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拡大同相写像とカントール集合となる極小集合について

加藤久男 (Hisao Kato)

筑波大学数学系 (Institute of Mathematics, University of Tsukuba)

1 Introduction.

All spaces considerd in this paper are assumed to be metric spaces. *Maps* are continuous functions. By a *compactum* we mean a nonempty compact metric space. A *continuum* is a connected nondegenerate compactum. A homeomorphism $f: X \to X$ of a compactum X with metric d is called *expansive* (see[4], [12] and [13]) if there is $c>0$ such that for any $x, y \in X$ and $x \neq y$, then there is an integer $n \in \mathbf{Z} = \{0, \pm 1, \pm 2, .., \}$ such that

$$d(f^n(x), f^n(y)) > c.$$

A homeomorphism $f: X \to X$ of a compactum X is *continuum-wise expansive* [5] if there is $c>0$ such that if A is a nondegenerate subcontinuum of X, then there is an integer $n \in \mathbf{Z}$ such that

$$\operatorname{diam} f^n(A) > c,$$

where $\operatorname{diam} B = \sup\{d(x,y) |\, x, y \in B\}$ for a set B. Such a positive number c is called an *expansive constant* for f. Note that each expansive homeomorphism is continuum-wise expansive, but the converse assertion is not true. There are many continuum-wise expansive homeomorphisms which are not expansive (eg., see [5], [6] and [8]). By the definitions, we see that expansiveness and continuum-wise expansiveness do not depend on the choice of the metric d of X. These notions have been extensively studied in the area of topological dynamics, ergodic theory and continuum theory (see [1]-[13]).

In [11], R. Mañé proved that minimal sets of expansive homeomorphisms are 0-dimensional. More generally, minimal sets of continuum-wise expansive homeomorphisms are 0-dimensional (see [5]). Also, for each continuum-wise expansive homeomorphism $f: X \to X$ of a compactum X with $\dim X > 0$, there is an f-invariant closed subset Y of X such that $\dim Y > 0$ and $f|Y : Y \to Y$ is weakly chaotic in the sense of Devaney (see [9]). In this paper, we prove the following result: If $f: X \to X$ is a continuum-wise expansive homeomorphism of a compactum X with $\dim X = 1$, then there is a Cantor set Z in X such that for some natural number N, $f^N(Z) = Z$ and $f^N|Z : Z \to Z$ is semiconjugate to the shift homeomorphism $\widetilde{\sigma} : \widetilde{\Sigma} \to \widetilde{\Sigma}$, where $\widetilde{\Sigma}$ is the Cantor set $\{0,1\}^{\mathbf{Z}}$. As a corollary, there is a family $\{C_\alpha |\ \alpha \in \Lambda\}$ of minimal sets C_α of f such that each C_α is a Cantor set, $\mathrm{Cl}\,(\bigcup\{C_\alpha |\ \alpha \in \Lambda\}) = Y$ is 1-dimensional and $f|Y : Y \to Y$ is weakly chaotic in the sense of Devaney. Also, we study infinite minimal sets of continuum-wise fully expansive homeomorphisms.

2 Continuum-wise expansive homeomorphisms and infinite minimal sets.

Let X be a compact metric space with metric d and $C(X)$ the hyperspace of all nonempty subcontinua of X with the Hausdorff metric d_H defined by

$$d_H(A,B) = \inf\{\epsilon > 0 \mid B \subset N(A,\epsilon), A \subset N(B,\epsilon)\}$$

for closed nonempty subsets A, B of X, where $N(A,\epsilon)$ denotes the ϵ-neighborhood of A in X. Let $f : X \to X$ be a homeomorphism. A nonempty closed subset M of X is a *minimal set* of f if M is f-invariant, i.e., $f(M) = M$, and no proper nonempty closed subset A of M is f-invariant. Note that a closed subset M of X is a minimal set of f if and only if for any $x \in M$,

$$M = \omega(x) = \{y \in X \mid \text{there is a sequence } n_1 < n_2 < \dots, \text{of natural numbers such that } \lim_{i\to\infty} f^{n_i}(x) = y\}.$$

Note that every homeomorphism of a compactum has a minimal set. If a minimal set M is a finite set, then M is a periodic orbit of some point $x \in X$, i.e., $M = \{x\ (= f^n(x)), f(x), f^2(x), .., f^{n-1}(x)\}$. If a minimal set M is an infinite set, then M is perfect. If an infinite minimal set M is 0-dimensional, then M is a Cantor set.

Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with $\dim X > 0$. Note that every minimal set of f is 0-dimensional (see [5, Theorem (5.2)]). Consider the following sets (see [9]):

(1) $\mathcal{I}(f) = \{A \mid A \text{ is an } f\text{-invariant closed subset of } X\}$.

(2) $\mathcal{M}_\infty(f)$ is the set of all infinite minimal sets of f. If $M \in \mathcal{M}_\infty(f)$, then M is a Cantor set.

(3) $\mathcal{I}^+(f) = \{A \in \mathcal{I}(f) \mid \dim A > 0\}$.

(4) $\mathcal{D}(f)$ is the set of all minimal elements in the partial order of $\mathcal{I}^+(f)$ by inclusion. Note that $\mathcal{D}(f) \neq \phi$ (see [9]).

Let Σ be the Cantor set, i.e., $\Sigma = \{0,1\}^\omega$. The shift map $\sigma : \Sigma \to \Sigma$ is defined by $\sigma(x_0, x_1, x_2, ...,) = (x_1, x_2, ...,)$ for each $(x_0, x_1, x_2, ...,) \in \Sigma$. Also, let $\widetilde{\Sigma} = \{0,1\}^{\mathbf{Z}} (= \{(x_n)_n \mid x_n \in \{0,1\}, n \in \mathbf{Z}\})$. The shift homeomorphism $\widetilde{\sigma} : \widetilde{\Sigma} \to \widetilde{\Sigma}$ is defined by $\widetilde{\sigma}((x_n)_n) = (x_{n+1})_n$ for $(x_n)_n \in \widetilde{\Sigma}$. Note that $\widetilde{\Sigma}$ is identified with the inverse limit of the inverse sequence $\{\Sigma, \sigma\}$ and $\widetilde{\sigma}$ is the homeomorphism induced by σ. Let $q : \widetilde{\Sigma} \to \Sigma$ be the natural projection. Then $q \cdot \widetilde{\sigma} = \sigma \cdot q$.

First, we obtain the following theorem.

(2.1) Theorem. *If $f : X \to X$ is a continuum-wise expansive homeomorphism of a compactum X with $\dim X = 1$, then there is a Cantor set Z in X such that for* *some natural number* N, Z *is* f^N*-invariant and* $f^N|Z : Z \to Z$ *is semiconjugate to the shift homeomorphism* $\widetilde{\sigma} : \widetilde{\Sigma} \to \widetilde{\Sigma}$, *i.e., there is an onto map* $p : Z \to \widetilde{\Sigma}$ *such that* $\widetilde{\sigma} \cdot p = p \cdot (f^N|Z)$.

For the proof of (2.1), we need the followings.

(2.2) Lemma (see [5, (2.4) and (2.5)]). *Let* $f : X \to X$ *be a continuum-wise expansive homeomorphism of a compactum* X *with* $\dim X > 0$. *Let* $c > 0$ *be an expansive constant of* f *and* $0 < 2\epsilon \le c$. *Then there is a positive number* $\delta \le \epsilon$ *satisfying the following conditions:*

(1) $\mathbf{V}^s(\delta;\epsilon) \neq \phi$ *or* $\mathbf{V}^u(\delta;\epsilon) \neq \phi$, *where*

$$\mathbf{V}^s(\delta;\epsilon) = \{A \in C(X) |\, \operatorname{diam} A \ge \delta, \textit{ and } \operatorname{diam} f^n(A) \le \epsilon \textit{ for each } n \ge 0\},$$
$$\mathbf{V}^u(\delta;\epsilon) = \{A \in C(X) |\, \operatorname{diam} A \ge \delta, \textit{ and } \operatorname{diam} f^{-n}(A) \le \epsilon \textit{ for each } n \ge 0\}.$$

In particular, if $A \in \mathbf{V}^s(\delta;\epsilon)$, *then* $\lim_{n\to\infty} \operatorname{diam} f^n(A) = 0$. *If* $A \in \mathbf{V}^u(\delta;\epsilon)$, *then* $\lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0$.

(2) For each $\gamma > 0$ *there is a natural number* $N = N(\gamma)$ *such that if* A *is a subcontinuum of* X *with* $\operatorname{diam} A \ge \gamma$, *then either* $\operatorname{diam} f^n(A) \ge \delta$ *for all* $n \ge N$ *or* $\operatorname{diam} f^{-n}(A) \ge \delta$ *for all* $n \ge N$.

(2.3) Lemma. *Let* X *be a 1-dimensional compactum. For any* $\epsilon > 0$ *there is a family* $\{U_1, U_2, .., U_m\}$ *of open subsets of* X *such that* $\operatorname{Cl}(U_i) \bigcap \operatorname{Cl}(U_j) = \phi\, (i \neq j)$, $\operatorname{diam} U_i < \epsilon$ *for each* i *and the diameters of components of* $X - (\bigcup_{i=1}^m U_i)$ *are less than* ϵ.

To prove the next theorem (2.5), we need the following lemma.

(2.4) Lemma. *Let* $f : X \to X$ *be a continuum-wise expansive homeomorphism of a compactum* X *with* $\dim X > 0$ *and* $Y \in \mathcal{D}(f)$. *Let* c, ϵ *and* δ *be positive numbers as in (2.2). Then for each* $0 < \gamma \le \delta$ *and nonempty open subset* V *of* Y *there is a natural number* $J = J(V, \gamma)$ *such that if* $A \subset Y$ *and* $A \in \mathbf{V}^u(\gamma;\epsilon)$, *then there is a natural number* $j = j(A)$ *such that* $1 \le j \le J$ *and* $f^j(A) \bigcap V \neq \phi$.

(2.5) Theorem. *Let* $f : X \to X$ *be a continuum-wise expansive homeomorphism of a compactum* X *with* $\dim X = 1$ *and* $Y \in \mathcal{D}(f)$. *Then there is a sequence* $M_1, M_2, ..,$ *of minimal sets of* $f|Y$ *such that each* M_n *is a Cantor set and* $\lim_{n\to\infty} d_H(Y, M_n) = 0$. *In particular,*

$$\operatorname{Cl}(\bigcup\{M |\, M \in \mathcal{M}_\infty(f|Y)\}) = Y.$$

(2.6) Proposition. *Let* $f : X \to X$ *be a continuum-wise expansive homeomorphism of a compactum* X *with* $\dim X > 0$ *and* $Y \in \mathcal{D}(f)$. *Then there is a sequence* $M_1, M_2, ..,$ *of minimal sets of* $f|Y$ *such that* $\lim_{n\to\infty} d_H(Y, M_n) = 0$.

Let $f : X \to X$ be a homeomorphism of a compactum X. Then f is *sensitive* if there is $c > 0$ such that if $x \in X$ and U is any neighborhood of x in X, there is $y \in U$ and a natural number $n \geq 1$ such that $d(f^n(x), f^n(y)) > c$. f is *topologically transitive* if there is a point $x \in X$ such that the orbit $\{x, f(x), f^2(x), .., \}$ of x is dense in X. Also, f is *weakly chaotic in the sense of Devaney* (see [9]) if f is sensitive, f is topologically transitive and $\mathrm{Cl}(\bigcup\{M \mid M \text{ is a minimal set of } f\}) = X$.

(2.7) Remark. In (2.5), we see that $f|Y : Y \to Y$ is weakly chaotic in the sense of Devaney (see [9]).

3 Infinite minimal sets of continuum-wise fully expansive homeomorphisms.

A homeomorphism $f : X \to X$ of a continuum X is *continuum-wise fully expansive* provided that for any $\epsilon > 0$ and $\delta > 0$, there is a natural number $N = N(\epsilon, \delta) > 0$ such that if A is a subcontinuum of X and $\operatorname{diam} A \geq \delta$, then either $d_H(f^n(A), X) < \epsilon$ for all $n \geq N$, or $d_H(f^{-n}(A), X) < \epsilon$ for all $n \geq N$. By the similar proofs as before, we obtain the following result.

(3.1) Theorem. *Let $f : X \to X$ be a continuum-wise fully expansive homeomorphism of a nondegenerate continuum X. Then*

(1) there is a Cantor set Z in X such that for some natural number N, Z is f^N-invariant and $f^N|Z : Z \to Z$ is semiconjugate to the shift homeomorphism $\widetilde{\sigma} : \widetilde{\Sigma} \to \widetilde{\Sigma}$, and

(2) there is a sequence $M_1, M_2, ..,$ of minimal sets of f such that each M_n is a Cantor set and $\lim_{n\to\infty} d_H(X, M_n) = 0$. In particular,

$$\mathrm{Cl}(\bigcup\{M \mid M \in \mathcal{M}_\infty(f)\}) = X.$$

(3.2) Example. Let $f : T^2 \to T^2$ be an Anosov diffeomorphism, say

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

on the 2-dimensional torus T^2. Then f is a continuum-wise fully expansive homeomorphism. Hence $\mathrm{Cl}(\bigcup\{M \mid M \in \mathcal{M}_\infty(f)\}) = T^2$.

(3.3) Problem. Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with $\dim X \geq 2$. In this case, are the conclusions of (2.1) and (2.5) true ?

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Hisao Kato
Institute of Mathematics
University of Tsukuba
Ibaraki, 305 Japan
e-mail: hisakato@sakura.cc.tsukuba.ac.jp