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The construction of chaotic maps on dendrites which commute to continuous maps with positive topological entropy on the unit interval

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1 Introduction

The purpose of this note is to introduce some results in [AC]. In [ACKY], a new space Z and the continuous map from Z to itself have been constructed by the geometrical method. The structure of Z changes corresponding to the behavior of a continuous map f from a finite graph to itself and the method of choosing an invariant subset of f. And it is shown that the space Z is a regular curve. The pointwise P-expansiveness plays an important role to decide the structure of the space Z. In this note, first we introduce that, for each continuous map f from the unit interval to itself, f has positive topological entropy if and only if f is pointwise P-expansive for some periodic orbit P of f.

The notion of chaos is important in the study of topological dynamical systems. The paper in which the word "chaos" first appeared was written by Li and Yorke [LY]. The word "chaos", however, is described by various definitions. One of those definitions is proposed by Devaney [D] as in Definition 1.1. Huang and Ye have showed that every chaotic map in the sense of Devaney from a compact metric space to itself is chaotic in the sense of Li-Yorke [HY].

Definition 1.1 Let f be a continuous map from a compact metric space (X, d) to itself. This map f is *chaotic in the sense of Devaney* if

(1) f is *topologically transitive*, that is, for any non-empty open sets U and V in X, there exists some non-negative integer k such that $f^k(U) \cap V \neq \emptyset$,

(2) the set of all periodic points of f is dense in X, and

(3) f has *sensitive dependence on initial conditions*, i.e., there exists a number $\delta > 0$ such that for every point x of X and every neighborhood V of x, there exists a point y of V and a non-negative integer n such that $d(f^n(x), f^n(y)) > \delta$.

In [BBCDS], it is shown that the above conditions (1) and (2) imply the condition (3). Furthermore in [BV], it is proved that, for continuous maps from the unit interval to itself, Condition (1) implies both Conditions (2) and (3), that is, continuous maps from the unit interval to itself are topologically transitive if and only if those are chaotic in the sense of Devaney. Every chaotic map in the sense of Devaney has positive topological entropy on the unit interval [BC]. However, the reverse is false, that is, every continuous map from the unit interval to itself with positive topological entropy is not necessarily chaotic in the sense of Devaney. So the following natural question arises : When f is a continuous map from the unit interval to itself having positive topological entropy, does there exist a chaotic map g from some good space Z to itself in the sense of Devaney which is semiconjugate to f and which has positive topological entropy?

$$\begin{array}{ccc} [0,1] & \xrightarrow{f} & [0,1] \\ \pi\downarrow & & \downarrow\pi \\ Z & \xrightarrow[g]{} & Z \end{array}$$

Sharkovsky's theorem is the well-known and impressive results about the co-existence of periods of periodic orbits of continuous maps from the unit interval to itself. The following is Sharkovsky ordering for positive integers :

$$3 \prec 5 \prec 7 \prec 9 \prec \cdots \prec 2\cdot 3 \prec 2\cdot 5 \prec \cdots \prec 2^2\cdot 3 \prec 2^2\cdot 5 \prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1$$

Theorem 1.2 [S] *Let f be a continuous map from the unit interval to itself. If f has a periodic orbit of period n and if $n \prec m$ in the above ordering, then f also has a periodic orbit of period m.*

As for continuous maps from the unit interval to itself, it is known that those have positive topological entropy if and only if there exists a periodic orbit with period except a power of 2 [BC,Theorem II.14 and Proposition VIII.34]. Hence the above question can be expressed as follows : When f is a continuous map from the unit interval to itself having a periodic orbit with period except a power of 2, does there exist a chaotic map from some good space to itself in the sense of Devaney which is semiconjugate to f and which has positive topological entropy? In this note, it is reported that if a continuous map f from the unit interval to itself has a periodic orbit with odd period, then there exists a chaotic map from a dendrite to itself in the sense of Devaney which is semiconjugate to f and which has positive topological entropy.

2 The elementary properties of pointwise P-expansive maps

A *dendrite* is a locally connected, uniquely arcwise connected continuum (see [N, Chapter X] for properties of dendrites). Let Y be a subspace of a dendrite X. We denote the minimum connected set containing Y by $[Y]$. Particularly, if $Y = \{x, y\}$, then express $[Y] = [x, y]$. Let $(x, y) = [x, y] \setminus \{x, y\}$ and $[x, y) = [x, y] \setminus \{y\}$. And write the closure of Y in X by $\mathrm{Cl}(Y)$. We denote the interior of Y in X by $\mathrm{Int}(Y)$ and $\mathrm{Bd}(Y) = \mathrm{Cl}(Y) \setminus \mathrm{Int}(Y)$. For any set A, $|A|$ means the cardinality of A.

Topological entropy is one of methods to measure how complicated a dynamical systems is. The definition is as follows :

Definition 2.1 Let f and (X, d) be as in Definition 1.1. And let n be a positive number, $Y \subset X$ and $\varepsilon > 0$. Define a new metric d_n on X by $d_n(x, y) = \max\{d(f^k(x), f^k(y)) | 0 \le k < n\}$. A set $E \subset Y$ is said to be *(n, ε, Y, f)-separated (by f)* if $d_n(x, y) > \varepsilon$ for any $x, y \in E$ with $x \neq y$. Denote $s_n(\varepsilon, Y, f)$ the biggest cardinality of any (n, ε, Y, f)-separated set in Y. Define

$$s(\varepsilon, Y, f) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon, Y, f).$$

Now we define the *topological entropy of f on the set Y* as

$$h(f, Y) = \lim_{\varepsilon\to 0} s(\varepsilon, Y, f).$$

$h(f) = h(f, X)$ is said to be a *topological entropy of f.*

Lemma 2.2 *Let f and (X, d) be as in* Definition 1.1 *and let $\varepsilon_0 > 0$. If $d(Y_0, Y_1) > \varepsilon_0$ and $Y_0 \cup Y_1 \subset f(Y_0) \cap f(Y_1)$ for some subspaces Y_0, Y_1 of X, then $h(f) \ge \log 2$.*

Denote $I = [0,1]$. Let $f : I \to I$ be a continuous map from I to itself with a periodic orbit P. We denote the set of all components of $I \setminus P$ contained in $[P]$ by $S(I,P)$.

Notice 2.3 In this note, we denote every periodic orbit P of $f : I \to I$ with period n by $P = \{p_0, p_1, \dots, p_{n-1}\}$ with $0 \le p_0 < p_1 < \dots < p_{n-1} \le 1$.

Definition 2.4 A continuous map $f : I \to I$ is *pointwise P-expansive* if for every element $C = (p_k, p_{k+1})$ of $S(I,P)$, there exists a positive integer ℓ such that $\big(f^\ell(p_k), f^\ell(p_{k+1})\big) \cap P \ne \emptyset$. Note that if f is pointwise P-expansive, then $|P| \ge 3$.

Lemma 2.5 [BC, Lemma I.4] *Let $f : I \to I$ be a continuous map and let $J_0, J_1, \dots, J_m$ be compact subintervals of I such that $J_{k+1} \subset f(J_k)(0 \le k \le m-1)$ and $J_0 \subset f(J_m)$. Then there exists a point x such that $f^{m+1}(x) = x$ and $f^k(x) \in J_k(0 \le k \le m)$.*

The following lemma is derived from Lemma 2.5 and the definition of pointwise P-expansive.

Lemma 2.6 *Let P be a periodic orbit of $f : I \to I$ as in* Notice 2.3. *If f is not pointwise P-expansive, then there exists a periodic orbit of f with period $\frac{n}{2}$, thus n is even. Hence, if n is odd or the supremum in the Sharkovsky ordering except a power of 2, then f is pointwise P-expansive.*

By the above lemmas, we see the following theorem.

Theorem 2.7 *Let $f : I \to I$ be a continuous map. The following statements are equivalent :*
(1) *f has positive topological entropy, and*
(2) *f is pointwise P-expansive for some periodic orbit P of f.*

3 The constructions of the dendrite $Z(f,P)$

In [ACKY], a regular curve Z has been constructed from a continuous map f from a finite graph to itself and an f-invariant subset of the finite graph. In this section, under some natural restriction, the dendrite $Z(f,P)$ is constructed from a continuous map $f : I \to I$ and a periodic orbit P of f. Let $P = \{p_0, p_1, \dots, p_{n-1}\}$ be a periodic orbit of f as in Notice 2.3 and suppose that f is pointwise P-expansive. Let C, C' be elements of $S(I,P)$. If $C' \cap f(C) \ne \emptyset$, then write $C \rightharpoonup C'$. And let C_i be an element of $S(I,P)$ satisfying $\{p_i, p_{i+1}\} = \mathrm{Bd}(C_i)$ for each $i = 0, 1, \dots, n-2$. Let $B_i = \{(x,y) | (x - i - \frac{1}{2})^2 + y^2 \le \frac{1}{4}\}$ be a disk in the two dimensional Euclidean space for each $i = 0, 1, \dots, n-2$. Since each element C_i of $S(I,P)$ can be matched off against each B_i, we put $\mathcal{A}_0 = \{B_i | C_i \in S(I,P)\}$. And write $P_0 = \{(0,0), (1,0), (2,0), \dots, (n-1,0)\}$ and $X_0 = \bigcup \mathcal{A}_0$ (see Figure 1).

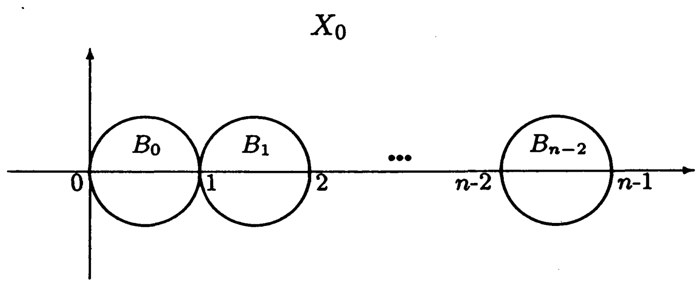

Figure 1.

Let $X(i) = \bigcup\{B_j | C_i \rightharpoonup C_j \in S(I,P)\}$ for each $i = 0, 1, \dots, n-2$ and let $h_i : X(i) \to B_i$ an embedding satisfying the following (i) and (ii) :

(i) $h_i(X(i)) \cap \mathrm{Bd}(B_i) = P_0 \cap \mathrm{Bd}(B_i)$.

(ii) If $f(p_i) = p_j$ and $f(p_{i+1}) = p_{j'}$, then $h_i((j,0)) = (i,0)$ and $h_i((j',0)) = (i+1,0)$.

Denote $D_k = \{(i_0, i_1, \dots, i_k) | C_{i_0} \rightharpoonup C_{i_1} \rightharpoonup \cdots \rightharpoonup C_{i_k}\}$, where C_{i_j} is an element of $S(I,P)$ for each $j = 0, 1, \dots, k$. And put $B_{i_0,i_1,\dots,i_k} = (h_{i_0} \circ h_{i_1} \circ \cdots \circ h_{i_{k-1}})(B_{i_k})$, where $(i_0, i_1, \dots, i_k) \in D_k$. Set $\mathcal{A}_k = \{B_{i_0,i_1,\dots,i_k} | (i_0, i_1, \dots, i_k) \in D_k\}$ and $X_k = \bigcup \mathcal{A}_k$. And denote $p_{i_0,i_1,\dots,i_{k-1},j_k} = (h_{i_0} \circ h_{i_1} \circ \cdots \circ h_{i_{k-1}})((j_k, 0))$, where $(i_0, i_1, \dots, i_{k-1}) \in D_{k-1}$ and $(j_k, 0) \in X(i_{k-1})$ (see Figure 2).

Moreover since the map f is pointwise P-expansive, we may assume that for any $\epsilon > 0$, there exists a positive integer k such that the diameter of each element $B_{i_0,i_1,\dots,i_k}$ of $\mathcal{A}_k$ is less than ϵ. Define $X_{\rightharpoonup} = \bigcap_{k=1}^{\infty} X_k$. Since any two points of $X_{\rightharpoonup}$ are separated in $X_{\rightharpoonup}$ by a third point of $X_{\rightharpoonup}$, we see that $X_{\rightharpoonup}$ is a dendrite by [N, Theorem 10.2, p.166].

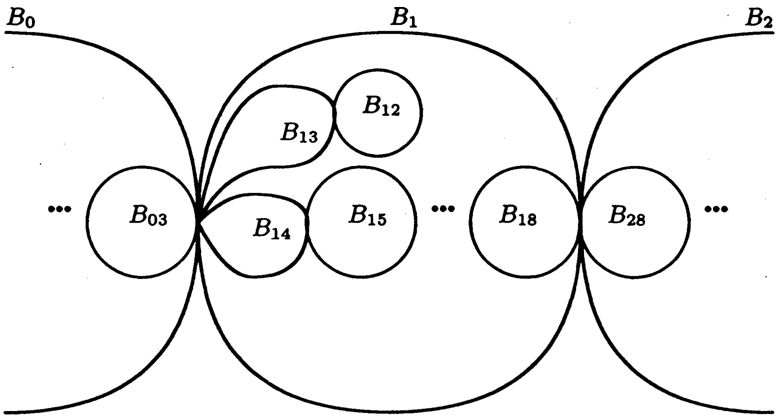

Figure 2.

Next let us define a map $\pi : I \to X_{\rightharpoonup}$ as (a) and (b) :

(a) $\pi(t) = \bigcap_{k=0}^{\infty} B_{i_0,i_1,\dots,i_k}$, if for each $k \geq 0$ there exists $C_{i_k} \in S(I,P)$ such that $f^k(t) \in \mathrm{Cl}(C_{i_k})$.

When there exists $m = \min\{k | f^k(t) \notin [P]\}$, define as the following :

$$(b)\ \ \pi(t) = \begin{cases} p_{i_0,i_1,\dots,i_{m-1},0} & \text{if } f^m(t) \in [0, p_0] \text{ and } m \neq 0 \\ p_0 & \text{if } f^m(t) \in [0, p_0] \text{ and } m = 0 \\ p_{i_0,i_1,\dots,i_{m-1},n-1} & \text{if } f^m(t) \in [p_{n-1}, 1] \text{ and } m \neq 0 \\ p_{n-1} & \text{if } f^m(t) \in [p_{n-1}, 1] \text{ and } m = 0 \end{cases}$$

This map π is well-defined and continuous by the natural construction. Indeed, for each element t of $I \setminus P$ and neighborhood V of $\pi(t)$ in $X_{\rightharpoonup}$, there exists some element $B_{i_0,i_1,\dots,i_k}$ of $\mathcal{A}_k$ such that $\pi(t) \in B_{i_0,i_1,\dots,i_k} \cap X_{\rightharpoonup} \subset V$. Then by the construction of $X_{\rightharpoonup}$, $\bigcap_{j=0}^{k} f^{-j}(\mathrm{Cl}(C_{i_j}))$ is a non-empty subset containing t and $\pi(\bigcap_{j=0}^{k} f^{-j}(\mathrm{Cl}(C_{i_j}))) \subset V$. Since $\bigcup\{\bigcap_{j=0}^{k} f^{-j}(\mathrm{Cl}(C_{i_j})) \mid \pi(t) \in \pi(\bigcap_{j=0}^{k} f^{-j}(\mathrm{Cl}(C_{i_j}))) \subset V\}$ is a neighborhood of t in I, π is continuous.

Set $Z(f,P) = \pi(I)$, which is a dendrite. Because every subcontinuum of a dendrite is a dendrite. Define a map $g : X_{\rightharpoonup} \to X_{\rightharpoonup}$ by $g(\bigcap_{k=0}^{\infty} B_{i_0,i_1,\dots,i_k}) = \bigcap_{k=1}^{\infty} B_{i_1,i_2,\dots,i_k}$, then the map g is well-defined and continuous. The map π is a semi-conjugacy between f and g, i.e. it is surjective and satisfies $\pi \circ f = g \circ \pi$. See [ACKY] for details. We notice that $g(p_{i_0,i_1,\ldots,i_m}) = p_{i_1,i_2,\ldots,i_m}$, thus, $g^m(p_{i_0,i_1,\ldots,i_m}) \in \pi(P)$.

Notice 3.1 Let P be a periodic orbit of f as in Notice 2.3. By the construction of $Z(f,P)$ and the pointwise P-expansiveness of f, we see that $\pi^{-1}(B_i \cap Z(f,P)) \subset C_{i-1} \cup \mathrm{Cl}(C_i) \cup C_{i+1}$ for each $B_i \in \mathcal{A}_0$. Particularly, it follows that $\pi^{-1}(\pi(p_i)) \subset C_{i-1} \cup \{p_i\} \cup C_i$.

4 The relationship between the cardinality of P and the chaoticity of g

In this section, we introduce the relationship between the cardinality of P and the behavior of $g : Z(f,P) \to Z(f,P)$ constructed in Section 3. The following lemmas are derived by the periodicity of P.

Lemma 4.1 *Let $f : I \to I$ be a continuous map and let P a periodic orbit of f as in* Notice 2.3. *For each element C of $S(I,P)$, there exists a natural number k such that $C_0 \subset f^k(C)$.*

Lemma 4.2 *Let $f : I \to I$ be a continuous map and let P a periodic orbit of f with odd period n as in* Notice 2.3. *If n is prime or the supremum in the Sharkovsky ordering, then $[P] \subset f^\ell(\mathrm{Cl}(C_0))$ for some ℓ.*

In the following theorem, the *topological mixing* means the following :

For every pair of non-empty open sets U and V, there exists a positive integer N such that $f^k(U) \cap V \neq \emptyset$ for all $k > N$.

Clearly if f is topologically mixing, then it is also topologically transitive.

Theorem 4.3 *Let $f : I \to I$ be a continuous map and let P a periodic orbit of f with odd period n as in* Notice 2.3. *If n is prime or the supremum in the Sharkovsky ordering, then g is topologically mixing and chaotic in the sense of Devaney, where $g : Z(f,P) \to Z(f,P)$ is the map constructed in* Section 3. *Moreover g has positive topological entropy.*

By Theorem 1.2 and 4.3, it is easy to prove the following main theorem.

Theorem 4.4 *Let $f : I \to I$ be a continuous map. If f has a periodic orbit with odd period, then there exists a chaotic map from a dendrite to itself in the sense of Devaney which is semiconjugate to f and has positive topological entropy.*

The following shows such example as $g : Z(f,P) \to Z(f,P)$ constructed in Section 3 is not chaotic in the sense of Devaney, when $|P|$ is the supremum in the Sharkovsky ordering but not odd.

Example 4.5 Let f be the piecewise linear function from $[0,5]$ to itself defined by

$$f(0) = 3, f(2) = 5, f(3) = 1, f(4) = 2, \text{ and } f(5) = 0.$$

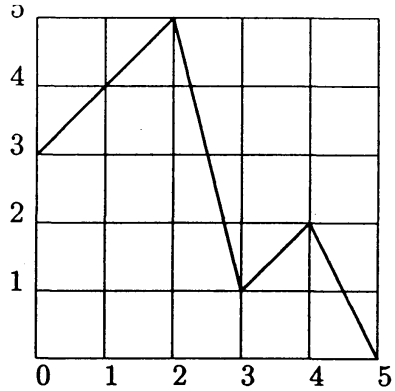

Figure 3.

Then $P = \{0, 1, \ldots, 5\}$ is a periodic orbit of f with a period 6 and it is the supremum in the Sharkovsky ordering. However, we see that $g : Z(f, P) \to Z(f, P)$ as in Section 3 is not chaotic in the sense of Devaney. Indeed, since $f^k([0, 2]) \subset [0, 5] \setminus (2, 3)$ for each $k \geq 0$, there exists some open subset U of $Z(f, P)$ such that $U \subset \pi([0, 2])$ and $\bigcup_{k \geq 0} g^k(U)$ is not dense in $Z(f, P)$.

The following provides such example as $g : Z(f, P) \to Z(f, P)$ costructed in Section 3 is not chaotic in the sense of Devaney, when $|P|$ is odd, but not the supremum in the Sharkovsky ordering.

Example 4.6 Let f be the piecewise linear function from $[0, 8]$ to itself defined by

$$f(0) = 3, f(5) = 8, f(6) = 1, f(7) = 2, \text{ and } f(8) = 0.$$

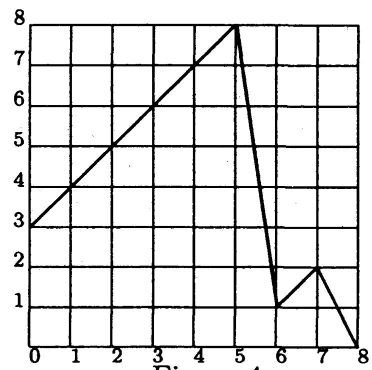

Figure 4.

Then $P = \{0, 1, 2, \ldots, 8\}$ is a periodic orbit of f with a period 9. Since $\{\frac{4}{3}, \frac{13}{3}, \frac{22}{3}\}$ is a periodic point of f with a perod 3, P is not the supremum in the Sharkovsky ordering. Let g and $Z(f, P)$ be as in Section 3. Since $f^k([0, 2]) \subset [0, 8] \setminus ((2, 3) \cup (5, 6))$ for each $k \geq 0$, there exists some open subset U of $Z(f, P)$ such that $V \subset \pi([0, 2])$ and $\bigcup_{k \geq 0} g^k(U)$ is not dense in $Z(f, P)$. It follows that g is not chaotic in the sense of Devaney.

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