Rightarrow r}{q \wedge r \Rightarrow r}\ \mathrm{R}}{p \wedge q \wedge r \Rightarrow r}\ \mathrm{R}}{p \wedge q \wedge r, p \wedge q \wedge r \Rightarrow r}\ \mathrm{S}}{(p \wedge q \wedge r) * (p \wedge q \wedge r) \Rightarrow r}\ \mathrm{Q}}{p, p \rightarrow (p \wedge q \wedge r) * (p \wedge q \wedge r) \Rightarrow r}\ \mathrm{P}$$

The left figure corresponds to the *sequential* process schedule PQRRRRS. Here, P, Q, R and S are (receiving), (parallel), (choice 1) and (communication 2), respectively. The right figure corresponds to the *concurrent* process schedule PQS((RR)||(RR)), where (RR)||(RR) means RR and RR are performed in parallel. PQS((RR)||(RR)) is a concurrent execution schedule of the sequential PQRRRRS, and represents a faster computation than that of PQRRRRS.

A figure with at least one infinite path is interpreated as an infinite (reactive) process schedule in CC. A finite figure, in which all processes end in initial sequents (i.e., the figure is a proof of CC), is interpreated as a failed process (abort process or dead-lock process) schedule in CC. For example, the process schedule PQRRRRS is a failed process. For any multiset Γ, the sequent $\Gamma \Rightarrow$ expresses the ability of process specification Γ to terminate successfully, and for any Γ and γ, the sequent $\Gamma \Rightarrow \gamma$ expresses the ability of process specification Γ to evolve into process γ. We also call this expression $\Gamma \Rightarrow \gamma$ specification.

We then have the following.

Observation 5.1 (Non Existence of Dead-lock Process Schedule) *For any specification Γ, there is no dead-lock process schedule of $\Gamma \Rightarrow$ in CC.*

Proof Suppose Γ is any multiset of formulas. By Theorem 2.1 (Cut-Elimination for $\mathrm{MILL_{xm}}$), the sequent $\Gamma \Rightarrow$ is not provable in the $\{\wedge, *, \rightarrow, !\}$-fragment of $\mathrm{MILL_{xm}}$. This can be shown by checking that for any rule of inference in the (cut-free part of) $\{\wedge, *, \rightarrow, !\}$-fragment of $\mathrm{MILL_{xm}}$, if none of the upper sequent(s) have empty conclusions, then the lower sequent has a non-empty conclusion. Then $\Gamma \Rightarrow$ is also unprovable in CC because CC is a subsystem of the $\{\wedge, *, \rightarrow, !\}$-fragment of $\mathrm{MILL_{xm}}$. Thus, $\Gamma \Rightarrow$ has no proof in CC. This means that there is no dead-lock process schedule of Γ in CC. ∎

The relevance principle is interpreted as follows.

Observation 5.2 (Computational Relevance Principle) *For any specifications α and β, if α and β have no message p in common, then both the specifications $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$ have no dead-lock process schedule in* CC.

The communication principle is interpreted as follows.

Observation 5.3 (Computational Communication Principle) *If for any reciving-action-free processes α, β and γ, either $\alpha \Rightarrow \gamma$ or $\beta \Rightarrow \gamma$ has no dead-lock process schedule in in* CC, *then the specification $\alpha * \beta \Rightarrow \gamma$ also has no dead-lock process schedule in* CC.

6 Process algebra with communication merge

In this section, we introduce an operational semantics for a process algebra with communication merge [8], and show that the operational semantics is analogous to a subsystem of $\mathrm{MILL_{xm}}$.

Prior to the detailed discussion, we introduce the language used in this paper for the process algebra with communication merge. Process terms are constructed from atomic actions $a, b, c \cdots$, $+$ (alternative composition operator) and $||$ (communication merge operator). We assume the set A of atomic actions, and the set P of process terms. $x + y$ represents the choice execution of two process terms x and y. $x||y$ represents the parallel execution of two process terms x and y. We also assume a communication function γ: $A \times A \longrightarrow A$, which produces for each pair of atomic actions v and w their communication $\gamma(v, w)$. This communication function is required to be commutative and associative. The expression $y \xrightarrow{v} y'$ shows that process term y can evolve into process term y' by the execution of action v, and $x \xrightarrow{v} \surd$ expresses the ability of process term x to terminate successfully by the execution of action v.

We introduce the following transition rules of the operational semantics for the process algebra with communication merge.

$$v \xrightarrow{v} \surd$$

$$\frac{x \xrightarrow{v} \surd}{x + y \xrightarrow{v} \surd}\ (+\text{-}1) \qquad \frac{y \xrightarrow{v} \surd}{x + y \xrightarrow{v} \surd}\ (+\text{-}2) \qquad \frac{x \xrightarrow{v} x'}{x + y \xrightarrow{v} x'}\ (+\text{-}3) \qquad \frac{y \xrightarrow{v} y'}{x + y \xrightarrow{v} y'}\ (+\text{-}4)$$

$$\frac{x \xrightarrow{v} \surd}{x||y \xrightarrow{v} y}\ (||\text{-}1) \qquad \frac{y \xrightarrow{v} \surd}{x||y \xrightarrow{v} x}\ (||\text{-}2) \qquad \frac{x \xrightarrow{v} x'}{x||y \xrightarrow{v} x'||y}\ (||\text{-}3) \qquad \frac{y \xrightarrow{v} y'}{x||y \xrightarrow{v} x||y'}\ (||\text{-}4)$$

$$\frac{x \xrightarrow{v} \surd \quad y \xrightarrow{w} y'}{x||y \xrightarrow{\gamma(v,w)} y'}\ (||\text{-}5) \qquad \frac{x \xrightarrow{v} x' \quad y \xrightarrow{w} \surd}{x||y \xrightarrow{\gamma(v,w)} x'}\ (||\text{-}6)$$

$$\frac{x \xrightarrow{v} \surd \quad y \xrightarrow{w} \surd}{x||y \xrightarrow{\gamma(v,w)} \surd}\ (||\text{-}7) \qquad \frac{x \xrightarrow{v} x' \quad y \xrightarrow{w} y'}{x||y \xrightarrow{\gamma(v,w)} x'||y'}\ (||\text{-}8)$$

The metavariables x, x', y and y' in the transition rules defined above range over the set P and the metavariables v and w range over the set A.

We consider an analogy of the transition rules above and the inference rules in $\mathrm{MILL_{xm}}$. $v \xrightarrow{v} \surd$ corresponds to an initial sequent of the form $\alpha \Rightarrow \alpha$. The transition rules for + (i.e., (+-1), (+-2), (+-3) and (+-4)) correspond to the rules ($\wedge$left1) and ($\wedge$left2), where we take $\wedge$ for +. The eight transition rules for $||$ correspond to the rules ($*$left), ($*$right), (mix) and (mingle), where we take the logical connective $*$ and the comma "," (of antecidents of sequents) for $||$. The analogy for the rules for $||$ is explained as follows. For the operational semantics, we add the constant named empty action $\varepsilon \in A$ and the transition rule $y \xrightarrow{\varepsilon} y$. Then $y \xrightarrow{\varepsilon} y$ corresponds to an initial sequent of the form $\alpha \Rightarrow \alpha$. We also assume ε is the unit of the communication function γ (i.e., $\gamma(v,\varepsilon) = \gamma(\varepsilon, v) = v$ for all $v \in A$). Here, for example, we consider the transition rule (||-1). This becomes the following using (||-5).

$$\frac{x \xrightarrow{v} \surd \quad y \xrightarrow{\varepsilon} y}{x||y \xrightarrow{\gamma(v,\varepsilon)} y}\ (||\text{-}5)$$

where $\gamma(v,\varepsilon) = v$. The rules (||-2), (||-3) and (||-4) can be simulated in the same way. Thus, the four rules for $||$ without γ (i.e., (||-1), (||-2), (||-3) and (||-4)) are redundant. The remaining four rules for $||$ with γ (i.e., (||-5), (||-6), (||-7) and (||-8)) are similar forms for (mix), (mingle) and ($*$right), where the associativity and commutativity of γ correspond to that of the multiset union operation for anticedents of sequents. Furthermore, an expression $x \xrightarrow{a} y$ ($a \in A$, $x, y \in P$) is interpreted as the sequent $x \Rightarrow a * y$ (x and y are considered to be formulas and a is a propositional variable). For example, $((a + (b||b))||c)||d \xrightarrow{b} c||d$ (where a, b, c and d are atomic actions) is interpreted as the sequent $((a \wedge (b * b)) * c) * d \Rightarrow b * (c * d)$ (where a, b, c and d are considered to be propositional variables). In this example, the following two analogous figures are generated by the operational semantics and $\mathrm{MILL_{xm}}$.

$$\cfrac{\cfrac{\cfrac{\cfrac{b \xrightarrow{\varepsilon} b \quad b \xrightarrow{b} \surd}{b||b \xrightarrow{b} \surd}\ (||\text{-}6)}{a + (b||b) \xrightarrow{b} \surd}\ (+\text{-}1)}{(a + (b||b))||c \xrightarrow{b} c}\ (||\text{-}1)}{((a + (b||b))||c)||d \xrightarrow{b} c||d}\ (||\text{-}3)
\qquad
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{b \Rightarrow b \quad b \Rightarrow b}{b, b \Rightarrow b}}{b * b \Rightarrow b}}{a \wedge (b * b) \Rightarrow b} \quad \cfrac{c \Rightarrow c \quad d \Rightarrow d}{c, d \Rightarrow c * d}}{a \wedge (b * b), c, d \Rightarrow b * (c * d)}}{(a \wedge (b * b)) * c, d \Rightarrow b * (c * d)}}{((a \wedge (b * b)) * c) * d \Rightarrow b * (c * d)}$$

Thus we can conclude the following.

Observation 6.1 (Simulation) *The operational semantics of the process algebra with communication merge can be simulated in* $\mathrm{MILL_{xm}}$.

7 Hilbert-style systems and Kripke models

In this section, we introduce Hilbert-style systems HMILL, $\mathrm{HMILL_x}$, $\mathrm{HMILL_c}$, $\mathrm{HMILL_m}$, $\mathrm{HMILL_{xm}}$, $\mathrm{HMILL_{xc}}$, $\mathrm{HMILL_{mc}}$, $\mathrm{HMILL_{xmc}}$ and HILL! for the Gentzen-style sequent calculi MILL, $\mathrm{MILL_x}$, $\mathrm{MILL_c}$, $\mathrm{MILL_m}$, $\mathrm{MILL_{xm}}$, $\mathrm{MILL_{xc}}$, $\mathrm{MILL_{mc}}$, $\mathrm{MILL_{xmc}}$ and ILL! respectively, and introduce Kripke models for HMILL, $\mathrm{HMILL_c}$, $\mathrm{HMILL_m}$, $\mathrm{HMILL_{mc}}$, and HILL!. Further the completeness theorem (with respect to the Kripke models) for these five systems is presnted. This theorem can be proved the same way in [14]. This framework for these Kripke models and a canonical model construction method for non-modal propositional substructural logics was originally proposed in [6, 7, 19, 11]. Informational or process algebraic interpretations of the framework were discussed in [21, 22, 5].

The axiom schemes and inference rules for a Hilbert-style system HMILL of MILL are as follows.

Non-modal part [2]:

[2]This axiomatization for non-modal part is due to Ishihara [11], which axiomatization corresponds to that for ILL of non-modal propositional intuitionistic linear logic. Moreover note that the rules (pref), (suff), (or), (residu), (ness) and the axiom scheme A1 are redundant.

A1: $\mathbf{1}$,

A2: $\alpha\to\top$,

A3: $\bot\to\alpha$,

A4: $\alpha\to\alpha$,

A5: $\alpha\wedge\beta\to\alpha$,

A6: $\alpha\wedge\beta\to\beta$,

A7: $(\gamma\to\alpha)\wedge(\gamma\to\beta)\to\gamma\to\alpha\wedge\beta$,

A8: $\alpha\to\alpha\vee\beta$,

A9: $\beta\to\alpha\vee\beta$,

A10: $\alpha\to\beta\to\alpha*\beta$,

A11: $(\alpha\to\gamma)\wedge(\beta\to\gamma)\to\alpha\vee\beta\to\gamma$,

A12: $(\beta\to\gamma)\to(\alpha\to\beta)\to(\alpha\to\gamma)$,

A13: $(\mathbf{1}\to\alpha)\to\alpha$,

A14: $\alpha\to\bot\to\beta$,

A15: $\alpha\to\mathbf{1}\to\alpha$,

A16: $(\alpha\to\beta\to\gamma)\to\alpha*\beta\to\gamma$,

A17: $(\alpha\to\beta\to\gamma)\to\beta\to\alpha\to\gamma$,

$$\frac{\alpha}{\mathbf{1}\to\alpha}\ (\text{ness}),$$

$$\frac{\alpha\quad\alpha\to\beta}{\beta}\ (\text{mp}),\qquad \frac{\beta\to\gamma}{(\alpha\to\beta)\to(\alpha\to\gamma)}\ (\text{pref}),\qquad \frac{\alpha\to\beta}{(\beta\to\gamma)\to(\alpha\to\gamma)}\ (\text{suff}),$$

$$\frac{\alpha\quad\beta}{\alpha\wedge\beta}\ (\text{adj}),\qquad \frac{\alpha\to\gamma\quad\beta\to\gamma}{\alpha\vee\beta\to\gamma}\ (\text{or}),\qquad \frac{\alpha\to\beta\to\gamma}{\alpha*\beta\to\gamma}\ (\text{residu}),$$

(no axiom schemes and inference rules for $\mathbf{0}$).

Modal part [3]:

A18: $!(\alpha\to\beta)\to!\alpha\to!\beta$,

A19: $!\alpha\to\alpha$,

A20: $!\alpha\to!!\alpha$,

A21: $(!\alpha\to!\alpha\to\beta)\to!\alpha\to\beta$,

A22: $!\alpha\to!\alpha\to!\alpha$,

$$\frac{\alpha}{!\alpha}\ (\text{!ness}).$$

We consider the following axiom schemes:

B1: $(\alpha\to\alpha\to\beta)\to\alpha\to\beta$,

B2: $\alpha\to\alpha\to\alpha$,

B3: $\mathbf{0}\to\mathbf{1}$,

B4: $\alpha\to!\beta\to\alpha$.

We can define the following Hilbert-style systems for $\mathrm{MILL_c}$, $\mathrm{MILL_m}$, $\mathrm{MILL_x}$, $\mathrm{MILL_{mc}}$, $\mathrm{MILL_{xc}}$, $\mathrm{MILL_{xm}}$, $\mathrm{MILL_{xmc}}$ and ILL!.

[3]Note that A22 is new and differ from that in ILL!. See [20, 18].

$\mathrm{HMILL_c} = \mathrm{HMILL} + \mathrm{B1}$,

$\mathrm{HMILL_m} = \mathrm{HMILL} + \mathrm{B2}$,

$\mathrm{HMILL_x} = \mathrm{HMILL} + \mathrm{B3}$,

$\mathrm{HMILL_{mc}} = \mathrm{HMILL} + \mathrm{B1} + \mathrm{B2}$,

$\mathrm{HMILL_{xc}} = \mathrm{HMILL} + \mathrm{B1} + \mathrm{B3}$,

$\mathrm{HMILL_{xm}} = \mathrm{HMILL} + \mathrm{B2} + \mathrm{B3}$,

$\mathrm{HMILL_{xmc}} = \mathrm{HMILL} + \mathrm{B1} + \mathrm{B2} + \mathrm{B3}$,

$\mathrm{HILL!} = \mathrm{HMILL} + \mathrm{B4}$.

We can show equivalence between Hilbert-style axiomatizations and Gentzen-style systems. Here we consider slight modifications of the sequent calculi. Such calculi have the exchange rule, and moreover, in the sequent expression $\Gamma \Rightarrow \gamma$ of the calculi, Γ denotes sequence of formulas. We use the same names for these modified calculi. Let Γ be a sequence $\gamma_1, \cdots, \gamma_n$ of formulas. Then $\Gamma \to \alpha$ is defined by $\gamma_1 \to \cdots \to \gamma_n \to \alpha$ ($\Gamma \to \alpha \equiv \alpha$ when $n = 0$).

Theorem 7.1 (Equivalence) (1) *If* $\Gamma \Rightarrow \alpha$ *(or* $\Gamma \Rightarrow$*) is provable in* MILL_z ($z \in$ {null, x, m, c, xc, xm, mc, xmc}), *then* $\Gamma \to \alpha$ *(or* $\Gamma \to \mathbf{0}$ *respectively) is also provable in* HMILL_z. (2) *If* $\Gamma \to \alpha$ *is provable in* HMILL_z ($z \in$ {null, x, m, c, xc, xm, mc, xmc}), *then* $\Gamma \Rightarrow \alpha$ *is also provable in* MILL_z.

Of course, we also have the same result for ILL! and HILL!.

We remark that B1, B2, B3 and B4 respectively correspond to (co), (mingle), (mix) and (!we). For example, we can present that $\mathbf{0} \to \mathbf{1}$ corresponds to (mix). $\mathbf{0} \to \mathbf{1}$ is provable in $\mathrm{MILL_x}$ and

$$\dfrac{\dfrac{\Gamma \to \mathbf{0} \quad \mathbf{0} \to \mathbf{1}}{\Gamma \to \mathbf{1}}\ (\text{cut}) \quad \dfrac{\Delta \to \gamma}{\mathbf{1} \to \Delta \to \gamma}\ (\text{ness})}{\Gamma \to \Delta \to \gamma}\ (\text{cut})$$

where (cut) is derivable using (suff) and (mp).

Next we define Kripke models for HMILL_z ($z \in \{\mathrm{null}, \mathrm{c}, \mathrm{m}, \mathrm{mc}\}$) and HILL!.

A *Kripke frame* for HMILL is a structure $\langle M, \dagger, \cap, \cdot, \varepsilon, \omega \rangle$ satisfying the following conditions:

1. $\langle M, \cap \rangle$ is a meet-semilattice with the greatest element ω,
2. $\cdot$ is a binary operation on M and $\varepsilon \in M$ such that

   C1: $\varepsilon \cdot x = x$ for all $x \in M$,
   C2: $\omega \cdot x = \omega$ for all $x \in M$,
   C3: $x \leq y$ implies $z \cdot x \leq z \cdot y$ for all $x, y, z \in M$ (where the order relation $x \leq y$ is defined by $x \cap y = x$),
   C4: $(x \cap y) \cdot z = (x \cdot z) \cap (y \cdot z)$ for all $x, y, z \in M$,
   C5: $(x \cdot y) \cap (x \cdot z) \leq x \cdot (y \cap z)$ for all $x, y, z \in M$,
   C6: $x \cdot (y \cdot z) \leq (x \cdot y) \cdot z$ for all $x, y, z \in M$,
   C7: $x \cdot \varepsilon \leq x$ for all $x \in M$,
   C8: $\omega \leq x \cdot \omega$ for all $x \in M$,
   C9: $x \leq x \cdot \varepsilon$ for all $x \in M$,
   C10: $(x \cdot y) \cdot z \leq x \cdot (y \cdot z)$ for all $x, y, z \in M$,
   C11: $(x \cdot z) \cdot y \leq (x \cdot y) \cdot z$ for all $x, y, z \in M$,

3. $\dagger$ is a unary operation on M such that

C12: $\dagger\varepsilon \le \varepsilon$,
C13: $\dagger(x \cdot y) \le \dagger x \cdot \dagger y$ for all $x, y \in M$,
C14: $x \le \dagger x$ for all $x \in M$,
C15: $\dagger\dagger x \le \dagger x$ for all $x \in M$,
C16: $x \le y$ implies $\dagger x \le \dagger y$ for all $x, y \in M$,
C17: $(x \cdot \dagger y) \cdot \dagger y \le x \cdot \dagger y$ for all $x, y \in M$,
C18: $\dagger x \cap \dagger y \le \dagger x \cdot \dagger y$ for all $x, y \in M$.

The additional conditions on $\langle M, \dagger, \cap, \cdot, \varepsilon, \omega\rangle$

C19: $(x \cdot y) \cdot y \le x \cdot y$ for all $x, y \in M$,

C20: $x \cap y \le x \cdot y$ for all $x, y \in M$,

C21: $x \le x \cdot \dagger y$ for all $x, y \in M$

respectively correspond to B1, B2 and B4.

A *valuation* $\models$ on a Kripke frame $\langle M, \dagger, \cap, \cdot, \varepsilon, \omega\rangle$ is a mapping which assigns a *filter* of M (i.e., a nonempty subset X of M such that $x, y, \in X$ iff $x \cap y \in X$) to each propositional variable. We will write $x \models p$ for $x \in \models (p)$. Each valuation $\models$ can be extended to a mapping from the set of all formulas to the power set of M by

1. $x \models \mathbf{1}$ iff $\varepsilon \le x$,
2. $x \models \top$ for all $x \in M$,
3. $x \models \bot$ iff $x = \omega$,
4. $x \models \alpha{\rightarrow}\beta$ iff $x \cdot y \le z$ and $y \models \alpha$ imply $z \models \beta$ for all $y, z \in M$,
5. $x \models \alpha \wedge \beta$ iff $x \models \alpha$ and $x \models \beta$,
6. $x \models \alpha \vee \beta$ iff $y \models \alpha$ or $y \models \beta$, and $z \models \alpha$ or $z \models \beta$ for some $y, z \in M$ with $y \cap z \le x$,
7. $x \models \alpha * \beta$ iff $y \models \alpha$ and $z \models \beta$ for some $y, z \in M$ with $y \cdot z \le x$,
8. $x \models\, !\alpha$ iff $y \models \alpha$ for some y with $\dagger y \le x$.

Let $\models$ be a valuation on a Kripke frame $\langle M, \dagger, \cap, \cdot, \varepsilon, \omega\rangle$. Then $\models (\alpha)$ is a filter for any formula α.

A *Kripke model* is a structure $\langle M, \dagger, \cap, \cdot, \varepsilon, \omega, \models\rangle$ such that

1. $\langle M, \dagger, \cap, \cdot, \varepsilon, \omega\rangle$ is a Kripke frame,
2. $\models$ is a valuation on $\langle M, \dagger, \cap, \cdot, \varepsilon, \omega\rangle$.

A formula α is *true* in a Kripke model $\langle M, \dagger, \cap, \cdot, \varepsilon, \omega, \models\rangle$ if $\varepsilon \models \alpha$, and *valid* in a Kripke frame $\langle M, \dagger, \cap, \cdot, \varepsilon, \omega\rangle$ if it is true for any valuation $\models$ on the Kripke frame.

Then we can prove the following using the same way in [14].

Theorem 7.2 (Completeness) *Let S be* HMILL$_z$ ($z \in$ {null, c, m, mc}) *or* HILL!. *Let C be a class of Kripke frames for S, $L(C) := \{\alpha \mid \alpha$ is valid in all frames of $C\}$ and $L := \{\alpha \mid \alpha$ is provable in $S\}$. Then $L = L(C)$.*

References

[1] A. R. Anderson and N.D.JR. Belnap, Entailment Vol.1, Princeton university press, 1975.

[2] S. Abramsky, Computational interpretations of linear logic, *Theoretical Computer Science* **111**, pp. 3–57, 1993.

[3] S. Abramsky, Proofs as processes, *Theoretical Computer Science* **135**, pp. 5–9, 1994.

[4] G. Bellin and P.J. Scott, On the π-calculus and linear logic, *Theoretical Computer Science* **135**, pp. 11–65, 1994.

[5] M. Dam, Process-algebraic interpretations of positive linear and relevant logics, *Journal of Logic and Computation* **4**, No. 6, pp. 939–973, 1994.

[6] K. Došen, Sequent systems and groupoid models. *I*, *Studia Logica* **47**, pp. 353–385, 1988.

[7] K. Došen, Sequent systems and groupoid models. *II*, *Studia Logica* **48**, pp. 41–65, 1988.

[8] W. Fokkink, Introduction to process algebra, Springer, Texts in Theoretical Computer Science, An EATCS Series, 2000.

[9] J-Y. Girard, Linear logic, *Theoretical Computer Science* **50**, pp. 1–102, 1987.

[10] R. Hori, H. Ono and H. Schellinx, Extending intuitionistic linear logic with knotted structural rules, *Notre Dame Journal of Formal Logic* **35**, pp. 219–242, 1994.

[11] H. Ishihara, A canonical model construction for substructural logics, *Journal of Universal Computer Science* **6**, pp. 155–168, 2000.

[12] N. Kamide, Substructural logics with mingle, *Journal of Logic, Language and Information* **11**, pp. 227–249, 2002.

[13] N. Kamide, Relevance principle for substructural logics with mingle and strong negation, *Journal of Logic and Computation* (to appear).

[14] N. Kamide, Kripke semantics for modal substructural logics, *Journal of Logic, Language and Information* **11**, pp. 453–470, 2002.

[15] N. Kamide, Classical linear logics with mix separation principle, *Mathematical Logic Quarterly* **49**, 2003 (to appear).

[16] M. Ohnishi and K. Matsumoto, A system for strict implication, *Annals of the Japan Association for Philosophy of Science* **2**, pp. 183–188, 1964.

[17] M. Okada. An introduction to linear logic: expressiveness and phase semantics, *MSJ Memoirs* **2**, pp. 255–295, 1998.

[18] H. Ono, Semantics for substructural logics, Substructural Logics (edited by K. Došen and P. Schroeder-Heister), Oxford University Press, pp. 259–291, 1993.

[19] H. Ono and Y. Komori, Logics without the contraction rule, *Journal of Symbolic Logic* **50**, pp. 169–201, 1985.

[20] A.S. Troelstra, Lectures on linear logic, CSLI Lecture Note 29, Stanford University, 1992.

[21] H. Wansing, Informational interpretation of substructural propositional logics, *Journal of Logic, Language and Information* **2**, pp. 285–308, 1993.

[22] H. Wansing, The logic of information structures, Lecture Notes in Artificial Intelligence 681,