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Absolute weak C-embedding in Hausdorff spaces

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Abstract

Answering a problem of A. V. Arhangel'skiĭ, we give a characterization of absolute weak C-embedding in Hausdorff spaces. We also introduce an alternative proof of the Bella-Yaschenko Theorem, which characterize absolute weak C-embedding in Tychonoff spaces.

All spaces are assumed to be T_1-spaces. Arhangel'skiĭ [2] says a subspace Y of a space X *weakly C-embedded* in X if every real-valued continuous function on Y can be extended to a real-valued function on X which is continuous at each point of Y. As was discussed in [2] (see also [5], [12]), some results of type of relative topological properties immediately follow from those of weak C-embeddings (see Section 2). It is obvious that C-embedding implies weak C-embedding. In fact, weak C-embedding is strictly weaker than z-embedding [12], where a subspace Y of a space X is said to be *z-embedded* in X if for every zero-set Z of Y there exists a zero-set Z' of X such that $Z' \cap Y = Z$.

For a space X and a subspace Y of X, the space X_Y denotes the set X with the topology consisting all sets of form $U \cup V$, where U is open in X and $V \subset X - Y$. Notice that a subspace Y of a space X is weakly C-embedded in X if and only if Y is C-embedded in X_Y. In [12], we characterize a subspace Y of a space X is weakly C-embedded in X if and only if every disjoint cozero-sets U_0 and U_1 of Y can be separated by disjoint open subsets in X.

Weak C-embedding plays an important role not only in the theory of relative topological properties but also in the extension theory of continuous functions themselves. For classical results related to absolute embedding of continuous functions in the realm of Tychonoff spaces, recall the following Theorem 1. A Tychonoff space Y is said to be *almost compact* if $|\beta Y - Y| \leq 1$, where βY denotes the Stone-Čech compactification of Y.

Theorem 1 (Blair [6], Blair-Hager [7], Hager-Johnson [11]). *Let Y be a Tychonoff space. Then, Y is z-embedded in every larger Tychonoff space if and only if Y is almost compact or Lindelöf.*

An alternative proof of Theorem 1 is recently given in [14] (see also Appendix

A corresponding result for weak C-embedding was recently obtained by Bella-Yaschenko [5] as follows. Assuming the normality of Y, Matveev-Pavlov-Taitir [13] also proved Theorem 2.

Theorem 2 (Bella-Yaschenko [5]). *Let Y be a Tychonoff space. Then, Y is weakly C-embedded in every larger Tychonoff space if and only if Y is almost compact or Lindelöf.*

Recently, Arhangel'skiĭ posed in [3, Problem 3.14] the following problem which motivates us to consider absolute weak C-embedding in the realm of Hausdorff spaces.

Problem (Arhangel'skiĭ [3]). *When a Hausdorff (Tychonoff) space Y is weakly C-embedded in every larger Hausdorff space X?*

We give a solution to this problem as follows:

Theorem 3 (main). *Let Y be a Hausdorff space. Then, Y is weakly C-embedded in every larger Hausdorff space if and only if either Y is compact or every real-valued continuous function on Y is constant.*

Corresponding to Theorems 2 and 3, we have other conclusion as follows:

Theorem 4. *Let Y be a regular space. Then, Y is weakly C-embedded in every larger regular space if and only if either Y is Lindelöf or for every two disjoint zero-sets of Y at least one of them is compact.*

Theorem 5. *Let Y be a T_1-space. Then, Y is weakly C-embedded in every larger T_1-space if and only if every real-valued continuous function on Y is constant.*

Remarks. (1) Related to absolute z-embedding in other classes of spaces, we easily have: *A Hausdorff (resp. regular, T_1) space Y is z-embedded in every larger Hausdorff (resp. regular, T_1) space X if and only if every real-valued continuous function on Y is constant.*

(2) A cardinal generalization of weak C-embedding is introduced in [12]: a subspace Y of a space X is said to be *weakly P-embedded* in X if every continuous pseudo-metric on Y can be extended to a pseudo-metric on X which is continuous at each point of $Y \times Y$. Motivated by the result due to Alò-Shapiro [1, pp183] that *a Tychonoff space Y is P-embedded in every larger Tychonoff space X if and only if Y is almost compact* which is a generalizetion of Theorem 1, we obtained in [12] the following: *A Tychonoff space Y is weakly P-embedded in every larger Tychonoff space X if and only if Y is*

almost compact or Lindelöf, which is a generalizetion of Theorem 2. Now, we have similar generalization to Theorems 3, 4 and 5 as follows:

Let Y be a Hausdorff space. Then, Y is weakly P-embedded in every larger Hausdorff space X if and only if either Y is compact or every real-valued continuous function on Y is constant.

Let Y be a regular space. Then, Y is weakly P-embedded in every larger regular space X if and only if either Y is Lindelöf or for every two disjoint zero-sets of Y at least one of them is compact.

Let Y be a T_1-space. Then, Y is weakly P-embedded in every larger T_1-space X if and only if every real-valued continuous function on Y is constant.

(3) In Theorems 1 and 2, it is known that "every larger Tychonoff space" can be replaced by "every larger Tychonoff space containing Y as a closed subspace". Similar replacements are possible for Theorems 3, 4 and 5 and all of the related results in this report. For, we have:

Let i be the one of $3\frac{1}{2}, 3, 2, 1$. Then, a T_i space Y is weakly C-embedded in every larger T_i-space if and only if Y is weakly C-embedded in every larger T_i-space containing Y as a closed subspace.

(4) Let $\mathcal{K}$ (resp. $\mathcal{T}_{3\frac{1}{2}}$) be the class of spaces consisting all compact Hausdorff (resp. all Tychonoff) spaces, for example, normal Hausdorff spaces, paracompact Hausdorff spaces, etc. We have:

Let $\mathcal{C}$ be a class of spaces with $\mathcal{K} \subset \mathcal{C} \subset \mathcal{T}_{3\frac{1}{2}}$. Then, a Tychonoff space Y is weakly C-embedded in every larger space X with $X \in \mathcal{C}$ if and only if Y is almost compact or Lindelöf.

(5) When we use weak C-embedding assuming Y to satisfy some separation axioms or have some covering properties, many known (or new) results immediately follow from Theorems 2, 3, 4 and 5. To show this, recall from [2, Thorem 11] and [12, Lemma 2.8] (and similar proofs to [14]) that: *Let i be the one of $3\frac{1}{2}, 3, 2, 1$. For a Y_i-space Y, Y is normal (or equivalently, strongly normal, internally normal) in a every larger T_i-space if and only if Y is normal, and Y is weakly C-embedded in every larger T_i-space.* Hence, when we use Theorems 2, 3, 4 and 5 assuming Y is normal, we have:

A Tychonoff space Y is strongly normal (equivalently, normal, internally normal) in every larger Tychonoff space if and only if Y is normal almost compact or Lindëlof (cf. [5], [13]).

A Hausdorff space Y is strongly normal (equivalently, normal, internally normal) in every larger Hausdorff space if and only if Y is compact.

A regular space Y is strongly normal (equivalently, normal, internally normal) in every larger regular space if and only if Y is normal almost compact or Lindëlof (cf. [13]).

A T_1-space Y is strongly normal (equivalently, normal, internally normal) in every larger T_1-space if and only if $|Y| \leq 1$.

Moreover, when we assume Y to be paracompact, the above facts provide some known results of Gordienko [10] (see [3, Theorem 7.5]) and [10] (see [2, Theorems 52 and 53] or [3, Theorem 7.10]).

Appendix: Alternative proofs of Theorems 1 and 2

Blair [6], Blair-Hager [7], Hager-Johnson [11] proved Theorem 1. Their proofs are obtained through several consequences under their own interests on real-compactness or rings of continuous functions, which seems to be not elementary. Bella-Yaschenko [5] proved Theorem 2 by the direct construction, but their proof is complicated. Hoshina and the auther [12] gave another proof to Theorem 2, but which depends on the technique of reducing this theorem to Theorem 1.

Now, we introduce alternative (and probably simple) proofs to Theorems 1 and 2 at a time, which was basically given in [14] only for Theorem 1.

Theorem ([5], [6], [7], [11]). *Let Y be a Tychonoff space. Then, the following statements are equivalent:*

(1) *Y is z-embedded in every larger Tychonoff space;*
(2) *Y is weakly C-embedded in every larger Tychonoff space;*
(3) *Y is almost compact or Lindelöf.*

For the proof, we use the following well-known facts:

(*a*) A Tychonoff space Y is Lindelöf if and only if for every compact subspace F of βY with $F \subset \beta Y - Y$ there exists a zero-set Z of βY such that $F \subset Z \subset \beta Y - Y$ (see [8, 3.12.25(b)]).

(*b*) The Tychonoff cube is a O_Z-space ($=$ a perfectly κ-normal space).

Proof of Theorem. To prove (1) $\Rightarrow$ (2), recall that z-embedding implies weak C-embedding ([12]). Indeed, assume that Y is z-embedded in X. Clearly, Y is z-embedded in X_Y. On the other hand, Y is always well-embedded in X_Y. Hence, Y is C-embedded in X_Y, equivalently, Y is weakly C-embedded in X. Since "(3) $\Rightarrow$ (1)" is easy to see (see [1]), the essential part is "(2) $\Rightarrow$ (3)".

To prove "(2) $\Rightarrow$ (3)", assume that Y is weakly C-embedded in every larger Tychonoff space. Suppose that Y is not almost compact. We shall show that Y is Lindelöf. To use the fact (a) above, let F be a compact subspace of βY with $F \subset \beta Y - Y$.

Claim. *For every $x \in F$, there exist an open neighborhood U_x of x in the subspace F and a zero-set Z_x of βY such that $U_x \subset Z_x \subset \beta Y - Y$.*

Proof of Claim. Let $x \in F$. Since $|\beta Y - Y| \geq 2$, pick up a point $y \in \beta Y - Y$ with $y \neq x$. Let $f : \beta Y \to [0,1]$ be a continuous function satisfying that $f(x) = 0$ and $f(y) = 1$. Let $Z = \beta Y/(F \cup \{y\})$ be the quotient space obtained from βY by identifying $F \cup \{y\}$ to a single point and $q : \beta Y \to Z$ be the natural quotient map. Since Z is Tychonoff, embed Z into the Tychonoff cube T. Since $q(Y)$ is homeomorhic to Y, $q(Y)$ is weakly C-embedded in T. Hence, $q(f^{-1}([0,1/2)) \cap Y)$ and $q(f^{-1}((1/2,1]) \cap Y)$ are separated by disjoint open subsets U_0 and U_1 in T, respectively. By the fact (b) above, there exist disjoint cozero-sets V_0 and V_1 of T such that $U_i \subset V_i$, $i = 0, 1$. Then, $q(x) \notin V_0$. Indeed, if $q(x) \in V_0$, then $y \in q^{-1}(V_0 \cap Z) \cap f^{-1}((1/2,1]) \subset \beta Y - Y$, a contradiction. Put $U_x = f^{-1}([0,1/3)) \cap F$ and $Z_x = f^{-1}([0,1/3]) - q^{-1}(V_0 \cap Z)$. These are the required sets. This completes the proof of Claim.

Finally, for some finite points $x_1, \ldots, x_n \in F$ with $F = \bigcup_{i=1}^n U_{x_i}$, put $Z = \bigcup_{i=1}^n Z_{x_i}$. Then, Z is a zero-set of βY and $F \subset Z \subset \beta Y - Y$. Hence Y is Lindelöf. This completes the proof. □

References

[1] R. A. Alò and H. L. Shapiro, *Normal Topological Spaces*, Cambridge University Press, Cambridge, 1974.

[2] A. V. Arhangel'skiĭ, *Relative topological properties and relative topological spaces*, Topology Appl. **70** (1996), 87–99.

[3] A. V. Arhangel'skiĭ, *From classic topological invariants to relative topological properties*, Sci. Math. Japon. **55** (2002), 153–201.

[4] A. V. Arhangel'skiĭ and H. M. M. Genedi, *Beginnings of the theory of relative topological properties*, in: General Topology. Spaces and Mappings (MGU, Moscow, 1989) 3–48 (in Russian).

[5] A. Bella and I. V. Yaschenko, *Lindelöf property and absolute embeddings*, Proc. Amer. Math. Soc. **127** (1999), 907–913.

[6] R. L. Blair, *On v-embedded sets in topological spaces*, TOPO 72—general topology and its applications, pp. 46–79. Lecture Notes in Math., Vol. 378, Springer, Berlin, 1974.

[7] R. L. Blair and A. W. Hager, *Extensions of zero-sets and of real-valued functions*, Math. Z. **136** (1974), 41–52.

[8] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.

[9] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.

[10] I. Ju. Gordienko, *A characterization of relative Lindelöf property by relative paracompactness*, in: General Topology. Spaces, Mappings and Functors (MGU, Moscow, 1992) 40–44 (in Russian).

[11] A. W. Hager and D. G. Johnson, *A note on certain subalgebras of* $C(X)$, Canad. J. Math. **20** (1968), 389–393.

[12] T. Hoshina and K. Yamazaki, *Weak C-embedding and P-embedding, and product spaces*, Topology Appl. **125** (2002), 233-247.

[13] M. V. Matveev, O. I. Pavlov and J. K. Tartir, *On relatively normal spaces, relatively regular spaces, and on relative property* (a), Topology Appl. **93** (1999), 121–129.

[14] K. Yamazaki, *A proof for the Blair-Hager-Johnson theorem on absolute z-embedding*, Comment. Math. Univ. Carolinae **43** (2002), 175–179.

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