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Determinants and Pfaffians

How to obtain N-soliton solutions from 2-soliton solutions

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Contents

1 Introduction

A method of obtaining N-soliton solution from 2-soliton solution is described. N-soliton solution of soliton equations are obtained by the following procedures;

1. Transform a soliton equation into a bilinear equation.
2. Solve the bilinear equation using a perturbational method. 2-soliton solutions are easily obtained by using the computer algebra (Mathematica,Reduce etc.).
3. Express the 2-soliton solutions by pfaffians (Determinants) .
4. Rewrite the bilinear equation using pfaffians and confirm that the bilinear equation is nothing but the pfaffian identities using the difference (or differential) formula for pfaffians.
5. Then the 2-soliton solution are easily extended to the N-soliton solutions.

To this end we study pfaffians.

2 Pfaffians

We expressed an entry (element) of a pfaffian by $\mathrm{pf}(a_1, a_2)$ of characters a_1, a_2. A 4th order pfaffian $\mathrm{pf}(a_1, a_2, a_3, a_4)$ is expanded by 6 entries,

$$\mathrm{pf}(a_1, a_2, a_3, a_4) = \mathrm{pf}(a_1, a_2)\mathrm{pf}(a_3, a_4) - \mathrm{pf}(a_1, a_3)\mathrm{pf}(a_2, a_4) + \mathrm{pf}(a_1, a_4)\mathrm{pf}(a_2, a_3).$$

Pfaffians are antisymmetric functions with respect to characters,

$$\mathrm{pf}(a, b) = -\mathrm{pf}(b, a), \quad \text{for any } a \text{ and } b,$$

from which we obtain antisymmetric properties of pfaffians, for example,

$$\mathrm{pf}(a_1, a_2, a_3, a_4) = -\mathrm{pf}(a_1, a_3, a_2, a_4).$$

A $2n$-th degree pfaffian is defined by the following expansion rule,

$$\begin{aligned}&\mathrm{pf}(a_1, a_2, \ldots, a_{2n})\\ &\quad = \sum_{j=2}^{n} \mathrm{pf}(a_1, a_j)(-1)^j \mathrm{pf}(a_2, \ldots, \hat{a}_j, \ldots, a_{2n}),\end{aligned}$$

where $\hat{a}_j$ represents elimination of character a_j.
For example, if $n = 3$, we have

$$\begin{aligned}
&\mathrm{pf}(a_1, a_2, a_3, a_4, a_5, a_6) \\
&= \sum_{j=2}^{6} \mathrm{pf}(a_1, a_j)(-1)^j \mathrm{pf}(a_2, \ldots, \hat{a}_j, \ldots, a_6) \\
&= \mathrm{pf}(a_1, a_2)\mathrm{pf}(a_3, a_4, a_5, a_6) - \mathrm{pf}(a_1, a_3)\mathrm{pf}(a_2, a_4, a_5, a_6) \\
&\quad + \mathrm{pf}(a_1, a_4)\mathrm{pf}(a_2, a_3, a_5, a_6) - \mathrm{pf}(a_1, a_5)\mathrm{pf}(a_2, a_3, a_4, a_6) \\
&\quad + \mathrm{pf}(a_1, a_6)\mathrm{pf}(a_2, a_3, a_4, a_5).
\end{aligned}$$

2.1 Determinants and Pfaffians

Pfaffians are related to determinants.
(i) Let A be a determinant of a $m \times m$ antisymmetric matrix defined by

$$A = \det |a_{j,k}|_{1 \leq j,k \leq m},$$

where $a_{j,k} = -a_{k,j}$ for $j, k = 1, 2, \ldots, m$.
If m is odd, A gives 0. On the other hand, if m is even, A gives a square of a pfaffian. This pfaffian has a degree $2m$ and is noted as $\mathrm{pf}(a_1, a_2, a_3, \cdots, a_{2m})$ with the entries $\mathrm{pf}(a_j, a_k) = a_{j,k}$ for $j, k = 1, 2, \ldots, m$,

$$\det |a_{j,k}|_{1 \leq j,k \leq m} = \mathrm{pf}(a_1, a_2, a_3, \cdots, a_{2m})^2.$$

For example,if $m = 4$, we have

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{vmatrix} = [a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}]^2$$
$$= [\mathrm{pf}(a_1, a_2, a_3, a_4)]^2.$$

(ii) Let E, A and B be a $m \times m$ unit matrix and $m \times m$ antisymmetric matrices respectively. Then the determinant $\det |E + AB|$ is a square of a pfaffian. This pfaffian is denoted as $\mathrm{pf}(a_1, a_2, a_3, \cdots, a_m, b_1, b_2, b_3, \cdots, b_m)$ with the entries $\mathrm{pf}(a_j, a_k) = a_{j,k}$, $\mathrm{pf}(b_j, b_k) = b_{j,k}$ and $\mathrm{pf}(a_j, b_k) = \delta_{j,k}$ for $j, k = 1, 2, \ldots, m$;

$$\det |E + AB| = \mathrm{pf}(a_1, a_2, a_3, \cdots, a_m, b_1, b_2, b_3, \cdots, b_m)^2.$$

This is because

$$\det|E+AB| = \det\begin{vmatrix} A & E \\ -E & B \end{vmatrix} = \mathrm{pf}(a_1, a_2, a_3, \cdots, a_m, b_1, b_2, b_3, \cdots, b_m)^2.$$

The pfaffian $\mathrm{pf}(a_1, a_2, a_3, \cdots, a_m, b_1, b_2, b_3, \cdots, b_m)$ plays a crucial role in expressing N-soliton solutions of coupled soliton equations.

2.2 Exterior algebra

Making use of exterior algebra, which is based on a concept of a vector exterior product $A \times B = -B \times A$, one can give a clearer definition of determinant and pfaffian. Let us introduce a *one-form* given by

$$\omega_i = \sum_{j=1}^{n} a_{j,k} x^j \quad (i = 1, 2, \cdots, 2n)$$

where x^j's satisfy the following antisymmetric commutation relations,

$$x_j \wedge x_k = -x_k \wedge x_j, \quad x_j \wedge x_j = 0, \quad j, k = 1, 2, \ldots, n.$$

Except the above relations, we obey the normal method of calculation. Coefficients $a_{j,k}$ are arbitrary complex functions.
A determinant $\det|a_{j,k}|_{1\le j,k\le n}$ is defined by means of exterior products of n one-forms.

$$\begin{aligned} &\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \ldots \wedge \omega_n \\ &\quad = \det|a_{j,k}|_{1\le j,k\le n} x^1 \wedge x^2 \wedge x^3 \ldots \wedge x^n. \end{aligned}$$

For example, if $n = 2$,

$$\begin{aligned} \omega_1 \wedge \omega_2 &= (a_{1,1}x^1 + a_{1,2}x^2) \wedge (a_{2,1}x^1 + a_{2,2}x^2) \\ &= (a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) x^1 \wedge x^2 \\ &= \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} x^1 \wedge x^2, \end{aligned}$$

which defines the 2×2 determinant $\det|a_{j,k}|_{1\le j,k\le 2}$.
Next let Ω be a *two-form* given by

$$\Omega = \sum_{1\le j,k,\le 2n} b_{j,k} x^j \wedge x^k, \quad b_{j,k} = -b_{k,j}.$$

A pfaffian with its (i,j) entry given by $b_{j,k}$ is defined by an n-tuple exterior product of Ω as

$$\Omega\wedge^n = (n!)\mathrm{pf}(b_1, b_2, b_3, \ldots, b_{2n})x_1 \wedge x_2 \wedge x_3 \ldots \wedge x_{2n},$$

where $n! = n(n-1)(n-2)\cdots 2 \times 1$.
From the above definition, one obtains an expansion formula of a pfaffian. For example, in the case $n = 2$, putting

$$\begin{aligned}\Omega = & b_{1,2}x^1 \wedge x^2 + b_{1,3}x^1 \wedge x^3 + b_{1,4}x^1 \wedge x^4 \\ & + b_{2,3}x^2 \wedge x^3 + b_{2,4}x^2 \wedge x^4 + b_{3,4}x^3 \wedge x^4\end{aligned}$$

we have

$$\begin{aligned}& \Omega \wedge \Omega \\ & = \{b_{1,2}x^1 \wedge x^2 + b_{1,3}x^1 \wedge x^3 + b_{1,4}x^1 \wedge x^4 \\ & \quad + b_{2,3}x^2 \wedge x^3 + b_{2,4}x^2 \wedge x^4 + b_{3,4}x^3 \wedge x^4\} \\ & \quad \wedge\{b_{1,2}x^1 \wedge x^2 + b_{1,3}x^1 \wedge x^3 + b_{1,4}x^1 \wedge x^4 \\ & \quad + b_{2,3}x^2 \wedge x^3 + b_{2,4}x^2 \wedge x^4 + b_{3,4}x^3 \wedge x^4\} \\ & = 2\{b_{1,2}b_{3,4} - b_{1,3}b_{2,4} + b_{1,4}b_{2,3}\}x^1 \wedge x^2 \wedge x^3 \wedge x^4. \qquad (1)\end{aligned}$$

On the other hand, from the definition, one has

$$\Omega \wedge \Omega = 2\mathrm{pf}(b_1, b_2, b_3, b_4)x^1 \wedge x^2 \wedge x^3 \wedge x^4. \qquad (2)$$

From eqs.(1) and (2), we have obtained the expansion expression

$$\mathrm{pf}(b_1, b_2, b_3, b_4) = b_{1,2}b_{3,4} - b_{1,3}b_{2,4} + b_{1,4}b_{2,3}.$$

2.3 Laplace expansions of determinants and Plücker relations

2.3.1 Laplace expansions of determinants

An n-th degree determinant given by $A = \det |a_{i,j}|_{1\le i,j\le n}$ can be expressed as a summation of products of r- and $(n-r)$-th degree determinants.
This expansion formula is called the Laplace expansion.

Let us show how the Laplace expansion is derived taking a 4th degree determinant an example. Let $\omega_j (j = 1, 2, 3, 4)$ be one-form,

$$\omega_j = \sum_{k=1}^{4} a_{j,k} x^k \quad (j = 1, 2, 3, 4)$$

Then from the definition, we have

$$\begin{aligned}&\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \\ &\qquad = \det |a_{j,k}|_{1 \le j,k \le 4} x^1 \wedge x^2 \wedge x^3 \wedge x^4.\end{aligned}$$

On the other hand,

$$\begin{aligned}\omega_1 \wedge \omega_2 &= (a_{1,1}x^1 + a_{1,2}x^2 + a_{1,3}x^3 + a_{1,4}x^4) \\ &\quad \wedge (a_{2,1}x^1 + a_{2,2}x^2 + a_{2,3}x^3 + a_{2,4}x^4) \\ &= \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} x^1 \wedge x^2 + \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} x^1 \wedge x^3 \\ &+ \begin{vmatrix} a_{1,1} & a_{1,4} \\ a_{2,1} & a_{2,4} \end{vmatrix} x^1 \wedge x^4 + \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} x^2 \wedge x^3 \\ &+ \begin{vmatrix} a_{1,2} & a_{1,4} \\ a_{2,2} & a_{2,4} \end{vmatrix} x^2 \wedge x^4 + \begin{vmatrix} a_{1,3} & a_{1,4} \\ a_{2,3} & a_{2,4} \end{vmatrix} x^3 \wedge x^4\end{aligned}$$

and

$$\begin{aligned}\omega_3 \wedge \omega_4 &= (a_{3,1}x^1 + a_{3,2}x^2 + a_{3,3}x^3 + a_{3,4}x^4) \\ &\quad \wedge (a_{4,1}x^1 + a_{4,2}x^2 + a_{4,3}x^3 + a_{4,4}x^4) \\ &= \begin{vmatrix} a_{3,1} & a_{3,2} \\ a_{4,1} & a_{4,2} \end{vmatrix} x^1 \wedge x^2 + \begin{vmatrix} a_{3,1} & a_{3,3} \\ a_{4,1} & a_{4,3} \end{vmatrix} x^1 \wedge x^3 \\ &+ \begin{vmatrix} a_{3,1} & a_{3,4} \\ a_{4,1} & a_{4,4} \end{vmatrix} x^1 \wedge x^4 + \begin{vmatrix} a_{3,2} & a_{3,3} \\ a_{4,2} & a_{4,3} \end{vmatrix} x^2 \wedge x^3 \\ &+ \begin{vmatrix} a_{3,2} & a_{3,4} \\ a_{4,2} & a_{4,4} \end{vmatrix} x^2 \wedge x^4 + \begin{vmatrix} a_{3,3} & a_{3,4} \\ a_{4,3} & a_{4,4} \end{vmatrix} x^3 \wedge x^4.\end{aligned}$$

Substituting of the above formulae into

$$\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 = (\omega_1 \wedge \omega_2) \wedge (\omega_3 \wedge \omega_4)$$

$$
\begin{aligned}
&\det |a_{j,k}|_{1\le j,k\le 4} x^1 \wedge x^2 \wedge x^3 \wedge x^4 \\
&= \left\{ \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} \begin{vmatrix} a_{3,3} & a_{3,4} \\ a_{4,3} & a_{4,4} \end{vmatrix} - \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} \begin{vmatrix} a_{3,2} & a_{3,4} \\ a_{4,2} & a_{4,4} \end{vmatrix} \right. \\
&\quad + \begin{vmatrix} a_{1,1} & a_{1,4} \\ a_{2,1} & a_{2,4} \end{vmatrix} \begin{vmatrix} a_{3,2} & a_{3,3} \\ a_{4,2} & a_{4,3} \end{vmatrix} + \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} \begin{vmatrix} a_{3,1} & a_{3,4} \\ a_{4,1} & a_{4,4} \end{vmatrix} \\
&\quad \left. - \begin{vmatrix} a_{1,2} & a_{1,4} \\ a_{2,2} & a_{2,4} \end{vmatrix} \begin{vmatrix} a_{3,1} & a_{3,3} \\ a_{4,1} & a_{4,3} \end{vmatrix} + \begin{vmatrix} a_{1,3} & a_{1,4} \\ a_{2,3} & a_{2,4} \end{vmatrix} \begin{vmatrix} a_{3,1} & a_{3,2} \\ a_{4,1} & a_{4,2} \end{vmatrix} \right\} \\
&\quad x^1 \wedge x^2 \wedge x^3 \wedge x^4 .
\end{aligned}
$$

From the definition, inside the parenthesis $\{\cdots\}$ is equal to the 4th degree determinant, which completes the proof of the Laplace expansion formula of 4th degree determinant.
From N one-forms,

$$\omega_j = \sum_{k=1}^{N} a_{j,k} x^j \quad (j = 1, 2, \ldots, N)$$

we generate an N-th degree determinant,

$$\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \ldots \wedge \omega_N = \det |a_{j,k}|_{1\le j,k\le N} x^1 \wedge x^2 \wedge \ldots \wedge x^N .$$

Decomposing the left hand side of the above equation into the product,

$$(\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_r) \wedge (\omega_{r+1} \wedge \omega_{r+2} \wedge \ldots \wedge \omega_N)$$

and rewriting the above equation into a sum of products of r-th and $(N-r)$-th degree determinants, we finally obtain the Laplace expansion theorem.

2.3.2 Plücker relations

The following identity holds for a summation of products of 2nd degree determinants.

$$\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \begin{vmatrix} a_0 & a_2 \\ b_0 & b_2 \end{vmatrix} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \begin{vmatrix} a_0 & a_3 \\ b_0 & b_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0.$$

which can be proved through direct expansion of each determinant. However, there is another way of proof. Let us consider a 4th degree determinant,

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ 0 & a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 & b_3 \end{vmatrix} = 0,$$

which is identically equal to 0. Then by means of the Laplace expansion theorem, the determinant is expanded as

$$0 = \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \begin{vmatrix} a_0 & a_2 \\ b_0 & b_2 \end{vmatrix} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \begin{vmatrix} a_0 & a_3 \\ b_0 & b_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix},$$

which is the simplest case of the Plücker relations.

2.4 Expressions of determinants and wronskians in terms of pfaffians

A determinant of n-degree,

$$B = \det |b_{j,k}|_{1\le j,k\le n},$$

is expressed by means of a pfaffian of $2n$-th degree as follows

$$\det |b_{j,k}|_{1\le j,k\le n} = \mathrm{pf}(b_1, b_2, \ldots, b_n, b_n^*, b_{n-1}^*, \ldots, b_2^*, b_1^*),$$

whose entries are defined by

$$\begin{aligned} &\mathrm{pf}(b_j, b_k) = \mathrm{pf}(b_j^*, b_k^*) = 0, \\ &\mathrm{pf}(b_j, b_k^*) = b_{j,k}, \quad \text{for } j, k = 1, 2, \ldots, n. \end{aligned}$$

For example, if n=2, we have

$$\begin{vmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{vmatrix} = \mathrm{pf}(b_1, b_2, b_2^*, b_1^*).$$

This is because

$$\begin{aligned} (r.h.s) &= -\mathrm{pf}(b_1, b_2^*)\mathrm{pf}(b_2, b_1^*) + \mathrm{pf}(b_1, b_1^*)\mathrm{pf}(b_2, b_2^*) \\ &= b_{1,1}b_{2,2} - b_{1,2}b_{2,1} = (l.h.s). \end{aligned}$$

Next, we consider a Wronskian, which often appears in the theory of linear ordinary differential equations. An n-th degree Wronskian$(f_1, f_2, \cdots, f_n)$ is defined by

$$Wr(f_1(x), f_2(x), \ldots, f_n(x)) = \det\left|\frac{\partial^{j-1} f_k(x)}{\partial x^{j-1}}\right|_{1\leq j,k\leq n}.$$

Let $f_i^{(m)}$ denote an m-th differential of $f_i = f_i(x)$ with respect to x,

$$f_i^{(m)} = \frac{\partial^m}{\partial x^m} f_i, \quad m = 0, 1, 2, \cdots.$$

We introduce a pfaffian(d_m, i),which represent $f_i^{(m)}$, defined by

$$\begin{aligned}
&\mathrm{pf}(d_m, i) = f_i^{(m)}, \quad i = 1, 2, \cdots, \\
&\mathrm{pf}(d_m, d_n) = 0, \quad m, n = 0, 1, 2, \cdots.
\end{aligned}$$

By employing the above notations, n-th degree Wronskian is expressed by $2n$-th degree pfaffian as

$$Wr(f_1(x), f_2(x), \ldots, f_n(x)) = \mathrm{pf}(d_0, d_1, d_2, \ldots, d_{n-1}, f_n, f_{n-1}, \cdots, f_1)$$

$$\begin{aligned}
&\mathrm{pf}(d_j, f_k) = \frac{\partial^j f_k}{\partial x^j}, \quad \text{for } j = 0, 1, \ldots \text{ and for } k := 1, 2, \ldots, n \\
&\mathrm{pf}(d_j, d_k) = 0, \quad \text{for } j, k = 0, 1, 2, \ldots\ .
\end{aligned}$$

For example, in the case of $n = 2$, we have

$$(l.h.s) = \begin{vmatrix} f_1 & \frac{\partial f_1}{\partial x} \\ f_2 & \frac{\partial f_2}{\partial x} \end{vmatrix} = f_1\frac{\partial f_2}{\partial x} - f_2\frac{\partial f_1}{\partial x}.$$

On the other hand,

$$\begin{aligned}
(r.h.s) = \mathrm{pf}(d_0, d_1, f_2, f_1) &= \mathrm{pf}(d_0, d_1)\mathrm{pf}(f_2, f_1) - \mathrm{pf}(d_0, f_2)\mathrm{pf}(d_1, f_1) \\
&\quad + \mathrm{pf}(d_0, f_1)\mathrm{pf}(d_1, f_2) \\
= f_1\frac{\partial f_2}{\partial x} - f_2\frac{\partial f_1}{\partial x},
\end{aligned}$$

which completes the proof.

2.5 Pfaffian identities

There are various kind of pfaffian identities. Let us derive most fundamental identities among them. We start with an expansion formula for $2m$-th degree pfaffian $\mathrm{pf}(a_1, a_2, a_3, \cdots, a_{2m})$,

$$\begin{aligned}&\mathrm{pf}(a_1, a_2, a_3, \cdots, a_{2m})\\&= \sum_{j=2}^{2m} (-1)^j \mathrm{pf}(a_1, a_j)\mathrm{pf}(a_2, \cdots, \hat{a}_j, \cdots, a_{2m}).\end{aligned}$$

Appending $2n$ characters $1, 2, 3, \cdots, 2n$ homogeneously to each pfaffian above, we obtain an extended expansion formula,

$$\begin{aligned}&\mathrm{pf}(a_1, a_2, \cdots, a_{2m}, 1, 2, \ldots, 2n)\mathrm{pf}(1, 2, \ldots, 2n)\\&= \sum_{j=2}^{2m} (-1)^j \mathrm{pf}(a_1, a_j, 1, 2, \ldots, 2n)\mathrm{pf}(a_2, \cdots, a_j, \cdots, a_{2m}, 1, 2, \ldots, 2n).\end{aligned} \tag{3}$$

Next expanding the following zero-valued pfaffian (m is odd),

$$0 = \mathrm{pf}(a_1, a_2, a_3, \cdots, \hat{a}_j, \cdots, a_m, 2n, 1, 1),$$

with respect to the final character 1, we obtain

$$\begin{aligned}&= \sum_{j=1}^{m} (-1)^{j-1} \mathrm{pf}(a_1, a_2, a_3, \cdots, \hat{a}_j, \cdots, a_m, 2n, 1)\mathrm{pf}(a_j, 1)\\&-\mathrm{pf}(a_1, a_2, a_3, \cdots, a_m, 1)(2n, 1).\end{aligned}$$

Therefore we have

$$\mathrm{pf}(a_1, a_2, a_3, \cdots, a_m, 1)(1, 2n) = \sum_{j=1}^{m} (-1)^{j-1} \mathrm{pf}(a_1, a_2, a_3, \cdots, \hat{a}_j, \cdots, a_m, 1, 2n).$$

Appending $2n-2$ characters $2, 3, \cdots, 2n-1$ homogeneously to each pfaffian again, we obtain an identity,

$$\begin{aligned}&\mathrm{pf}(a_1, a_2, a_3, \cdots, a_m, 1, 2, 3, \cdots, 2n-1)(1, 2, \cdots, 2n)\\&= \sum_{j=1}^{m} (-1)^{j-1} \mathrm{pf}(a_j, 1, 2, \cdots, 2n-1)\mathrm{pf}(a_1, a_2, a_3, \cdots, \hat{a}_j, \cdots, a_m, 1, 2, \cdots, 2n).\end{aligned} \tag{4}$$

For example, in the case $m = 2$, eq(3) is written as

$$\begin{aligned} &\mathrm{pf}(a_1, a_2, a_3, a_4, 1, 2, \ldots, 2n)\mathrm{pf}(1, 2, \ldots, 2n) \\ &\quad = \mathrm{pf}(a_1, a_2, 1, 2, \ldots, 2n)\mathrm{pf}(a_3, a_4, 1, 2, \ldots, 2n) \\ &\quad -\mathrm{pf}(a_1, a_3, 1, 2, \ldots, 2n)\mathrm{pf}(a_2, a_4, 1, 2, \ldots, 2n) \\ &\quad +\mathrm{pf}(a_1, a_4, 1, 2, \ldots, 2n)\mathrm{pf}(a_2, a_3, 1, 2, \ldots, 2n). \end{aligned} \tag{5}$$

In the case $m = 3$, eq(4) is written as

$$\begin{aligned} &\mathrm{pf}(a_1, a_2, a_3, 1, 2, 3, \cdots, 2n-1)(1, 2, \cdots, 2n) \\ &\quad = \mathrm{pf}(a_1, 1, 2, \cdots, 2n-1)\mathrm{pf}(a_2, a_3, 1, 2, \cdots, 2n) \\ &\quad -\mathrm{pf}(a_2, 1, 2, \cdots, 2n-1)\mathrm{pf}(a_1, a_3, 1, 2, \cdots, 2n) \\ &\quad +\mathrm{pf}(a_3, 1, 2, \cdots, 2n-1)\mathrm{pf}(a_1, a_2, 1, 2, \cdots, 2n). \end{aligned} \tag{6}$$

These are examples of the pfaffian identities which we prove later.
We show later that the pfafian identity (5) includes both Jacobi idenity and Plücker relation.

2.5.1 Jacobi identities for determinants

The Jacobi identity for determinants is expressed as

$$D\,D\begin{pmatrix} i & j \\ k & l \end{pmatrix} = D\begin{pmatrix} i \\ k \end{pmatrix} D\begin{pmatrix} j \\ l \end{pmatrix} - D\begin{pmatrix} i \\ l \end{pmatrix} D\begin{pmatrix} j \\ k \end{pmatrix}, \qquad i < j,\ k < l, \tag{7}$$

where D is a n-th degree determinant and the minor determinant $D\begin{pmatrix} j \\ k \end{pmatrix}$ is obtained by eliminating j-th row and k-th column from D. The minor determinant $D\begin{pmatrix} i & j \\ k & l \end{pmatrix}$ is obtained by eliminating i, j row and k, l column from D.
For example, if $n = 3$ and $i = 1, j = 2, k = 1, l = 2$ we have

$$\begin{aligned} &\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} \begin{vmatrix} a_{3,3} \end{vmatrix} \\ &= \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} - \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{3,2} & a_{3,3} \end{vmatrix}, \end{aligned}$$

which we express by the pfaffians

$$\begin{aligned}
&\mathrm{pf}(a_1, a_2, a_3, a_3^*, a_2^*, a_1^*)\mathrm{pf}(a_3, a_3^*) \\
&\quad = \mathrm{pf}(a_1, a_3, a_3^*, a_1^*)\mathrm{pf}(a_2, a_3, a_3^*, a_2^*) \\
&\qquad -\mathrm{pf}(a_1, a_3, a_3^*, a_2^*)\mathrm{pf}(a_2, a_3, a_3^*, a_1^*).
\end{aligned}$$

Arranging the characters we obtain

$$\begin{aligned}
&\mathrm{pf}(a_1, a_2, a_2^*, a_1^*, a_3, a_3^*)\mathrm{pf}(a_3, a_3^*) \\
&\quad = \mathrm{pf}(a_1, a_2, a_3, a_3^*)\mathrm{pf}(a_2^*, a_1^*, a_3, a_3^*) \\
&\qquad -\mathrm{pf}(a_1, a_2^*, a_3, a_3^*)\mathrm{pf}(a_2, a_1^*, a_3, a_3^*) \\
&\qquad +\mathrm{pf}(a_1, a_1^*, a_3, a_3^*)\mathrm{pf}(a_2, a_2^*, a_3, a_3^*),
\end{aligned} \tag{8}$$

which is nothing but the Pfaffian identity (5) for $n = 1$、$a_3 = a_2^*, a_4 = a_1^*, 1 = a_3, 2 = a_3^*$. The first term in the r.h.s is identically equal to zero ($\mathrm{pf}(a_j, a_k) = 0$).
The Plücker relation is obtained by putting the l.h.s of eq.(5) to be zero.

2.6 Proof of the pfaffian identities

In order to prove the pfaffian identities, we start with the following simple identity after Ohta (Y.Ohta:*Bilinear Theory of Soliton*,PhD Thesis (Faculty of Engineering, Tokyo Univ. 1992)).

$$\begin{aligned}
&\sum_{j=0}^{M}(-1)^j \mathrm{pf}(b_0, b_1, \ldots, \hat{b}_j, \ldots, b_M)\mathrm{pf}(b_j, c_0, c_1, \ldots, c_N) \\
&= \sum_{k=0}^{N}(-1)^k \mathrm{pf}(b_0, b_1, \ldots, b_M, c_k)\mathrm{pf}(c_0, c_1, \ldots, \hat{c}_k, \ldots, c_N)
\end{aligned} \tag{9}$$

The proof of eq.(9) is quite simple. Expanding pfaffian $\mathrm{pf}(b_j, c_0, c_1, \ldots, c_N)$ on the left hand side with respect to the first character b_j and $\mathrm{pf}(b_0, b_1, \ldots, b_M, c_k)$ on the right hand side with respect to the final character c_k,we obtain

$$\begin{aligned}
&\sum_{j=0}^{M}(-1)^j \sum_{k=0}^{N}(-1)^k \mathrm{pf}(b_0, b_1, \ldots, \hat{b}_j, \ldots, b_M) \\
&\qquad \times \mathrm{pf}(b_j, c_k)\mathrm{pf}(c_0, c_1, \ldots, \hat{c}_k, \ldots, c_N)
\end{aligned}$$

$$= \sum_{k=0}^{N}(-1)^k \sum_{j=0}^{M}(-1)^j \mathrm{pf}(b_0, b_1, \ldots, \hat{b}_j, \ldots, b_M)$$
$$\times \mathrm{pf}(b_j, c_k)\mathrm{pf}(c_0, c_1, \ldots, \hat{c}_k, \ldots, c_N) \tag{10}$$

which is nothing but a trivial identity obtained by interchanging the sums over j and k.
As a special case of the identity (9), we select $M = 2n, N = 2n + 2$ and characters b_j, c_k as follows;

$$b_0 = a_1, b_1 = 1, b_2 = 2, b_3 = 3, \ldots, b_M = 2n,$$

$$c_0 = a_2, c_1 = a_3, c_2 = a_4, c_3 = 1, c_4 = 2, c_5 = 3, \ldots, c_N = 2n.$$

Since the above choice makes summands on the left hand side of eq.(9) 0 except j=0, the left hand side is equal to

$$= \mathrm{pf}(1, 2, \ldots, 2n)\mathrm{pf}(a_1, a_2, a_3, a_4, 1, 2, 3, \ldots, 2n). \tag{11}$$

On the other hand, the right hand side of eq.(9) is equal to

$$\begin{aligned}
&= \sum_{k=0}^{N}(-1)^k \mathrm{pf}(b_0, b_1, \ldots, b_M, c_k)\mathrm{pf}(c_0, c_1, \ldots, \hat{c}_k, \ldots, c_N) \\
&= \sum_{k=0}^{2}(-1)^k \mathrm{pf}(b_0, b_1, \ldots, b_M, c_k)\mathrm{pf}(c_0, c_1, \ldots, \hat{c}_k, \ldots, c_N) \\
&+ \sum_{k_1=1}^{2n}(-1)^{k_1} \mathrm{pf}(a_1, 1, 2, \ldots, 2n, k_1)\mathrm{pf}(a_2, a_3, a_4, 1, 2, \ldots, \hat{k}_1, \ldots, 2n), \\
&\qquad\qquad (k = k_1 + 2), \\
&= \mathrm{pf}(b_0, b_1, \ldots, b_M, c_0)\mathrm{pf}(c_1, c_2, c_3, \ldots, c_N) \\
&-\mathrm{pf}(b_0, b_1, \ldots, b_M, c_1)\mathrm{pf}(c_0, c_2, c_3, \ldots, c_N) \\
&-\mathrm{pf}(b_0, b_1, \ldots, b_M, c_2)\mathrm{pf}(c_0, c_1, c_3, \ldots, c_N) \\
&= \mathrm{pf}(a_1, 1, 2, \ldots, 2n, a_2)\mathrm{pf}(a_3, a_4, 1, 2, \ldots, 2n) \\
&-\mathrm{pf}(a_1, 1, 2, \ldots, 2n, a_3)\mathrm{pf}(a_2, a_4, 1, 2, \ldots, 2n) \\
&+\mathrm{pf}(a_1, 1, 2, \ldots, 2n, a_4)\mathrm{pf}(a_2, a_3, 1, 2, \ldots, 2n)
\end{aligned} \tag{12}$$

Hence, eq.(9) results in eq.(5).

The identity (3) is obtained from eq.(9) by choosing $M = 2n, N = 2n + 2m - 2$ (m is odd) and characters b_j, c_k as follows;

$$b_0 = a_1, b_1 = 1, b_2 = 2, b_3 = 3, \dots, b_M = b_{2n} = 2n,$$

$$c_0 = a_2, c_1 = a_3, c_2 = a_4, c_3 = a_5, \dots, c_{2m-2} = a_{2m},$$

$$c_{2m-1} = 1, c_{2m} = 2, c_{2m+1} = 3, \dots, c_N = c_{2n+2m-2} = 2n.$$

Then eq.(9) results in the following equation.

$$\begin{aligned} &\mathrm{pf}(1,2,\dots,2n)\mathrm{pf}(a_1,a_2,a_3,\dots,a_{2m},1,2,3,\dots,2n) \\ &= \sum_{k_1=2}^{2m} (-1)^{k_1}\mathrm{pf}(a_1,a_{k_1},1,2,\dots,2n) \\ &\quad \times \mathrm{pf}(a_2,a_3,\dots,\hat{a}_{k_1},\dots,a_{2m},1,2,\dots,2n) \end{aligned} \tag{13}$$

which is the pfaffian identity (3).

The identity (4) is obtained from eq.(9) by choosing $M = 2n - 2, N = 2n + 2m - 1$ (m is odd) and characters b_j, c_k as follows;

$$b_0 = 1, b_1 = 2, b_2 = 3, b_3 = 4, \dots, b_M = b_{2n-2} = 2n - 1,$$

$$c_0 = a_1, c_1 = a_2, c_2 = a_3, c_3 = a_4, \dots, c_{m-1} = a_m,$$

$$c_m = 1, c_{m+1} = 2, c_{m+2} = 3, \dots, c_N = c_{2n+m-1} = 2n.$$

Then eq.(9) results in the following equation,

$$\begin{aligned} &\mathrm{pf}(1,2,\dots,2n)\mathrm{pf}(a_1,a_2,a_3,\dots,a_m,1,2,3,\dots,2n-1) \\ &= \sum_{j=1}^{m}(-1)^{j-1}\mathrm{pf}(a_j,1,2,\dots,2n-1) \\ &\quad \times \mathrm{pf}(a_1,a_2,a_3,\dots,\hat{a}_j,\dots,a_m,1,2,\dots,2n) \end{aligned}$$

which is the pfaffian identity (4).
We have observed that almost all bilinear soliton equations result in the pfaffian identities eqs.(3) and (4).

2.7 Expansion formulae for the pfaffian $(a_1, a_2, 1, 2, \cdots, 2n)$

The pfaffian $(a_1, a_2, 1, 2, \cdots, 2n)$ is, if $(a_1, a_2) = 0$, expanded in the following forms (i),(ii);

(i) $$(a_1, a_2, 1, 2, \cdots, 2n) = \sum_{1 \le j < k \le 2n} (-1)^{j+k-1} (a_1, a_2, j, k)(1, 2, \cdots, \hat{j}, \cdots, \hat{k}, \cdots, 2n)$$

(ii) $$(a_1, a_2, 1, 2, \cdots, 2n) = \sum_{j=2}^{2n} (-1)^j [(a_1, a_2, 1, j)(2, 3, \cdots, \hat{j}, \cdots, 2n) + (1, j)(a_1, a_2, 2, 3, \cdots, \hat{j}, \cdots, 2n)]$$

Let us prove (i) first. Expanding the pfaffian $(a_1, a_2, 1, 2, \cdots, 2n)$ first with respect to a_1 and next a_2, we have

$$\begin{aligned}(a_1, a_2, 1, 2, \cdots, 2n) &= \sum_{j=1}^{2n} \sum_{k=1}^{2n} (a_1, j)(a_2, k)(1, 2, \cdots, \hat{j}, \cdots, \hat{k}, \cdots, 2n) \\ &= \sum_{1 \le j < k \le 2n} (-1)^{j+k} [(a_1, j)(a_2, k) - (a_1, k)(a_2, j)](1, 2, \cdots, \hat{j}, \cdots, \hat{k}, \cdots, 2n).\end{aligned}$$

Noticing the relation $(a_1, a_2) = 0$, we obtain

$$= \sum_{1 \le j < k \le 2n} (-1)^{j+k-1} (a_1, a_2, j, k)(1, 2, \cdots, \hat{j}, \cdots, \hat{k}, \cdots, 2n).$$

In order to prove (ii), we expand the pfaffian $(a_1, a_2, 1, 2, \cdots, 2n)$ with respect to the character 1.

$$\begin{aligned}(a_1, a_2, 1, 2, \cdots, 2n) &= (1, a_1)(a_2, 2, \cdots, 2n) - (1, a_2)(a_1, 2, \cdots, 2n) \\ &\quad + \sum_{j=2}^{2n} (-1)^j (1, j)(a_1, a_2, 2, 3, \cdots, \hat{j}, \cdots, 2n).\end{aligned}$$

Next, pfaffians $(a_2, 2, \cdots, 2n)$ and $(a_1, 2, \cdots, 2n)$ are expanded as

$$\begin{aligned}&= (1, a_1) \sum_{j=2}^{2n} (-1)^j (a_2, j)(2, 3, \cdots, \hat{j}, \cdots, 2n) \\ &\quad -(1, a_2) \sum_{j=2}^{2n} (-1)^j (a_1, j)(2, 3, \cdots, \hat{j}, \cdots, 2n) \\ &\quad + \sum_{j=2}^{2n} (-1)^j (1, j)(a_1, a_2, 2, 3, \cdots, \hat{j}, \cdots, 2n).\end{aligned}$$

Noticing the relation $(a_1, a_2) = 0$, we obtain

$$= \sum_{j=2}^{2n}(-1)^j[(a_1, a_2, 1, j)(2, 3, \cdots, \hat{j}, \cdots, 2n) + (1, j)(a_1, a_2, 2, 3, \cdots$$

If we consider a pfaffian $(b_1, b_2, 1, 2, \cdots, 2n)$ instead of $(1, 2, \cdots, 2n)$ in the expansion formula (i), this formula can be generalized as follows;

(iii) $(a_1, a_2, b_1, b_2, 1, 2, \cdots, 2n)$

$$= \sum_{j=1}^{2n} \sum_{k=j+1}^{2n} (a_1, a_2, j, k)(b_1, b_2, 1, 2, 3, \because, \hat{j}, \cdots, \hat{k}, \cdots, 2n),$$

where $(a_j, a_k) = (a_j, b_k) = (b_j, b_k) = 0$, for $j, k = 1, 2$.
We make use of these expansion formulae as pfaffian difference (differential) formulae later.

2.8 Difference formula for pfaffians

In order to show that bilinear soliton equations result in the pfaffian identities, we study difference formula for pfaffians.
We consider a $2n$-th degree pfaffian with special entries,

$$\mathrm{pf}(a_1, a_2, a_3, \ldots, a_{2n})_\alpha \tag{14}$$

whose entries $\mathrm{pf}(a_i, a_j)_\alpha$ are given by summation of pfaffians.

$\mathrm{pf}(a_j, a_k)_\alpha = \mathrm{pf}(a_j, a_k) + \lambda\, \mathrm{pf}(d_0, d_1, a_j, a_k)$

where λ is a parameter and $\mathrm{pf}(d_0, d_1) = 0$.
The pfaffian (14) obeys the usual expansion rule,

$$\begin{aligned}&\mathrm{pf}(a_1, a_2, a_3, \ldots, a_{2n})_\alpha \\ &= \sum_{j=2}^{2n}(-1)^j \mathrm{pf}(a_1, a_j)_\alpha \mathrm{pf}(a_2, a_3, \ldots, \hat{a}_j, \ldots, a_{2n})_\alpha.\end{aligned} \tag{15}$$

Then the following formula holds for arbitrary n,

$$\begin{aligned}\mathrm{pf}(a_1, a_2, a_3, \ldots, a_{2n})_\alpha &= \mathrm{pf}(a_1, a_2, a_3, \ldots, a_{2n}) \\ &+\lambda\, \mathrm{pf}(d_0, d_1, a_1, a_2, a_3, \ldots, a_{2n}).\end{aligned} \tag{16}$$

Let us prove the formula (16) by induction. Obviously, the formula holds if $n = 1$. We suppose that the formula holds for an arbitrary $(2n-2)$-th degree pfaffian,

$$\begin{aligned}&\mathrm{pf}(a_2, a_3, \cdots, \hat{a}_j, \cdots, 2n)_\alpha \\ &\quad = \mathrm{pf}(a_2, a_3, \cdots, \hat{a}_j, \cdots, 2n) + \lambda\ \mathrm{pf}(d_0, d_1, a_2, a_3, \cdots, \hat{a}_j, \cdots, 2n)\end{aligned} \tag{17}$$

Expanding the left hand side in eq.(16), we have

$$\begin{aligned}&\mathrm{pf}(a_1, a_2, a_3, \ldots, a_{2n})_\alpha \\ &= \sum_{j=2}^{2n} (-1)^j \mathrm{pf}(a_1, a_j)_\alpha \mathrm{pf}(a_2, a_3, \ldots, \hat{a}_j, \ldots, a_{2n})_\alpha.\end{aligned} \tag{18}$$

Employing eq.(17) we have

$$\begin{aligned}&= \sum_{j=2}^{2n} (-j)^j [\mathrm{pf}(a_1, a_j) + \lambda \mathrm{pf}(d_0, d_1, a_1, a_j)][\mathrm{pf}(a_2, a_3, \cdots, \hat{a}_j, \cdots, a_{2n}) \\ &\qquad + \lambda\ \mathrm{pf}(d_0, d_1, a_2, a_3, \cdots, \hat{a}_j, \cdots, a_{2n})],\end{aligned}$$

whose coefficient in λ^0 is obviously $\mathrm{pf}(a_1, a_2, \cdots, a_{2n})$. Coefficient in λ^1 is $\mathrm{pf}(d_0, d_1, a_1, a_2, \cdots, a_{2n})$ due to the expansion formula (ii).
Expanding the following zero-valued pfaffian, we obtain

$$\begin{aligned}0 &= \mathrm{pf}(d_0, d_1, a_1, a_2, \cdots, a_{2n}, d_0, d_1) \\ &= \sum_{j=2}^{2n} \mathrm{pf}(d_0, d_1, a_1, a_j)\mathrm{pf}(a_2, a_3, \cdots, \hat{a}_j, \cdots, a_{2n}, d_0, d_1),\end{aligned}$$

from which we find that coefficient in λ^2 is zero. Therefore, we have

$$\mathrm{pf}(a_1, a_2, a_3, \ldots, a_{2n})_\alpha = \mathrm{pf}(a_1, a_2, a_3, \ldots, a_{2n}) + \lambda \mathrm{pf}(d_0, d_1, a_1, a_2, a_3, \ldots, a_{2n}),$$

which completes the proof.

2.9 Difference formula for determinants

We have the pfaffian expression for a determinant,

$$\det |a_{j,k}|_{1\le j,k\le n} = \mathrm{pf}(a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*) \tag{19}$$

where $\mathrm{pf}(a_j, a_k) = \mathrm{pf}(a_j^*, a_k^*) = 0,\ \mathrm{pf}(a_j, a_k^*) = a_{j,k}$. If entries of a determinant are expressed by pfaffians,

$$\mathrm{pf}(a_j, a_k^*)_\alpha = \mathrm{pf}(a_j, a_k^*)' + \mathrm{pf}(d_\gamma, a_j, a_k^*, d_\delta^*)', \tag{20}$$
$$\mathrm{pf}(a_j, a_k)_\alpha = \mathrm{pf}(a_j^*, a_k^*)' = \mathrm{pf}(d_\gamma, d_\delta^*)' = 0, \tag{21}$$

we obtain, by using the difference formula for paffians,

$$\begin{aligned}&\mathrm{pf}(a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*)_\alpha \\ &\quad = \mathrm{pf}(a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*,)' \\ &\quad + \mathrm{pf}(d_\gamma, a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*, d_\delta^*)'.\end{aligned} \tag{22}$$

Suppose that the entries have the following properties;

$$\mathrm{pf}(a_j, a_k^*)' = \mathrm{pf}(a_j, a_k^*)c_k/c_j, \tag{23}$$
$$\mathrm{pf}(d_\gamma, a_k^*)' = \mathrm{pf}(d_\gamma, a_k^*)c_k, \tag{24}$$
$$\mathrm{pf}(a_j, d_\delta^*)' = \mathrm{pf}(a_j, d_\delta^*)/c_j \tag{25}$$

where all $c_j(\neq 0,)\ j = 1, 2, \ldots, n$ are parameters. Then we have the following relation ,

$$\begin{aligned}&\mathrm{pf}(d_\gamma, a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*, d_\delta^*)' \\ &\quad = \mathrm{pf}(d_\gamma, a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*, d_\delta^*).\end{aligned} \tag{26}$$

Accordingly the difference formula for the determinant is expressed by

$$\begin{aligned}&\mathrm{pf}(a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*)_\alpha \\ &= \mathrm{pf}(a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*) \\ &+ \mathrm{pf}(d_\gamma, a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*, d_\delta^*),\end{aligned} \tag{27}$$

provided that

$$\mathrm{pf}(a_j, a_k^*)_\alpha = [\mathrm{pf}(a_j, a_k^*) + \mathrm{pf}(d_\gamma, a_j, a_k^*, d_\delta^*)]c_k/c_j, \tag{28}$$
$$\mathrm{pf}(a_j, a_k)_\alpha = \mathrm{pf}(a_j^*, a_k^*)_\alpha = 0, \tag{29}$$
$$\mathrm{pf}(d_\gamma, d_\delta^*) = 0. \tag{30}$$

3 Pfaffian Solutions to the Discrete KdV Equation

3.1 Discretization of the KdV equation

The KdV equation

$$\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0$$

is transformed into the bilinear form

$$D_x(D_t+D_x^3)f\cdot f=0 \tag{31}$$

through the logarithmic transformation

$$\begin{aligned} u &= 2\frac{\partial^2}{\partial x^2}\log f \\ &= \frac{D_x^2 f\cdot f}{f^2}. \end{aligned}$$

We rewite the above equation as

$$(D_t+D_x^3)f\cdot f_x=0. \tag{32}$$

A semi-discrete KdV equation is obtained by discretizing the spatial part of the bilinear equation (32),

$$D_t f_n\cdot f_{n+1}+\frac{1}{\epsilon}(f_{n+1}f_n-f_{n+2}f_{n-1})=0, \tag{33}$$

where ϵ is a spatial interval.

Replacing the differential bilinear operator D_t by a corresponding difference operator and taking a gauge-invariance of the bilinear equation into account, we obtain a discrete KdV equation,

$$\begin{aligned} &f_n^{m+1}f_{n+1}^m-f_n^m f_{n+1}^{m+1} \\ &\quad +q_0(f_{n+1}^{m+1}f_n^m-f_{n+2}^{m+1}f_{n-1}^m)=0, \end{aligned} \tag{34}$$

where $q_0=\delta/\epsilon$, δ being a time-interval.

3.2 Soliton solution to the discrete KdV equation

It is easy to obtain 2-soliton solution to the discrete bilinear KdV equation (34) by using a perturbational method. We find

$$f_n^m = 1 + a_1 \exp\eta_1 + a_2 \exp\eta_2 + a_{1,2}a_1a_2 \exp(\eta_1 + \eta_2), \tag{35}$$
$$\exp\eta_j = \Omega_j^m P_j^n, \tag{36}$$
$$\Omega_j = \frac{1 + q_0/P_j}{1 + q_0 P_j}, \quad \text{for} \quad j = 1, 2, \tag{37}$$
$$a_{1,2} = (P_1 - P_2)^2/(P_1P_2 - 1)^2. \tag{38}$$

where a_1, a_2 are arbitrary parameters. Hereafter we choose the parameters to be $a_j = 1/(p_j^2 - 1)$ for $j = 1, 2$.
We express 2-soliton solution to the discrete KdV equation by a pfaffian,

$$f_n^m = \mathrm{pf}(a_1, a_2, a_2^*, a_1^*), \tag{39}$$
$$\mathrm{pf}(a_j, a_k) = \mathrm{pf}(a_j^*, a_k^*) = 0, \tag{40}$$
$$\mathrm{pf}(a_j, a_k^*) = \delta_{j,k} + \exp[(\eta_j + \eta_k)/2]/(P_jP_k - 1), \tag{41}$$

which is equal to the following determinant expression,

$$f_n^m = \begin{vmatrix} 1 + \exp(\eta_1)/(P_1^2 - 1) & \exp[(\eta_1 + \eta_2)/2]/(P_1P_2 - 1) \\ \exp[(\eta_1 + \eta_2)/2]/(P_1P_2 - 1) & 1 + \exp(\eta_2)/(P_2^2 - 1) \end{vmatrix}.$$

4 Pfaffian identities of the discrete bilinear Kdv equation

We are going to show that the bilinear equation results in the pfaffian identity

$$\begin{aligned} &\mathrm{pf}(a_1, a_2, a_3, a_4, 1, 2, \ldots, 2n)\mathrm{pf}(1, 2, \ldots, 2n) \\ &\quad = \mathrm{pf}(a_1, a_2, 1, 2, \ldots, 2n)\mathrm{pf}(a_3, a_4, 1, 2, \ldots, 2n) \\ &\quad -\mathrm{pf}(a_1, a_3, 1, 2, \ldots, 2n)\mathrm{pf}(a_2, a_4, 1, 2, \ldots, 2n) \\ &\quad +\mathrm{pf}(a_1, a_4, 1, 2, \ldots, 2n)\mathrm{pf}(a_2, a_3, 1, 2, \ldots, 2n). \end{aligned} \tag{42}$$

In order to show that bilinear discrete KdV eq.(34) results in the pfaffian identiy (42), we have to express $f_{n+1}^m, f_{n-1}^m, f_n^{m+1}, etc$ by pfaffians.

To this end we start with the simplest case of f_n^m, 1-soliton solution. Let us introduce a pfaffian entry pf(j,k^*), for $j,k=1,2,\cdots$, by

$$\mathrm{pf}(j,k^*)=\delta_{j,k}+\exp\left[(\eta_j+\eta_k)/2\right]/(P_jP_k-1),$$

where $\delta_{j,k}$ is a Kronecker's delta.
Then 1-soliton solution is expressed by the pfaffian,

$$f_n^m=1+\exp(\eta_1)/(P_1^2-1)=\mathrm{pf}(1,1^*).$$

We have

$$\mathrm{pf}(1,1^*)_{n+1}=f_{n+1}^m=1+P_1\exp(\eta_1)/(P_1^2-1),$$

wich is rewritten as

$$\begin{aligned}&=1+\exp(\eta_1)/(P_1^2-1)+(P_1-1)/(P_1^2-1)\exp(\eta_1)\\&=f_n^m+\frac{1}{P_1+1}\exp(\eta_1).\end{aligned}$$

The last term in the above expression is expressed by a pfaffian,

$$\frac{1}{P_1+1}\exp(\eta_1)=\mathrm{pf}(d_p,1,1^*,d_0^*),$$

by introducing the following pfaffian entries;

$$\begin{aligned}&\mathrm{pf}(d_p,j)=0,\\&\mathrm{pf}(d_p,j^*)=\frac{1}{P_j+1}\exp(\eta_j/2),\\&\mathrm{pf}(d_p,d_0^*)=0,\\&\mathrm{pf}(j,d_0^*)=-\exp(\eta_j/2),\\&\mathrm{pf}(j^*,d_0^*)=0,\quad\text{for}\quad j=1,2,\cdots.\end{aligned}$$

Because we have

$$\begin{aligned}\mathrm{pf}(d_p,1,1^*,d_0^*)&=-\mathrm{pf}(d_p,1^*)\mathrm{pf}(1,d_0^*)\\&=\frac{1}{P_j+1}\exp(\eta_j).\end{aligned}$$

Accordingly we find that the difference of the pfaffian is expressed by a sum of pfaffians;

$$\mathrm{pf}(1,1^*)_{n+1}=\mathrm{pf}(1,1^*)+\mathrm{pf}(d_p,1,1^*,d_0^*).$$

Following the same procedure we obtain

$$
\begin{aligned}
\mathrm{pf}(1,1^*)_{n-1} &= \mathrm{pf}(1,1^*) + \mathrm{pf}(d_p,1,1^*,d_n^*),\\
\mathrm{pf}(1,1^*)^{m+1} &= \mathrm{pf}(1,1^*) + \mathrm{pf}(d_q,1,1^*,d_n^*),\\
\mathrm{pf}(1,1^*)_{n+2}^{m+1} &= \mathrm{pf}(1,1^*) + \mathrm{pf}(d_q,1,1^*,d_0^*)/q_0,\\
\mathrm{pf}(1,1^*)_{n+1}^{m+1} &= \mathrm{pf}(1,1^*) + \mathrm{pf}(d_p,1,1^*,d_0^*) - \mathrm{pf}(d_q,1,1^*,d_0^*),\\
&= \mathrm{pf}(1,1^*) - \mathrm{pf}(d_q,1,1^*,d_n^*)/q_0 - \mathrm{pf}(d_p,1,1^*,d_n^*),
\end{aligned}
$$

where we have introduced pfaffian entries as follow

$$
\begin{aligned}
&\mathrm{pf}(d_p,d_q) = 0,\\
&\mathrm{pf}(d_p,d_n^*) = 0,\\
&\mathrm{pf}(d_q,j) = 0,\\
&\mathrm{pf}(d_q,j^*) = \frac{1}{P_j+1/q_0}\exp(\eta_j/2),\\
&\mathrm{pf}(d_q,d_n^*) = 0,\\
&\mathrm{pf}(d_q,d_0^*) = 0,\\
&\mathrm{pf}(j,d_n^*) = \frac{1}{P_j}\exp(\eta_j/2),\\
&\mathrm{pf}(d_n^*,d_0^*) = 0,\\
&\mathrm{pf}(j^*,d_0^*) = 0, \quad \text{for} \quad j = 1,2,\cdots.
\end{aligned}
$$

In the above pfaffian representations $\mathrm{pf}(1,1^*)_{n+1}^{m+1}$ is not uniquely determined. We fixed it by using 2-soliton solution. We have, for 2-soliton solution,

$$
f_n^m = \mathrm{pf}(1,2,2^*,1^*).
$$

We assume that the pfaffian representations of $\mathrm{pf}(1,2,2^*,1^*)_{n+1}^{m+1}$ has the following form

$$
\begin{aligned}
\mathrm{pf}(1,2,2^*,1^*)_{n+1}^{m+1} &= \mathrm{pf}(1,2,2^*,1^*) + c_1\mathrm{pf}(d_p,1,2,2^*,1^*,d_0^*)\\
&+c_2\mathrm{pf}(d_q,1,2,2^*,1^*,d_0^*) + c_3\mathrm{pf}(d_p,1,2,2^*,1^*,d_n^*)\\
&+c_4\mathrm{pf}(d_q,1,2,2^*,1^*,d_n^*) + c_5\mathrm{pf}(d_q,d_q,1,2,2^*,1^*,d_n^*,d_0^*),
\end{aligned}
$$

where $c_1, c_2, \cdots, c_5$ are parameters to be determined. By using a computer algebra (Mathematica,Reduce etc.), we determine the parameters,

$$
\mathrm{pf}(1,2,2^*,1^*)_{n+1}^{m+1}
$$

$$= \mathrm{pf}(1,2,2^*,1^*) + [-\mathrm{pf}(d_p,1,2,2^*,1^*,d_0^*) + \mathrm{pf}(d_q,1,2,2^*,1^*,d_0^*)$$
$$+q_0 * \mathrm{pf}(d_p,1,2,2^*,1^*,d_n^*) - \mathrm{pf}(d_q,1,2,2^*,1^*,d_n^*)$$
$$-\mathrm{pf}(d_q,d_q,1,2,2^*,1^*,d_n^*,d_0^*)]/(q_0-1)$$

and confirm the pfaffian expressions;

$$\mathrm{pf}(1,2,2^*,1^*)_{n-1} = \mathrm{pf}(1,2,2^*,1^*) + \mathrm{pf}(d_p,1,2,2^*,1^*,d_n^*),$$
$$\mathrm{pf}(1,2,2^*,1^*)^{m+1} = \mathrm{pf}(1,2,2^*,1^*) + \mathrm{pf}(d_q,12,2^*,,1^*,d_n^*),$$
$$\mathrm{pf}(1,2,2^*,1^*)^{m+1}_{n+2} = \mathrm{pf}(1,2,2^*,1^*) + \mathrm{pf}(d_q,1,2,2^*,1^*,d_0^*)/q_0.$$

Substituting these expressions into the discrete bilinear KdV equation (34) we find that eq.(34) is reduced to the pfaffian identity

$$\begin{aligned}
&\mathrm{pf}(d_p,d_q,1,2,2^*,1^*,d_n^*.d_0^*)\mathrm{pf}(1,2,2^*,1^*) \\
&\quad = \mathrm{pf}(d_p,d_q,1,2,2^*,1^*)\mathrm{pf}(1,2,2^*,1^*,d_n^*,d_0^*) \\
&\quad -\mathrm{pf}(d_p,1,2,2^*,1^*,d_n^*)\mathrm{pf}(d_q,1,2,2^*,1^*,d_0^*) \\
&\quad +\mathrm{pf}(d_p,1,2,2^*,1^*,d_0^*)\mathrm{pf}(d_q,1,2,2^*,1^*,d_n^*),
\end{aligned} \tag{43}$$

where the first term in the right hand side is identically equal to zero ($\mathrm{pf}(d_p,d_q) = \mathrm{pf}(1,2) = 0$).

Accordingly we have shown that the discrete bilinear KdV equation (34) results in the pfaffian identity for 2-soliton solution.
The pfaffian identity (43) for 2-soliton solution is easily extended to the pfaffian identity for N-soliton solution,

$$\begin{aligned}
&\mathrm{pf}(d_p,d_q,1,2,\cdots,N,N^*,\cdots,2^*,1^*,d_n^*.d_0^*)\mathrm{pf}(1,2,\cdots,N,N^*,\cdots,2^*,1^*) \\
&\quad = \mathrm{pf}(d_p,d_q,1,2,\cdots,N,N^*,\cdots,2^*,1^*)\mathrm{pf}(1,2,\cdots,N,N^*,\cdots,2^*,1^*,d_n^*,d_0^*) \\
&\quad -\mathrm{pf}(d_p,1,2,\cdots,N,N^*,\cdots,2^*,1^*,d_n^*)\mathrm{pf}(d_q,1,2,\cdots,N,N^*,\cdots,2^*,1^*,d_0^*) \\
&\quad +\mathrm{pf}(d_p,1,2,\cdots,N,N^*,\cdots,2^*,1^*,d_0^*)\mathrm{pf}(d_q,1,2,\cdots,N,N^*,\cdots,2^*,1^*,d_n^*),
\end{aligned}$$

Thus we have shown that the N-soliton solution expressed by the pfaffian,

$$f_n^m = pf(1,2,\cdots,N,N^*,\cdots,2^*,1^*)$$

satisfies the discrete bilinear KdV equation (34).