数理解析研究所講究録 1301 巻 2003 年 55-68

Transducers as Discrete Twiners

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Abstract
Twiners on groupoids are used by the author to build a model of the second order linear logic. Discrete twiners, which are a special case of twiners with an additional condition, are employed to interpret variable types in this model.

In this note, we consider twiners on the category of automata. In particular, discrete twiners turn out to be a generalization of transducers. One may regard this as an evidence that twiners are available in various contexts of mathematics and theoretical computer science.

1 Category of automata

An automaton M over an alphabet A is a pair of a set $|M|$ of states and a transition function $\delta : |M| \times A^* \to |M|$. Whenever we say simply automata in this note, they mean deterministic automata, where the alphabet A and the set $|M|$ of states are allowed to be infinite. For the moment, we leave the matters of initial and final states. In other words, the transition function gives a left action of the monoid A^* on the set $|M|$.

Remark: In this note, against the convention in the automata theory, we write operations from left, as in ordinary mathematics. For example, the action given by transition is written from left: $\delta(q, u) = u \cdot q$

An automaton may be regarded as a functor. Let us regard the monoid A^* as the category that has only one object $\bullet$, and where morphisms have one-to-one correspondence to the words $u \in A^*$. An identity morphism is the empty word ε and composition of morphisms is simply concatenation of two words: $u \circ v = uv$. Then the data of an automaton $(|M|, \delta)$ provide a functor from category A^* into the large category **Set** of all sets: the functor carries the unique object $\bullet$ to the set $|M|$, and each morphism u in A^* to the endofunction on $|M|$ carrying $q \in |M|$ to $u \cdot q$.

In general, a functor from a category **C** into **Set** is called a *presheaf* on **C** [10]. So an automaton over A is exactly a presheaf on A^*. Let us start with a basic properties of presheaves.

Remark: We use the same symbol A^* as a monoid, as a category and, later, even as a presheaf. The intended usage will be clear from the context.

We denote by $\mathbf{Set}^{\mathbf{C}}$ the large category of all presheaves on $\mathbf{C}$ and all natural transformations. Adapting a definition in [2], we define the following.

1.1 Definition
A *covering* of a category $\mathbf{C}$ is a pair of a category $\mathbf{E}$ and a functor $\mathbf{E} \xrightarrow{\varphi} \mathbf{C}$ subject to the condition that, for every morphism $\varphi(X) \xrightarrow{h} y$ in $\mathbf{C}$, there is a unique pair of an object Y and a morphism $X \xrightarrow{f} Y$ in $\mathbf{E}$ such that $\varphi(f) = h$.

We let $\mathrm{Cov}(\mathbf{C})$ denote the large category of all coverings of $\mathbf{C}$, where a morphism $(\mathbf{E}, \varphi) \xrightarrow{\theta} (\mathbf{E}', \varphi')$ is a functor $\mathbf{E} \xrightarrow{\theta} \mathbf{E}'$ rendering the diagram

$$\begin{tikzcd} \mathbf{E} \arrow[rr, "\theta"] \arrow[dr, "\varphi"'] & & \mathbf{E}' \arrow[dl, "\varphi'"] \\ & \mathbf{B} & \end{tikzcd}$$

commutative. We note that the diagram is requested to strictly commute, not up to a natural isomorphism.

Another fundamental concept, which plays an important role in this note, is Grothendieck construction of presheaves [10]. The concept is defined in more general contexts, but we use it for presheaves only. In this case, a Grothendieck construction is called also a category of elements.

1.2 Definition
Let $\mathbf{C} \xrightarrow{F} \mathbf{Set}$ be a presheaf.

The *Grothendieck construction* of F is the category, denoted by $\int_{x \in \mathbf{C}} F(x)$ or $\mathrm{Gr}(F)$ in symbols, defined as follows: The objects are all pairs of an object $x \in \mathbf{C}$ and an element a of the set $F(x)$. A morphism $(x, a) \xrightarrow{h} (x', a')$ in $\mathrm{Gr}(F)$ is a morphism $x \xrightarrow{h} x'$ in $\mathbf{C}$ satisfying $F(h) : a \mapsto a'$.

We note that there is a canonical functor from $\int_{x \in \mathbf{C}} F(x)$ into $\mathbf{C}$ projecting on the first component of (x, a).

1.3 Proposition
Let $\mathbf{C}$ *be a category.*

Equivalence $\mathrm{Cov}(\mathbf{C}) \cong \mathbf{Set}^{\mathbf{C}}$ *between large categories holds*

(*Proof*) Given a covering $\mathbf{E} \xrightarrow{\varphi} \mathbf{C}$, we define presheaf F. The set $F(x)$ is $\varphi^{-1}(x)$, i.e., the collection of all objects $X \in \mathbf{E}$ satisfying $\varphi(X) = x$. Given a morphism $x \xrightarrow{h} y$ in $\mathbf{C}$, the function $F(h)$ is defined to carry each $X \in \varphi^{-1}(x)$ to the object $Y \in \varphi^{-1}(y)$ uniquely determined by the definition of covering. Conversely, given a presheaf $F \in \mathbf{Set}^{\mathbf{C}}$, the Grothendieck construction $\mathrm{Gr}(F)$ with the canonical projection gives a covering on $\mathbf{C}$. $\square$

We return to automata. Applying this proposition to an automaton $M \in \mathbf{Set}^{A^*}$, we obtain a covering $\int M$ of A^*. The objects of the covering are the states $q \in |M|$, if we omit writing the unique object in $(\bullet, q)$. A morphism is $q \xrightarrow{u} q'$ provided $\delta(u, q) = q'$ holds. Hence the covering $\int M$ is nothing but the presentation of M in the form of a labeled transition graph.

1.4 Definition (of presheaf A^*)
A presheaf A^* on category A^* is the functor $\mathrm{Hom}_{A^*}(\bullet, \text{-})$ represented by the unique object. Namely, the presheaf A^* carries $\bullet$ to the set A^*, endowed with left action defined by concatenation: $u \cdot v =_{def} uv$.

As a labeled transition graph, the automaton A^* is the full infinite tree with a single root, the successors of each state having one-to-one correspondence to the letters in A.

Given an arbitrary automaton M over A, the homset $\mathrm{Hom}(A^*, M)$ in $\mathbf{Set}^{A^*}$ is bijective to the set $|M|$ of states. This is an immediate consequence of the Yoneda lemma [9, 10]. Likewise, if we let κA^* denote direct sum of cardinality κ copies of A^*, the homset $\mathrm{Hom}(\kappa A^*, M)$ is bijective to the set of all κ-tuples of states of M. Later we show that the projective objects in $\mathbf{Set}^{A^*}$ are exactly of the form κA^*.

2 Twiners on automata

Twiners are introduced by the author to give a model of second order linear logic [6]. They are generalization of normal functors [4] and analytic functors [7, 8]. By definition, twiners are the pseudo-functors on groupoids which are equivalent to the 2-functors of the shape $\int_{x \in G} \mathrm{Hom}_{\mathbf{Gpoid}}(\varphi(x), \text{-})$. Here $\mathbf{Gpoid}$ is the 2-category of all groupoids, and G is a groupoid endowed with a pseudo-functor $G \xrightarrow{\varphi} \mathbf{Gpoid}$. There is a characterization of twiners via preservation of bicategorical universal properties.

In this section, we define twiners on categories of automata, and verify a characterization theorem similar to the one for twiners on groupoids.

A *finitely presentable* object in a category $\mathbf{C}$ is an object X subject to the condition that the representable functor $\mathrm{Hom}_{\mathbf{C}}(X, \text{-}) : \mathbf{C} \to \mathbf{Set}$ preserves filtered colimits [1]. A *finitely generated* object is defined likewise except that the representable functor is requested to preserve filtered colimits of the diagrams consisting of monomorphisms only. For instance, in the category of groups, the terminology coincides with the standard notion. Later we give a characterization of these types of objects in the category $\mathbf{Set}^{A^*}$ of automata. For the moment, however, these abstract properties are sufficient.

A *locally finitely presentable* category is such that every object is a filtered colimit of finitely presentable objects [1]. The category $\mathbf{Set}^{\mathbf{C}}$ of presheaves is locally finitely presentable. Indeed, finitely presentable presheaves in $\mathbf{Set}^{\mathbf{C}}$ are finite colimits of representable presheaves $\mathrm{Hom}_{\mathbf{C}}(x, \text{-})$. Employing the Yoneda lemma, every presheaf is a colimit of representable presheaves, thus a filtered colimit of finitely presentable ones.

We note that limits and colimits in $\mathbf{Set}^{\mathbf{C}}$ are pointwise. Namely, for a diagram F_i of presheaves, there is an isomorphism $(\varprojlim F_i)(x) \cong \varprojlim F_i(x)$ natural in $x \in \mathbf{C}$, and likewise for colimits.

A *locally continuous* functor $\mathbf{C} \xrightarrow{F} \mathbf{D}$ is such that, for each object X of $\mathbf{C}$. the induced functor $\mathbf{C}/X \xrightarrow{F} \mathbf{D}/F(X)$ between slice categories preserves all limits. An *accessible* functor F is a functor preserving all filtered colimits [1].

Now we define twiners between categories of automata. This is a straightforward reformulation of preservation of universal properties for normal functors [4] and for twiners [6].

2.1 Definition
A *twiner* $\mathbf{Set}^{A^*} \xrightarrow{F} \mathbf{Set}^{B^*}$ is a locally continuous, accessible functor.

Our first theorem is the following characterization theorem. Therein we identify automata as presheaves and their representations as coverings (i.e., as labeled transition graphs).

2.2 Theorem
Let $\mathbf{Set}^{A^*} \xrightarrow{F} \mathbf{Set}^{B^*}$ be a functor between categories of automata.

The following are equivalent

(i) *F is a twiner.*

(ii) *F is naturally isomorphic to a functor of the shape $\int_{x \in T} \mathrm{Hom}_{\mathbf{Set}^{A^*}}(\varphi(x), \text{-})$ for some covering $T \to B^*$, this category T endowed with a functor $T^{\mathrm{op}} \xrightarrow{\varphi} \mathbf{Set}^{A^*}$ satisfying that $\varphi(x)$ is finitely presentable for every object $x \in T$.*

(*Proof*) (i)$\Rightarrow$(ii). Forgetting the left action of B^*, F is regarded as a functor $\mathbf{Set}^{A^*} \to \mathbf{Set}$, which is still locally continuous and accessible since (co)limits of presheaves are pointwise. Then the Grothendieck construction $\mathrm{Gr}(F)$ satisfies the normal form property [5]: For each object (X, a), there is a morphism

$(Z, c) \xrightarrow{k} (X, a)$ that is an initial object in the slice category $\mathrm{Gr}(F)/(X, a)$. To prove this, we employ Freyd's adjoint functor theorem[9]. First $\mathrm{Gr}(F)/(X, a)$ has all limits since F is locally continuous. Moreover the collection of all isomorphism classes of $(Y, b) \to (X, a)$ with finitely presentable automaton $Y \in \mathbf{Set}^{A^*}$ forms the solution set, for F preserves filtered colimits and $\mathbf{Set}^{A^*}$ is locally finitely presentable. Therefore the category $\mathrm{Gr}(F)/(X, a)$ has an initial object $(Z, c) \xrightarrow{k} (X, a)$, which we call a normal form of (X, a).

Let us construct category T. The objects are the collection of all isomorphism classes of normal forms. A morphism is $(Z, c) \xrightarrow{u/k} (Z', c')$ for $u \in B^*$ and $Z' \xrightarrow{k} Z$ in $\mathbf{Set}^{A^*}$ provided $(Z', c') \xrightarrow{k} (Z,\ u \cdot c)$ is a normal form. This T is a covering of B^* by $u/k \mapsto u$. Moreover $T^{\mathrm{op}} \xrightarrow{\varphi} \mathbf{Set}^{A^*}$ is given by $(X, a) \mapsto X$ and $u/k \mapsto k$. If we define $\int_{(Z,c) \in T} \mathrm{Hom}_{\mathbf{Set}^{A^*}}(Z, Y) \xrightarrow{\theta_Y} FY$ by $(Z, c, f) \mapsto Ff(c)$, the normal form property implies that θ_Y is an isomorphism natural in Y. We must verify that θ_Y commutes also with left actions of B^*. For an arbitrary (Z, c, f), we have $u \cdot (Z, c, f) = (Z', c', fk)$ provided $(Z, c) \xrightarrow{u/k} (Z', c')$ is a morphism of T. Thus we conclude $\theta_Y(u \cdot (Z, c, f)) = F(fk)(c') = Ff(u \cdot c) = u \cdot Ff(c) = u \cdot \theta_Y((Z, c, f))$.

(ii)$\Rightarrow$(i). Since (co)limits of presheaves are pointwise, it suffices to show that $\sum_{x \in |T|} \mathrm{Hom}_{\mathbf{Set}^{A^*}}(\varphi(x), \text{-})$ is locally continuous and accessible. This is immediate, since each $\mathrm{Hom}_{\mathbf{Set}^{A^*}}(\varphi(x), \text{-})$ preserves all limits and all filtered colimits (the latter since $\varphi(z)$ is finitely presentable), and since the disjoint sum preserves all colimits and connected limits. $\square$

3 Discrete twiners and transducers

Twiners are still a little unwieldy, since finitely presentable automata can be complicated one. So we enforce a further condition.

A *projective* object in a category $\mathbf{C}$ is an object X such that the representable functor $\mathrm{Hom}_{\mathbf{C}}(X, \text{-})$ preserves epimorphisms. In other terms, X is projective iff, for each epimorphism $A \xrightarrow{e} B$ and each morphism $X \xrightarrow{g} B$ in $\mathbf{C}$, there is a factoring morphism $X \xrightarrow{f} A$ such that $ef = g$.

In general, a morphism $F \xrightarrow{f} G$ in category $\mathbf{Set}^{\mathbf{C}}$ is epi iff the function $f(x)$ is onto for every object x in $\mathbf{C}$. In particular, a morphism $M \xrightarrow{f} N$ between automata over A is epi iff the underlying function $|M| \xrightarrow{f} |N|$ between the set of states is onto.

Let us note that an automaton is isomorphic to κA^* for some cardinal κ iff every state q has a transition $q_0 \xrightarrow{u} q$ for unique $u \in A^*$ from a root q_0. Here we call a state q_0 a root if there is no transition $p \xrightarrow{u} q_0$ except the empty transition $q_0 \xrightarrow{\epsilon} q_0$.

3.1 Proposition
An object in $\mathbf{Set}^{A^*}$ *is projective iff it is isomorphic to a presheaf* κA^* *for some cardinal* κ.

(*Proof*) First κA^* is clearly projective since $\mathrm{Hom}(\kappa A^*, M)$ is the set of κ-tuples of states. Conversely, let us assume M is projective. We consider κA^* where κ is the cardinality of $|M|$. Identifying κ with $|M|$, the states of κA^* are written $\langle q, u\rangle$ with $q \in |M|$ and $u \in A^*$. Transitions have the form $\langle q, v\rangle \xrightarrow{u} \langle q, uv\rangle$. Let us consider an epimorphism $\kappa A^* \xrightarrow{e} M$ defined by $\langle q, v\rangle \mapsto v \cdot q$. By projectivity, there is a morphism $M \xrightarrow{f} \kappa A^*$ such that ef equals an identity on M. Let us denote the underlying function of f by $q \mapsto \langle \tilde{q}, v_q\rangle$. Since $ef = 1$, we must have $v_q \cdot \tilde{q} = q$. If there is no root q_0 with $q_0 \xrightarrow{u} q$, we have an infinite sequence $\cdots \xrightarrow{u_n} p_n \xrightarrow{u_{n-1}} p_{n-1} \longrightarrow \cdots \xrightarrow{u_1} p_1 \xrightarrow{u_0} q$ of transitions where none of p_i is a root and none of u_i is an empty word Then $v_q = u_0 u_1 \cdots u_{n-1} v_{p_n}$ for every n. Take n greater than the length of the word v_q. Contradiction. So there must be at least one transition $q_0 \xrightarrow{u} q$ from a root q_0. We suppose there are two transitions $q_0 \xrightarrow{u} q$ and $q_0' \xrightarrow{u'} q$ from roots q_0 and q_0'. For root q_0, in general, we must have $f(q_0) = \langle q_0, \varepsilon\rangle$ ($\because$ $\tilde{q}_0 \xrightarrow{v_{q_0}} q_0$ implies $v_{q_0} = \varepsilon$). Applying f to the two transitions above, we have $\langle q_0, \varepsilon\rangle \xrightarrow{u} \langle \tilde{q}, v_q\rangle$ and $\langle q_0', \varepsilon\rangle \xrightarrow{u'} \langle \tilde{q}, v_q\rangle$. Hence we must have $q_0 = \tilde{q} = q_0'$ and $u = v_q = u'$. So M is isomorphic to κA^* where κ is the number of roots. □

As an immediate corollary, an object in $\mathbf{Set}^{A^*}$ is both finitely presentable and projective iff it is isomorphic to nA^* where n is a finite cardinal.

3.2 Definition
A *discrete* twiner $\mathbf{Set}^{A^*} \xrightarrow{F} \mathbf{Set}^{B^*}$ is a twiner preserving epimorphisms.

3.3 Corollary
A discrete twiner $\mathbf{Set}^{A^*} \xrightarrow{F} \mathbf{Set}^{B^*}$ *is isomorphic to* $\int_{x \in T} \mathrm{Hom}_{\mathbf{Set}^{A^*}}(\varphi(x), \text{-})$ *where* $T^{\mathrm{op}} \xrightarrow{\varphi} \mathbf{Set}^{A^*}$ *carries every object* x *to a finitely presentable, projective object.*

Now we want to show that discrete twiners between automata are regarded as an extension of transducers. Let us start with definition. There we identify each non-negative integer n as the set $\{0, 1, \ldots, n-1\}$.

3.4 Definition
A *polyvalent transducer* of input alphabet A and output alphabet B is an automaton over B where each state is endowed with a non-negative integer. We let (z, n) denote the state z endowed with non-negative integer n. Moreover,

to each transition $z \xrightarrow{b} w$ with $z = (z, n)$, $w = (w, m)$, and a word $b \in B$ of length 1, two functions $m \xrightarrow{k} n$ and $m \xrightarrow{v} A^*$ are associated. We denote such a transition by $(z, n) \xrightarrow{b/k,v} (w, m)$.

Ordinary transducers are a special case of polyvalent transducers where all states are endowed with integer 1. Next we define cascade product, as for ordinary transducers.

3.5 Definition
Let T be a polyvalent transducer of input alphabet A and output alphabet B, and let M be an automaton over A.

The *cascade product* $M \circ T$ is the automaton over B defined as follows: The states are $(z, q_0, q_1, \ldots, q_{n-1})$ where (z, n) is a state of T and q_i are states of M. The transition is generated from all $(z, q_0, q_1, \ldots, q_{n-1}) \xrightarrow{b} (w, q'_0, q'_1, \ldots, q'_{m-1})$ where $(z, n) \xrightarrow{b/k,v} (w, m)$ is a transition of T and $q_{k_i} \xrightarrow{v_i} q'_i$ is a transition of M for each $i \in m$.

Graphically these definitions may be depicted as follows: For example, let us suppose $z = (z, 2)$, $w = (w, 3)$, and $3 \xrightarrow{k} 2$ carries $0, 1, 2$ to $1, 0, 1$ respectively. Then

$$\begin{tikzcd}
z \arrow[r, "b"] & w \\
q_0 \arrow[dr, "v_1"] & q'_0 \\
q_1 \arrow[ur, "v_0"] \arrow[dr, "v_2"'] & q'_1 \\
 & q'_2
\end{tikzcd}$$

is the transition of the cascade product $M \circ T$. Polyvalent transducers somehow support multiple outputs. For example, if we are in the state z of T, then the firing of b in T may induce two transitions $q_1 \xrightarrow{v_0} q'_0$ and $q_1 \xrightarrow{v_2} q'_0$ in M.

We note that the pair of $m \xrightarrow{k} n$ and $m \xrightarrow{v} A^*$ is regarded as a morphism from mA^* into nA^* between presheaves. In fact, the Yoneda lemma implies $\mathrm{Hom}_{\mathbf{Set}^{A^*}}(mA^*, nA^*) \cong (nA^*)^m$. Therefore, reading (z, n) as (nA^*, z) and $u/k, v$ as u/f where $mA^* \xrightarrow{f} nA^*$ corresponds to k, v, the definition of polyvalent transducer T is exactly equal to the construction of the covering T in the characterization theorem 2.2. Hence the following theorem is obvious.

3.6 Theorem
Let T be a polyvalent transducer of input alphabet A and output alphabet B, and let M be an automaton over A.

We have $M \circ T \cong \int_{x \in T} \mathrm{Hom}_{\mathbf{Set}^{A^*}}(\varphi(x),\, M)$ *as automata over* B, *where* $T^{\mathrm{op}} \xrightarrow{\varphi} \mathbf{Set}^{A^*}$ *is given by* $(z, n) \mapsto nA^*$.

This theorem asserts that polyvalent transducers are exactly discrete twiners between categories of automata.

It is well-known that the language of the past linear temporal formulas are recognized by finite aperiodic automata. As pointed out in [11], this is best understood in terms of transducers.

We define the past linear temporal logic and its semantics. The formulas φ are generated by

$$\varphi \quad ::= \quad X \mid \neg\varphi \mid \varphi \vee \varphi \mid \ominus\varphi \mid \diamondminus\varphi \mid \varphi \,\mathsf{S}\, \varphi$$

where X ranges over a given set V of propositional variables. The temporal operator $\ominus$ stands for previously, $\diamondminus$ for once-upon-a-time, and S for since, as seen from the semantics defined shortly.

Let A be the powerset $\mathcal{P}ow(V)$. We define relation $(w, i) \models \varphi$ where w is a word in A^*, i a non-negative integer less than the length $|w|$, and φ a formula.

(i) $(w, i) \models X$ iff, for $w = a_{n-1} \cdots a_1 a_0$, the i-th letter $a_i \in A$ contains X. The word w is read from right to left on the contrary to the ordinary fashion, since we adopt the convention to write all operations from left following the standard mathematical notations.

(ii) $(w, i) \models \neg\varphi$ iff not $(w, i) \models \varphi$.

(iii) $(w, i) \models \varphi \vee \psi$ iff $(w, i) \models \varphi$ or $(w, i) \models \psi$.

(iv) $(w, i) \models \ominus\varphi$ iff $i \geq 1$ and $(w, i - 1) \models \varphi$.

(v) $(w, i) \models \diamondminus\varphi$ iff there is $j \leq i$ such that $(w, j) \models \varphi$.

(vi) $(w, i) \models \varphi \,\mathsf{S}\, \psi$ iff there is $j \leq i$ such that $(w, j) \models \psi$ and $(w, k) \models \varphi$ for every $k = j, j + 1, \ldots, i$.

We write simply $w \models \varphi$ if $(w, n - 1) \models \varphi$ where $n = |w|$, that is, $n - 1$ is the leftmost position of the word. The language $L(\varphi)$ recognized by the formula φ is the set of all words $w \in A^*$ satisfying $w \models \varphi$.

We must take into consideration of initial and final states of automata. In the rest of this section, we equip each automaton M with an initial state $q_0 \in |M|$ and a set of final states $F \subseteq |M|$. We consider only finite automata. The language $L(M)$ recognized by an automaton M is defined as usual. If T is an ordinary transducer, a state (z, q) is a final state of cascade product $M \circ T$ iff q is a final state of M.

Let us consider a formula $\varphi[X_1, X_2, \ldots, X_n]$ where X_i are all propositional variables occurring in φ. For an n-tuple of formulas $\tau_1, \tau_2, \ldots, \tau_n$, we can form a substituted formula $\varphi[\tau_1/X_1, \ldots \tau_n/X_n]$. The aim is to explain the language $L(\varphi[\vec{\tau}/\vec{X}])$ from the automata recognizing $L(\varphi)$ and $L(\tau_i)$.

To the substitution $[\vec{\tau}/\vec{X}]$, we associate a transducer T with input and output alphabets both A. Let $N_1, N_2, \ldots, N_n$ be the automaton subject to the condition that $L(N_i) = L(\tau_i)$. The states of T are all n-tuples $\langle q_1, q_2, \ldots, q_n\rangle$ of states $q_i \in |N_i|$. The transition

$$\langle q_1, q_2, \ldots, q_n\rangle \xrightarrow{a/b} \langle q_1', q_2', \ldots, q_n'\rangle$$

for $a, b \in A$ is defined iff $q_i \xrightarrow{a} q_i'$ is a transition in N_i for every i and b is the set of all X_i where q_i' is a final state of N_i (recall that A is the powerset $Pow(V)$ of the set of all propositional variables.) We note that the output word b always has length 1. The initial state of T is the tuple of the initial states of N_i. As usual, we define the sequential function $\sigma_T : A^* \to A^*$ from the transducer T. The following theorem is stated in [11].

3.7 Theorem
We consider a formula $\varphi[\tau_1/X_1, \tau_2/X_1, \ldots, \tau_n/X_n]$. Let M be the automaton satisfying $L(\varphi) = L(M)$, and let T be the transducer associated to the substitution $[\vec{\tau}/\vec{X}]$ as above.

Then $L(\varphi[\vec{\tau}/\vec{X}])$ is equal to $L(M \circ T)$.

(*Proof*) We note that $w \in L(M \circ T)$ if and only if $\sigma_T(w) \in L(M)$. So it suffices to show that $w \models \varphi[\vec{\tau}/\vec{X}]$ if and only if $\sigma_T(w) \models \varphi$. We verify that $(w, i) \models \varphi[\vec{\tau}/\vec{X}]$ is equivalent to $(\sigma_T(w), i) \models \varphi$ by induction on construction of φ. From this, the claim follows since the sequential function σ_T does not change the lengths of words by definition of T.

The base case is that $\varphi = X_k$. We have $(\sigma_T(w), i) \models X_k$ iff the i-th letter of $\sigma_T(w)$ contains X_k, that is, the output of the i-th transition in the transducer T contains X_k. By definition of T, it happens exactly when the automaton N_k reaches a final state after the i-th transition, that is, (w, i) is recognized by N_k. Hence $(w, i) \models \varphi$ follows and vice versa. The induction step is straightforward. $\square$

This gives a modular way to verify that the formulas of the past linear temporal logic is recognized by aperiodic automata. It is easy to see that cascade product preserves aperiodicity. In comparison to more primitive proofs, e.g., [3], the proof above reflects directly the construction of formulas in the shape of automata.

The theorem asserts the substitution is regarded as an application of (a special case of) discrete twiners. This is somehow reminiscent of the semantics of second order linear logic, where formulas are interpreted as discrete twiners [6]. In both semantics, substitution involves application of discrete twiners. However they are dual in some sense. In the semantics of linear logic, a formula $\varphi[\vec{X}]$ is interpreted as a discrete twiner and the substitution $\varphi[\vec{\tau}/\vec{X}]$ as an application. On the contrary, in the semantics of the past linear temporal logic, the substitution $[\vec{\tau}/\vec{x}]$ is interpreted as a discrete twiner whereas $\varphi[\vec{\tau}/\vec{X}]$ as its application.

4 Weaker conditions for twiners

In section 2, twiners are defined as those functors which are locally continuous and accessible, that is, those functors which preserves all limits in slice categories and all filtered colimits. With a more elaborate proof, we can verify that preservation of weaker universal properties suffices. Namely functors preserving (infinite) pullbacks and filtered colimits turn out to be twiners.

We need analyses in depth of finitely presentable automata. Finitely presentable objects in categories of presheaves are finite colimits of representable presheaves. In the category $\mathbf{Set}^{A^*}$ of automata, A^* is a unique representable presheaf. Hence we have the following syntactic characterization of finitely presentable automata.

Let $\rho_1, \rho_2, \ldots, \rho_n$ be a finite number of fresh symbols (standing for roots). The terms are of the shape $u\rho_i$ for $u \in A^*$ and $i = 1, 2, \ldots, n$. Moreover, a finite number of equational axioms $u\rho_i = u'\rho_{i'}$ are given. Two terms are regarded to be equal iff equality is derived with the usual reflexive, symmetric, and transitive laws from equalities of the shape $vu\rho_i = vu'\rho_{i'}$ with $v \in A^*$ where $u\rho_i = u\rho_{i'}$ is one of axioms. Let us denote by $[w\rho_j]$ the class of which is a member $w\rho_j$. We have the automata where the states are all classes $[w\rho_j]$ and the action of A^* is given by $u \cdot [w\rho_j] = [uw\rho_j]$.

We prove that a colimit of each finite diagram where all objects are A^* can be written in this syntactic way. To each object in the finite diagram, we associate one root ρ_i. Let us denote by A^*_i the object A^* corresponding to the root ρ_i, in order to distinguish the occurrences of the same A^* in the diagram. By the Yoneda lemma, a morphism from A^* to A^* must correspond to a word $u \in A^*$. To the morphism $A^*_i \xrightarrow{u} A^*_j$, we associate an equational axiom $\varepsilon\rho_i = u\rho_j$ where ε is the empty word. Since the diagram is finite, the number of roots and the number of equations are finite. The colimit of the diagram is isomorphic to the syntactically constructed automaton. Conversely, the automaton defined from finite roots and finite equational axioms are finite colimits of A^*.

Remark: It is incorrectly stated in [1] that finitely presentable objects in the category $\mathbf{Aut}$ of automata are finite automata. Since $\mathbf{Aut}$ is the model category of a finite limit sketch, the finitely presentable objects should be finite colimits of reflection of finite presheaves, where the reflection is a left adjoint of inclusion of $\mathbf{Aut}$ into the category $\mathbf{Set}^{\mathbf{A}}$ for the underlying category $\mathbf{A}$ of the sketch.

Next we give a graphic characterization of finitely presentable automata as labeled transition graph. A *bulb* in au automaton is a state q subject to the condition that there is a transition $q \xrightarrow{v} p$ for some $v \in A^*$ whenever there is a transition $p \xrightarrow{u} q$ for some $u \in A^*$. If we take the equality between states determined by mutual reachability, the component of a bulb is a root of the obtained directed acyclic graph. A *degree* $dg(q)$ of a state q is the number of incoming transitions $p \xrightarrow{a} q$ with words $a \in A$ of length 1. We note that the degree of a state may be infinite in general.

For example, the degree of a state q in a finitely presentable, projective automaton nA^* is equal to 1 unless q is a root; in which case the degree is equal to 0. Moreover the bulbs in nA^* is the finite number of roots.

4.1 Lemma
An automaton M is finitely presentable in $\mathbf{Set}^{A^*}$ *iff its transition graph* $\int M$ *satisfies the following two conditions:*

(i) *There are finitely many components of bulbs in M, and for every state q there is a bulb q_0 with a transition $q_0 \xrightarrow{u} q$ for some $u \in A^*$.*

(ii) *For each state q, the degree $dg(q)$ is finite. Moreover, there are only finitely many states with degree 2 or more.*

(*Proof*) We verify that every finitely presentable automaton M satisfies the two conditions. The condition (i) is obvious since M is obtained from nA^* with identification of states.

For (ii), we add equational axioms $u\rho_i = u'\rho_{i'}$ one by one. We prove that the sum of number $dg(q) - 1$ is finite where q ranges over all states, ignoring the states with degree 0 (there are only a finite number of degree 0 states since (i) is satisfied). At the first stage, M is nA^*, so the sum of $dg(q) - 1$ equals 0.

Let q and q' be the states corresponding to $u\rho_i$ and $u'\rho_{i'}$. We identify q and q' with all other states intact. After identification, the graph may be non-deterministic, namely, there may be two transitions $q \xrightarrow{a} p_1$ and $q \xrightarrow{a} p_2$ with the same letter a from a single state. If this is the case, we identify p_1 and p_2. We repeat this procedure as far as there is a non-deterministic state. Of course, we must impose a certain fairness condition for the order of applications of the procedure to ensure that all non-deterministic transition $q \xrightarrow{a} p_1$ and $q \xrightarrow{a} p_2$ are handled eventually.

After the first identification of two states corresponding $u\rho_i$ and $u'\rho_{i'}$, the sum of $dg(q)-1$ may increase by 1. However, during all other consequent identifications, the sum does not increase at all (it may decrease). Hence, by an addition of one equational axiom, the sum of $dg(q) - 1$ increases at most by 1. Since the number of equational axioms is finite, the condition (ii) must be fulfilled. □

Finitely generated objects of $\mathbf{Set}^{A^*}$ are syntactically constructed from nA^* as finitely presentable objects, except that there may be infinitely many equational axioms $u\rho_i = u'\rho_{i'}$. Hence, graphically, finitely presentable automata are characterized by the condition (i) alone of the preceding lemma.

The next shows that finitely generated subobjects of finitely presentable automata are finitely presentable.

4.2 Corollary
Let M be a finitely presentable automaton.

If a subautomaton $N \subseteq M$ is finitely generated, then N is finitely presentable.

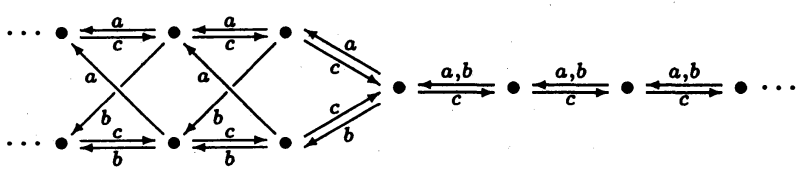
(*Proof*) Obvious, since N satisfied the condition (ii) of the lemma as a subgraph of M. □

4.3 Lemma
An epimorphism $M \xrightarrow{e} M$ for a finitely presentable automaton M is an isomorphism.

(*Proof*) Each component of bulbs contains only finitely many states. Indeed, by the condition (ii) of the graphic characterization of finitely presentable automata, there is no infinite path $q_0 \xrightarrow{a_0} q_1 \xrightarrow{a_1} q_2 \cdots$ of pairwise distinct bulbs ($\because$ for sufficiently large n, all the states reachable from q_n must have degree 1). So the total number of bulbs is finite.

If $e(q)$ is a bulb, then, for every state p having a transition $p \xrightarrow{u} q$, also $e(p)$ is a bulb in the same component as $e(q)$. This fact implies that it is impossible for e to carry a component of bulbs to a component of non-bulbs, since there are only finitely many components of bulbs. So e carries bulbs to bulbs. Since the number of bulbs is finite and e is a surjection, e must give a permutation on the finite set of bulbs. Let e^n be an identity on bulbs. Then e^n turns out to be an identity on all states, for every state is reachable from a bulb. Thus e is an isomorphism. □

Remark: This lemma fails for finitely generated automata. The following automaton M over alphabet $\{a, b, c\}$ is finitely generated, consisting a single component of bulbs:



The right shift of states gives an endomorphism on M. In particular, two states in the left of the fork $\succ$ are sent to the same state. This morphism is epi, but not an isomorphism.

In the next theorem, infinite pullbacks mean the limits of the set of morphisms $Y_i \xrightarrow{f_i} X$ where i is allowed to range over infinite sets.

4.4 Theorem
If a functor $\mathbf{Set}^{A^*} \xrightarrow{F} \mathbf{Set}^{B^*}$ *preserves infinite pullbacks and filtered colimits, then* F *is a twiner.*

(*Proof*) It suffices to prove that, for the induced set-valued functor $\mathbf{Set}^{A^*} \xrightarrow{F} \mathbf{Set}$, the Grothendieck construction $\mathrm{Gr}(F)$ satisfies the normal form property. We note that $\mathrm{Gr}(F)$ has infinite pullbacks, for F preserves infinite pullbacks.

First of all, let us call an object (Z, c) in $\mathrm{Gr}(F)$ is minimal iff every morphism $(V, d) \xrightarrow{f} (Z, c)$ is an epimorphism (note that, in general, $(V, d) \xrightarrow{f} (Z, c)$ is epi in the Grothendieck construction iff the underlying $V \xrightarrow{f} Z$ is epi). Then, for each object (X, a), there is $(Z, c) \to (X, a)$ such that Z is finitely presentable and (Z, c) is minimal. Indeed, since every X is a filtered colimit of finitely presentable objects and F preserves it, there is $(Z', c') \to (X, a)$ with finitely presentable Z'. Then we take the pullback (Z, c) of all $(Z'', c'') \to (Z', c')$ wher Z'' is a subobject of Z'. Then (Z, c) is minimal. We note that Z is finitely generated; otherwise writing Z as a filtered colimit of monomorphisms of finitely generated objects, there is a strict subobject $(V, d) \to (Z, c)$, contradicting minimality. As a finitely generated subobject of finitely presentable automaton Z', the automaton Z is finitely presentable. For simplicity, we refer such (Z, c) a finitely presented minimal object.

Towards the normal form property, let us take an arbitrary object (X, a). We take the pullback (P', d') of all finitely presentable minimal $(Z, c) \to (X, a)$. Then we choose a finitely presented minimal $(P, d) \to (P', d')$. If there is $(Z, c) \xrightarrow{f} (P, d)$ from finitely presented minimal (Z, c), the morphism f must be epi by minimality. By construction of (P, d), there is also a morphism $(P, d) \xrightarrow{g} (Z, c)$, which must be epi by minimality. Hence there is an epimorphism $Z \xrightarrow{gf} Z$ on finitely presented automaton Z. Thus gf must be an isomorphism, and so is f. Therefore every morphism $(Z, c) \xrightarrow{f} (P, d)$ on every finitely presentable minimal objects (Z, c) are isomorphisms.

Hence we can verify that $(P, d) \to (X, a)$ is a weak normal form as in [5]. In this paper, existence of weak normal forms is proved from several properties of minimal objects. In place, we can use finitely presentable minimal objects, since they satisfy similar properties. Thus $\mathrm{Gr}(F)$ has the weak normal form property. However [5] shows also that if a category has the weak normal form property and has binary pullbacks then it enjoys the normal form property. So $\mathrm{Gr}(F)$ has the normal form property. $\square$

Remark In the theorem, we can further weaken the condition to that F preserves countable pullbacks. In the proof above, there may be uncountable subobjects Z'' of Z', even if Z' is finitely presentable. But we can restrict Z'' to finitely generated subobjects and the number of such subobjects of finitely presentable automaton Z' is countable. So we can prove the theorem using countable pullbacks only.

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