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Variants of subtlety in $P_\kappa\lambda$

— comparison of ideals defined by combinatorial principles —

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概 要

A type of subtlety for $P_\kappa\lambda$ called "A-subtle" is presented and compared with Menas' notion "M-subtle". Using it we prove almost ineffability is consistencywise stronger than Shelah property although $P_\kappa\lambda$ is A-subtle for every $\lambda \geq \kappa$ if κ is subtle. The following are also shown (i) "$\{x \in P_\kappa\lambda : x \cap \kappa < |x|\}$ is A-subtle" has rather strong consequences. (ii) The subtle ideals are not λ-saturated, and completely ineffable ideal is not precipitous. (iii) $NAIn_{\kappa,\lambda} = NIn_{\kappa,\lambda}$ and $I_{\kappa,\lambda}$ does not have the partition property if $\lambda^{<\kappa} = 2^\lambda$. (iv) We can not prove "κ is $\lambda^{<\kappa}$-ineffable whenever κ is λ-ineffable".

1 Notations and basic facts

Throughout this paper κ denotes a regular uncountable cardinal and λ a cardinal $\geq \kappa$. Let A be a set and a a set of ordinals with $|a| \leq |A|$. For any such pair (a, A), $P_a A$ denotes the set $\{x \subset A : |x| < |a|\}$. Thus $P_\kappa\lambda$ denotes the set $\{x \subset \lambda : |x| < \kappa\}$ and, for $x \in P_\kappa\lambda$, $P_{x\cap\kappa}x = \{s \subset x : |s| < |x \cap \kappa|\}$.
Combinatorial properties originated for regular uncountable cardinals have been translated into $P_\kappa\lambda$ in [10], [6] and [7]. For $x \in P_\kappa\lambda$, $\hat{x}$ denotes the set $\{y \in$

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$P_\kappa\lambda : x \subset y\}$. We say $X \subset P_\kappa\lambda$ is *unbounded* if $X \cap \hat{x} \neq \emptyset$ for all $x \in P_\kappa\lambda$. Let $I_{\kappa\lambda} = \{X \subset P_\kappa\lambda : X \text{ is not unbounded}\}$.
We say I is an ideal on $P_\kappa\lambda$ if the following hold:

1. $I \subset PP_\kappa\lambda$,
2. $\emptyset \in I$ and $P_\kappa\lambda \notin I$,
3. $I_{\kappa,\lambda} \subset I$,
4. I is κ-*complete*; $\cup X \in I$ for any $X \subset I$ with $|X| < \kappa$.

Thus $I_{\kappa,\lambda}$ is the minimal ideal on $P_\kappa\lambda$.
We say $X \subset P_\kappa\lambda$ is *closed* if $\cup D \in X$ for any $\subset$-chain $D \subset X$ with $|D| < \kappa$. X is called *club* if it is closed and unbounded.

Fact 1.1. *Let $X \subset P_\kappa\lambda$. Then, X is club if and only if there exists $f : \lambda \times \lambda \to \lambda$ such that $C_f := \{x \in P_\kappa\lambda : f``(x \times x) \subset x$ and $x \cap \kappa \in \kappa\} \subset X$.*

We say X is *stationary* if $X \cap C \neq \emptyset$ for any club C. Let $NS_{\kappa,\lambda} = \{X \subset P_\kappa\lambda : X$ is nonstationary$\}$. Let $I^+ = PP_\kappa\lambda \backslash I$. For $X \in I^+$ $I \restriction X$ denotes the set $\{Y \subset P_\kappa\lambda : Y \cap X \in I\}$, which is an ideal on $P_\kappa\lambda$ extending I. We say an ideal I is *normal* if I is closed under diagonal unions; $\nabla X := \{x \in P_\kappa\lambda : x \in \bigcup\{X_\alpha : \alpha \in x\}\} \in I$ for any $X = \{X_\alpha : \alpha < \lambda\} \subset I$. Note that I is normal if and only if every regressive function on $X \in I^+$ is constant on some $Y \in P(X) \cap I^+$, where a function f is said to be *regressive* if $f(x) \in x$ for any x in dom$(f) \setminus \{\emptyset\}$.
The relation $\prec$ is defined by $y \prec z$ if $y \in P_{z\cap\kappa}z$. An ideal I is *strongly normal* if for any $X \in I^+$ and $f : X \to P_\kappa\lambda$ such that $f(x) \prec x$ for all $x \in X$ there exists $Y \in P(X) \cap I^+$ such that $f \restriction Y$ is constant. This is equivalent to the following: for any $\{X_s : s \in P_\kappa\lambda\} \subset I$, $\nabla_\prec X_s := \{x : x \in \bigcup\{X_s : s \prec x\}\} \in I$. Clearly every strongly normal ideal is normal. A filter F on $P_\kappa\lambda$ and an ideal I on $P_\kappa\lambda$ are *dual* to each other if the following holds:

$$X \in F \text{ if and only if } P_\kappa\lambda \setminus X \in I \text{ for every } X \subset P_\kappa\lambda.$$

The dual filter of I will be denoted by I^*.
For $f : P_\kappa\lambda \to P_\kappa\lambda$, let $C_f = \{x \in P_\kappa\lambda : f''P_{x\cap\kappa}x \subset P_{x\cap\kappa}x\}$. We define an ideal $WNS_{\kappa,\lambda}$ by:

$$X \in WNS_{\kappa,\lambda} \text{ if and only if } X \subset P_\kappa\lambda \text{ and } X \cap C_f = \emptyset \text{ for some } f : P_\kappa\lambda \to P_\kappa\lambda.$$

The following are well-known [5], [8] and [1].

Fact 1.2. (1) *$NS_{\kappa,\lambda}$ is the minimal normal ideal on $P_\kappa\lambda$.*
(2) *$WNS_{\kappa,\lambda}$ is the minimal strongly normal ideal on $P_\kappa\lambda$ and $NS_{\kappa,\lambda} \subsetneq WNS_{\kappa,\lambda}$.*
(3) *$WNS_{\kappa,\lambda}$ is proper if and only if κ is Mahlo or $\kappa = \nu^+$ with $\nu^{<\nu} = \nu$.*
(4) *If κ is Mahlo, then $\{x \in P_\kappa\lambda : x \cap \kappa = |x \cap \kappa|$ is inaccessible$\} \in WNS^*_{\kappa,\lambda}$.*
(5) *If $h : P_\kappa\lambda \to \lambda$ is a bijection, then $WNS_{\kappa,\lambda} = NS_{\kappa,\lambda} \restriction \{x : h``P_{x\cap\kappa}x = x\}$.*
(6) *If $\{s_\alpha : \alpha < \lambda^{<\kappa}\}$ is an enumeration of $P_\kappa\lambda$ and $f : P_\kappa\lambda \to P_\kappa\lambda^{<\kappa}$ is defined by $f(x) = \{\alpha : s_\alpha \prec x\}$, then $f_*(WNS_{\kappa,\lambda}) := \{X \subset P_\kappa\lambda^{<\kappa} : f^{-1}(X) \in WNS_{\kappa,\lambda}\} = WNS_{\kappa,\lambda^{<\kappa}}$ and $\{x \in P_\kappa\lambda : f(x) \cap \lambda = x\} \in WNS^*_{\kappa,\lambda}$.*

All the notions defined above for $P_\kappa\lambda$ can be naturally translated into $P_{x\cap\kappa}x$ if $x\cap\kappa$ is regular uncountable. For instance, $X \subset P_{x\cap\kappa}x$ is unbounded if for any $y \in P_{x\cap\kappa}x$ there is $z \in X$ such that $y \subset z$, and $I_{x\cap\kappa,x}$ denotes the set $\{X \subset P_{x\cap\kappa}x : X$ is not unbounded in $P_{x\cap\kappa}x\}$ which is a $x \cap \kappa$-complete ideal on $P_{x\cap\kappa}x$.

First we observe a type of subtlety for $P_\kappa\lambda$: $X \subset P_\kappa\lambda$ is *A-subtle* if for any $\{S_x \subset P_{x\cap\kappa}x : x \in P_\kappa\lambda\}$ and $C \in WNS^*_{\kappa,\lambda}$, there exist $y \prec z$ both in $C \cap X$ such that $S_y = S_z \cap P_{y\cap\kappa}y$.

The following are shown in §2:

Theorem 1.3. (1) *κ is subtle if and only if $P_\kappa\lambda$ is A-subtle for every $\lambda \geq \kappa$.*
(2) *If $X \subset P_\kappa\lambda$ is A-subtle, there exists $\{S_x \subset P_{x\cap\kappa}x : x \in X\}$ such that $\{x \in X : S_x = S \cap x\} \in WNS^+_{\kappa,\lambda}$ for every $S \subset P_\kappa\lambda$.*
(3) *If κ is weakly Mahlo and $2^{\alpha^{<\kappa}} \leq \lambda$ for every $\alpha < \lambda$, there exists $\{S_x \subset P_{x\cap\kappa}x : x \in P_\kappa\lambda$ and $x \cap \kappa$ is regular$\}$ such that S_x is a club in $P_{x\cap\kappa}x$ and $\{x : S_x \not\subset C\} \in NS_{\kappa,\lambda}$ for any club $C \subset P_\kappa\lambda$.*

The last statement is false for $\lambda = \kappa$.
In the next section we study large cardinal aspects of A-subtlety to prove an analogue of Baumgartner's theorem [4] for regular uncountable cardinals:

Theorem 1.4. *If $X \subset P_\kappa\lambda$ is A-subtle, then $\{x \in X : X \cap P_{x\cap\kappa}x$ is Π^m_n-indescribable$\}$ is A-subtle for every m, $n < \omega$.*

Thus our subtlety takes an appropriate place in $P_\kappa\lambda$ combinatorics. The next follows immediately.

Corollary 1.5. *If $cf(\lambda) \geq \kappa$ and $X \subset P_\kappa\lambda$ is almost ineffable, then $\{x \in X : X \cap P_{x\cap\kappa}x$ is Shelah$\}$ is almost ineffable.*

Hence "κ is almost λ-ineffable" is an essentially stronger hypothesis than "κ is λ-Shelah". Kamo [12] already proved:

Fact 1.6. (Kamo) If κ is λ-ineffable, then $\{x \in P_\kappa\lambda : P_{x\cap\kappa}x$ is not almost ineffable$\}$ is not ineffable.

Thus we have the same hierarchy of combinatorial properties for $P_\kappa\lambda$ as for regular uncountable cardinals. Our proof is applicable for Kamo's theorem and more simple than his.
Another corollary is:

Corollary 1.7. *If $\{x \in P_\kappa\lambda : x \cap \kappa < |x|\}$ is A-subtle, then $V \neq L[U]$*

Note that $L \models$ "κ is subtle" if κ is subtle.
In §4 we turn to saturation of subtle ideals and show:

Theorem 1.8. (1) *The ideals of non-subtle sets are not λ-saturated.*
(2) *The ideal of non-completely ineffable sets is not precipitous*

The last section is devoted to almost ineffability and ineffability. Our results might be surprising comparing with Kamo's theorem:

Theorem 1.9. *Suppose that $\lambda^{<\kappa} = 2^\lambda$. Then, X is almost ineffable if and only if X is ineffable.*

As a corollary we get:

Corollary 1.10. (1) *We can not prove in ZFC that κ is $\lambda^{<\kappa}$-ineffable whenever κ is λ-ineffable.*
(2) *If $\lambda^{<\kappa} = 2^\lambda$, $I_{\kappa,\lambda}$ does not have the partition property*

2 The Subtle ideals on $P_\kappa\lambda$.

Menas [16] tried to introduce subtlety into $P_\kappa\lambda$ as follows (we call *M-subtle* in this paper):

Definition 2.1. *Let $X \subset P_\kappa\lambda$. X is M-subtle if for any $\{S_x \subset x : x \in P_\kappa\lambda\}$ and a club $C \subset P_\kappa\lambda$ there exist $y \subsetneq z$ both in $C \cap X$ such that $S_y = S_z \cap y$.*

This is not an essential generalization as proved in the same paper.

Fact 2.2. *For every $\lambda \geq \kappa$, $P_\kappa\lambda$ is M-subtle if and only if κ is subtle.*

We present a new definition of subtlety for $P_\kappa\lambda$ and use the word "*A-subtle*" for it. In place of the filter of club sets we use $WNS_{\kappa,\lambda}$.

Definition 2.3. *For $X \subset P_\kappa\lambda$, X is A-subtle if for any $\{S_x \subset P_{x\cap\kappa}x : x \in P_\kappa\lambda\}$ and $C \in WNS^*_{\kappa\lambda}$, there are $y \prec z$ both in $C \cap X$ such that $S_y = S_z \cap P_{y\cap\kappa}y$.*
Set $I_M = \{X \subset P_\kappa\lambda : X$ is not M-subtle$\}$ and $I_A = \{X \subset P_\kappa\lambda : X$ is not A-subtle$\}$.

Remark 2.4. *If $\lambda^{<\kappa} = \lambda$, then $\{x \in P_\kappa\lambda : h''P_{x\cap\kappa}x \subset x\} \in WNS^*_{\kappa,\lambda}$ for any bijection $h : P_\kappa\lambda \to \lambda$. Thus, in this case, $X \subset P_\kappa\lambda$ is A-subtle if and only if for any $\{S_x \subset x : x \in P_\kappa\lambda\}$ and $C \in WNS^*_{\kappa,\lambda}$ there are $y \prec z$ both in $C \cap X$ such that $S_y = S_z \cap y$.*

We say κ is *λ-subtle* if $P_\kappa\lambda$ is A-subtle (in $P_\kappa\lambda$). If $P_\kappa\lambda$ is A-subtle, then it is M-subtle. So κ is assumed to be subtle in the rest of this section.
We collect several facts for subtle ideals:

Proposition 2.5. (1) $I_M \subset I_A$.
(2) *I_M is a normal ideal on $P_\kappa\lambda$.*
(3) *I_A is a strongly normal ideal on $P_\kappa\lambda$*
(4) *$\{x \in P_\kappa\lambda : x \cap \kappa$ is Mahlo$\} \in I^*_M$.*
(5) *If $\kappa \leq \delta < \lambda$ and $X \subset P_\kappa\lambda$ is A-subtle, then $X \restriction \delta := \{x \cap \delta : x \in X\} \subset P_\kappa\delta$ is A-subtle.*
(6) *If κ is Mahlo and $\{x \in P_\kappa\lambda : X \cap P_{x\cap\kappa}x$ is A-subtle$\} \in WNS^+_{\kappa,\lambda}$, then X is A-subtle.*
(7) *Let $\{s_\alpha : \alpha < \delta\}$ be an enumeration of $P_\kappa\lambda$ and $f(x) = \{\alpha : s_\alpha \prec x\}$ for $x \in P_\kappa\lambda$. Then, $f''X \subset P_\kappa\delta$ is A-subtle if and only if $X \subset P_\kappa\lambda$ is A-subtle.*
(8) *If λ is regular, $X \subset P_\kappa\lambda$ and $\{\alpha < \lambda : X \cap P_\kappa\alpha$ is M-subtle$\} \in NS^+_\lambda$, then X is M-subtle.*

Proof. We only show (3). Let $X \subset P_\kappa\lambda$ be subtle and $f : X \to P_\kappa\lambda$ such that $f(x) \prec x$ for every $x \in X$. Suppose to the contradiction that $f^{-1}(\{a\})$ is not subtle for any $a \in P_\kappa\lambda$. For $a \in P_\kappa\lambda$, we fix $\{S^a_x \subset P_{x\cap\kappa}x : x \in P_\kappa\lambda\}$ and $D_a \in WNS^*_{\kappa,\lambda}$ such that $S^a_y \neq S^a_z \cap P_{y\cap\kappa}y$ for any $y \prec z$ both in $D_a \cap f^{-1}(\{a\})$.
Let $h : P_\kappa\lambda \times P_\kappa\lambda \to P_\kappa\lambda$ be a bijection and set $T_x = h''(\{f(x)\} \times S^{f(x)}_x) \cap P_{x\cap\kappa}x$. Note that $C = \{x \in P_\kappa\lambda : h''(P_{x\cap\kappa}x \times P_{x\cap\kappa}x) \subset P_{x\cap\kappa}x\} \in WNS^*_{\kappa,\lambda}$. Since X is A-subtle and $E = C \cap \Delta_\prec D_a \in WNS^*_{\kappa,\lambda}$, there exist $y \prec z$ both in $E \cap X$ such that $T_y = T_z \cap P_{y\cap\kappa}y$. Then, $f(y) = f(z)$ and $S^{f(y)}_y = S^{f(z)}_z \cap P_{y\cap\kappa}y$. Set $a = f(y) = f(z)$. We have $y \prec z$ are both in $D_a \cap f^{-1}(\{s\})$ and $S^a_y = S^a_z$, which contradicts our assumption. □

A natural question arises:

Question 2.6. *Can it be proved that $I_M \subsetneq I_A$?*

It turns out that "A-subtle" is neither an essential generalization.

Theorem 2.7. *If κ is subtle, then $P_\kappa\lambda$ is A-subtle.*

Proof. Let $S_x \subset P_{x\cap\kappa}x$ for $x \in P_\kappa\lambda$ and $D \in WNS^*_{\kappa,\lambda}$. Since $\kappa^{<\kappa} = \kappa$, $WNS_{\kappa^+,\lambda}$ is proper.
We first show $\{x \in P_{\kappa^+}\lambda : D \cap P_\kappa x \in WNS^*_{\kappa,x}\} \in WNS^*_{\kappa^+,\lambda}$. Let $f : P_\kappa\lambda \to P_\kappa\lambda$ such that $C_f \subset D$. If $\{x \in P_{\kappa^+}\lambda : f''P_\kappa x \subset P_\kappa x\} \notin WNS^*_{\kappa^+,\lambda}$, $X := \{x \in P_{\kappa^+}\lambda : \kappa \subset x \wedge \exists y_x \in P_\kappa x\,(f(y) \notin P_\kappa x)\} \in WNS^+_{\kappa^+,\lambda}$. Note that $\kappa = |x \cap \kappa|$ for every $x \in X$. By strong normality we have $y \in P_\kappa\lambda$ such that $Y := \{x \in X : y_x = y\} \in WNS^+_{\kappa^+,\lambda}$. $Y \cap \widehat{f(y)} = \emptyset$. Contradiction. Thus $Z := \{x \in P_{\kappa^+}\lambda : \kappa \subset x \wedge f \restriction P_\kappa x : P_\kappa x \to P_\kappa x\} \in WNS^*_{\kappa^+,\lambda}$. For $x \in Z$ $D_x := \{s \in P_\kappa x : f''P_{s\cap\kappa}x \subset P_{s\cap\kappa}x\} \in WNS^*_{\kappa,x}$. For every $x \in Z$ $D \cap P_\kappa x \in WNS^*_{\kappa,x}$ sincce $D_x \subset C_f \cap P_\kappa x \subset D$.
Note that κ is subtle if and only if $P_\kappa\kappa$ is A-subtle. Thus $P_\kappa y$ is A-subtle for every $y \in Z$. Now we consider $\{S_x : x \in P_\kappa y\}$ and $D \cap P_\kappa y$. There exist $x_1 \prec x_2$ both in $D \cap P_\kappa y$ such that $S_{x_1} = S_{x_2} \cap P_{x_1\cap\kappa}x_1$. □

The following observations suggest two ideals may be the same. The first appeas in [16].

Fact 2.8. *If $X \subset P_\kappa\lambda$ is M-subtle and $S_x \subset x$ for each $x \in X$, then for any club $C \subset P_\kappa\lambda$ there exist x, y both in $C \cap X$ such that $x \subset y$, $x \cap \kappa < y \cap \kappa$, and $S_x = S_y \cap x$.*

Proposition 2.9. *If κ is subtle, then $X = \{x \in P_\kappa\lambda : x \cap \kappa = |x|\}$ is A-subtle.*

Proof. Note that $X \notin WNS_{\kappa,\lambda}$ ([1]) . Let $f : P_\kappa\lambda \to P_\kappa\lambda$ and $S_x \subset P_{x\cap\kappa}x$ for all $x \in P_\kappa\lambda$. We build a chain $\langle x_\alpha : \alpha < \kappa\rangle$ as follows:
Choose $x_0 \in X \cap C_f$ arbitrally and $x_{\alpha+1} \in X \cap C_f$ so that $x_\alpha \prec x_{\alpha+1}$. For limit α let $x_\alpha = \bigcup\{x_\beta : \beta < \alpha\}$.
Set $x = \bigcup\{x_\alpha : \alpha < \kappa\}$. Then, $x \cap \kappa = |x| = \kappa$, $P_{x\cap\kappa}x = \bigcup\{P_{x_\alpha\cap\kappa}x_\alpha : \alpha < \kappa\}$, and there exists a club $D \subset \kappa$ such that for every $\alpha \in D$ $x_\alpha \cap \kappa = \alpha = |x_\alpha|$. Note that $x_\alpha \in C_f$ if α is regular. Let $g : P_{x\cap\kappa}x \to \kappa$ be any bijection. Then, we have a club $E \subset \{\alpha \in D : g``P_\alpha x_\alpha \subset \alpha\}$. Since $E$ is subtle in $\kappa$, there exist regular $\beta < \gamma$ both in $E$ such that $g``S_{x_\beta} = g``S_{x_\gamma} \cap \beta$. Since β and γ are regular, x_β and x_γ belong to C_f. □

S. Baldwin[3] and others observed the consistency strength of the stationarity of $\{x \in P_\kappa\lambda : x \cap \kappa < |x|\}$, the complement of the set we mentioned now.

Fact 2.10. ([3],[14],[1])
(1) *If κ is weakly inaccesible and λ Ramsey $> \kappa$, then $\{x \in P_\kappa\lambda : x \cap \kappa < |x|\}$ is stationary.*
(2) *If $\{x \in P_\kappa\kappa^+ : x \cap \kappa < |x|\}$ is stationary, then $0^\dagger$ exists.*
(3) *If $P_\kappa\lambda$ is Shelah, then $\{x \in P_\kappa\lambda : x \cap \kappa < |x|\} \in NSh^*_{\kappa,\lambda}$.*

Corollary 2.11. *Suppose that $0^\dagger$ does not exist and $S = \{x \in P_\kappa\lambda : h``P_{x\cap\kappa}x = x\}$ where $h : P_\kappa\lambda \to \lambda$ is a bijection. Then, $I_M \restriction S = I_A$. Thus $I_M = I_A$ if I_M is strongly normal.*

The same relation holds between subtlety and its another weakening, which relates to "ethereal" introduced by Kunen: $X \subset \kappa$ is *ethereal* if for any $\langle S_\alpha \in [\alpha]^\alpha : \alpha < \kappa\rangle$ and a club $C \subset \kappa$, there are $\beta < \gamma$ both in $C \cap X$ such that $|S_\beta \cap S_\gamma| = |\beta|$.

Definition 2.12. *Let $X \subset P_\kappa\lambda$. We say X is weakly subtle if for any $\{S_x \in I^+_{x\cap\kappa,x} : x \in P_\kappa\lambda\}$ and a club $C \subset P_\kappa\lambda$, there are $y \prec z$ both in $C \cap X$ such that $S_y \cap S_z \in I^+_{y\cap\kappa,y}$. Let $I_W = \{X \subset P_\kappa\lambda : X$ is not weakly subtle$\}$.*

We have the following:

Proposition 2.13. (1) *I_W is a normal ideal.*
(2) *If $P_\kappa\lambda$ is weakly subtle, then κ is ethereal.*
(3) *$I_W \restriction S = I_A$ where $S = \{x \in P_\kappa\lambda : h``P_{x\cap\kappa}x = x\}$ for a bijection $h : P_\kappa\lambda \to \lambda$.*

Three subtle ideals are interesting from the view of the diamond principles for $P_\kappa\lambda$. The following two cardinal version of diamond principle by Jech is wellknown.

Definition 2.14. *Let $X \subset P_\kappa\lambda$. Then,*
$\diamondsuit_0(X)$: there exist $\{S_x \subset x : x \in X\}$ such that $\{x \in X : S_x = S \cap x\}$ is stationary for any $S \subset \lambda$. We simply write $\diamondsuit_0$ for $\diamondsuit_0(P_\kappa\lambda)$. Let $J_0 = \{X \subset P_\kappa\lambda : \diamondsuit_0(X)$ does not hold$\}$.

The following are known (see [9], [10]):

Fact 2.15. (1) *J_0 is a normal ideal on $P_\kappa\lambda$.*
(2) *$L \models$ "$\diamondsuit_0(X)$ for any $X \subset P_{\omega_1}\lambda$".*
(3) *If $2^{<\kappa} < \lambda$, then J_0 is proper.*
(4) *If $\diamondsuit_0$ holds, then $NS_{\kappa,\lambda}$ is not 2^λ saturated.*

In the context of I_A and I_W another version of diamond arises.

Definition 2.16. *Let $X \subset P_\kappa\lambda$. Then,*
$\diamondsuit_1(X)$: there exists $\{S_x \subset P_{x\cap\kappa}x : x \in X\}$ such that $\{x \in X : S_x = S \cap P_{x\cap\kappa}x\}$ is stationary for any $S \subset P_\kappa\lambda$.
$\diamondsuit_2(X)$: there exists $\{S_x \subset P_{x\cap\kappa}x : x \in X\}$ such that $\{x \in X : S_x = S \cap P_{x\cap\kappa}x\} \in WNS^+_{\kappa,\lambda}$ for any $S \subset P_\kappa\lambda$.
J_1 and J_2 are similarly defined as J_0.

Of course $J_0 \subset J_1 \subset J_2$ and it is easily seen:

Lemma 2.17. (1) *J_1 is a normal ideal on $P_\kappa\lambda$.*
(2) *J_2 is a strongly normal ideal on $P_\kappa\lambda$*
(3) *If $\lambda^{<\kappa} = \lambda$ and $h : P_\kappa\lambda \to \lambda$ is a bijection, then $J_2 = J_0 \restriction S$ with $S = \{x : h''P_{x\cap\kappa}x = x\}$.*
(4) *$L \models$ "$J_2 = WNS_{\omega_1,\lambda}$".*
(5) *If J_2 is proper, then any ideal $\subset WNS_{\kappa,\lambda}$ is not $2^{\lambda^{<\kappa}}$ saturated.*

Theorem 2.18. *If κ is subtle, then J_2 is proper.*

Proof. We show $\diamondsuit_2(X)$ holds for every A-subtle $X \subset P_\kappa\lambda$.
By induction on $\prec$ we define $S_x \subset P_{x\cap\kappa}x$ for $x \in X$ as well as $C_x \subset P_{x\cap\kappa}x$.
If x is a $\prec$ minimal element of X, then $S_x = C_x = \emptyset$.

Suppose S_y and C_y is defined for every $y \in X \cap P_{x\cap\kappa}x$. If there exist $S \subset P_{x\cap\kappa}x$ and $C \in WNS^*_{x\cap\kappa,x}$ such that $S_y \neq S \cap P_{y\cap\kappa}y$ for any $y \in C$, then let S_x and C_x be any such S and C. We say this is the substantial case. Otherwise let $S_x = C_x = P_{x\cap\kappa}x$. To show $\{S_x : x \in X\}$ is a witness of $\diamondsuit_2(X)$, let $S \subset P_\kappa\lambda$, $D \in WNS^*_{\kappa,\lambda}$, and $S_x \neq S \cap P_{x\cap\kappa}x$ for any $x \in X \cap D$. Since $X \cap D$ is A-subtle, we may assume that $D \cap P_{x\cap\kappa}x \in WNS^*_{x\cap\kappa,x}$ for every $x \in X \cap D$. Thus $S \cap P_{x\cap\kappa}x$ and $D \cap P_{x\cap\kappa}x$ witness the substantial case occurs for every $x \in X \cap D$. However $X \cap D$ is subtle hence there exist $y \prec z$ both in $X \cap D$ such that $S_y = S_z \cap P_{y\cap\kappa}y$. In particular $y \in D \cap P_{z\cap\kappa}z$. Contradiction. □

Note that $\diamondsuit_\kappa$ holds if κ is ethereal and $\kappa^{<\kappa} = \kappa$ [13].

Question 2.19. (1) *Does $\diamondsuit_1$ hold if $P_\kappa\lambda$ is weakly subtle?*
(2) *If κ is ethereal, then $P_\kappa\lambda$ is weakly subtle?*

Let Reg$= \{x \in P_\kappa\lambda : x \cap \kappa \text{ is regular}\}$. We conclude this section by:

Proposition 2.20. *Suppose κ is weakly Mahlo and $2^{\alpha^{<\kappa}} \leq \lambda$ for every $\alpha < \lambda$. Then, there exists $\{S_x : x \in P_\kappa\lambda, x \cap \kappa$ is regular$\}$ such that*

1. *$S_x \subset P_{x\cap\kappa}x$ club,*
2. *for every club $C \subset P_\kappa\lambda$ $\{x : S_x \not\subset C\} \in NS_{\kappa,\lambda} \restriction Reg$.*

Proof. Let $\{C_\alpha : \kappa^+ \leq \alpha < \lambda\} = \{X : \exists\beta < \lambda\ X \text{ is a club of } P_\kappa\beta\}$ be an enumeration and C_α a club of $P_\kappa\beta(\alpha)$. Set $B = \{x : \forall\alpha \in x\ \beta(\alpha) \in x\}$. Then, $B \in NS^*_{\kappa,\lambda}$. Let $S_x = \{y \in P_{x\cap\kappa}x : \forall\alpha \in x\ y \cap \beta(\alpha) \in C_\alpha\}$.
For every α $\{x \in P_\kappa\lambda : C_\alpha \cap P_{x\cap\kappa}(x \cap \beta(\alpha)) \text{ is a club of } P_{x\cap\kappa}(x \cap \beta(\alpha))\} \in (NS_{\kappa,\lambda} \restriction \text{Reg})^*$. Thus, $\{x \in P_\kappa\lambda : \{z \in P_{x\cap\kappa}x : z \cap \beta(\alpha) \in C_\alpha\} \text{ is a club of } P_{x\cap\kappa}x\} \in (NS_{\kappa,\lambda} \restriction \text{Reg})^*$. Hence $A = \{x \in P_\kappa\lambda : S_x \text{ is a club of } P_{x\cap\kappa}x\} \in (NS_{\kappa,\lambda} \restriction \text{Reg})^*$.
Pick any club $C \subset P_\kappa\lambda$. We have $f : \lambda \times \lambda \to \lambda$ with $C_f \subset C$. Define g by $C_f \restriction \beta = C_{g(\beta)}$ for $\beta < \lambda$. Note that $\beta = \beta(g(\beta))$.
Let $x \in A \cap B \cap C_f$. For every $\beta \in x$ $g(\beta) \in x$. Hence for every $y \in S_x$ and $\beta \in x$ we have $y \cap \beta \in C_{g(\beta)} = C_f \restriction \beta$. If $\{\xi, \zeta\} \subset y$, then $\{\xi, \zeta, f(\xi, \zeta)\} \subset \beta$ for some $\beta \in x$. Choose $z \in C_f$ such that $y \cap \beta = z \cap \beta$. Then, $f(\xi, \zeta) \in z \cap \beta = y \cap \beta$. Hence $y \in C_f$. We have shown that $S_x \subset C_f \subset C$ for every $x \in A \cap B \cap C_f$. □

Remark 2.21. *This is false for $\lambda = \kappa$.*

3 Subtlety and large cardinals

Recall that $X \subset \kappa$ is Π^m_n*-indescribable* if for any $R \subset V_\kappa$ and Π^m_n sentence φ such that $(V_\kappa, \in, R) \models \varphi$, there exists $\alpha \in X$ such that $(V_\alpha, \in, R \cap V_\alpha) \models \varphi$.

Fact 3.1. (1)*Suppose that λ is weakly compact. Then, $X \subset P_\kappa\lambda$ is M-subtle if and only if $\{\alpha < \lambda : X \cap P_\kappa\alpha$ is not M-subtle$\}$ is not Π^1_1-indescribable.*
(2) *I_M is a normal ideal on $P_\kappa\lambda$ such that $\{x \in P_\kappa\lambda : x \cap \kappa$ is not Π^m_n-indescribable$\} \in I_M$ for every $m, n < \omega$.*

Carr[7] defined $P_\kappa\lambda$-version of indescribability.

Definition 3.2. *A sequence $\langle V_\alpha(\kappa, \lambda) : \alpha \leq \kappa \rangle$ is recursively defined as follows:*

$$\begin{aligned} V_0(\kappa,\lambda) &= \lambda \\ V_{\alpha+1}(\kappa,\lambda) &= P_\kappa(V_\alpha(\kappa,\lambda)) \cup V_\alpha(\kappa,\lambda) \\ V_\alpha(\kappa,\lambda) &= \bigcup\{V_\beta(\kappa,\lambda) : \beta < \alpha\} \qquad \text{if } \alpha \text{ is a limit ordinal} \end{aligned}$$

This definition can be carried out for $x \in P_\kappa\lambda$ if $x \cap \kappa$ is inaccessible. For such x we consider the structure $(V_{x\cap\kappa}(x \cap \kappa, x), \in)$ in the same way as $(V_\kappa(\kappa, \lambda), \in)$.

Definition 3.3. *We say $X \subset P_\kappa\lambda$ is Π^m_n-indescribable if for any $R \subset V_\kappa(\kappa, \lambda)$ and Π^m_n sentence φ such that $(V_\kappa(\kappa,\lambda), \in, R) \models \varphi$, there exists $x \in X$ such that $x \cap \kappa = |x \cap \kappa|$ and $(V_{x\cap\kappa}(x \cap \kappa, x), \in, R \cap V_{x\cap\kappa}(x \cap \kappa, x)) \models \varphi$.*

Lemma 3.4. *If $X \subset P_\kappa\lambda$ is A-subtle and $S_x \subset P_{x\cap\kappa}x$ for $x \in P_\kappa\lambda$, then $\{x \in X : \{y \in X \cap P_{x\cap\kappa}x : S_y = S_x \cap P_{y\cap\kappa}y\}$ is not Π^m_n-indescribable for some $m, n\}$ is not A-subtle.*

Proof. Otherwise, by κ-completeness of I_A, $Y := \{x \in X : \{y \in X \cap P_{x\cap\kappa}x : S_y = S_x \cap P_{y\cap\kappa}y\}$ is not Π^m_n-indescribable$\}$ is subtle for some $m, n < \omega$. We may assume that $x \cap \kappa$ is inaccessible for all $x \in Y$. For $x \in Y$ there exist $R_x \subset V_{x\cap\kappa}(x \cap \kappa, x)$ and a Π^m_n sentence φ_x such that $(V_{x\cap\kappa}(x\cap\kappa, x), \in, R_x) \models \varphi_x$ while $(V_{y\cap\kappa}(y \cap \kappa, y), \in, R_x \cap V_{y\cap\kappa}(y \cap \kappa, y)) \models \neg\varphi_x$ for any $y \in X \cap P_{x\cap\kappa}x$ with $S_y = S_x \cap P_{x\cap\kappa}x$. By κ-completeness again, we can assume for all $x \in Y$ $\varphi_x = \varphi$ for some φ.
Since Y is subtle, there are $y \prec z$ both in Y such that $R_y = R_z \cap V_{y\cap\kappa}(y \cap \kappa, y)$ and $S_y = S_z \cap P_{y\cap\kappa}y$. Then, $y \in X \cap P_{z\cap\kappa}z$, $\quad S_y = S_z \cap P_{y\cap\kappa}y \quad$ and $\quad (V_{y\cap\kappa}(y \cap \kappa, y), \in, R_z \cap V_{y\cap\kappa}(y \cap \kappa, y)) \models \varphi_z$, which is a contradiction. □

As a corollary we have:

Theorem 3.5. *$\{x \in P_\kappa\lambda : P_{x\cap\kappa}x$ is not Π^m_n-indescribable$\} \in I_A$ for every $m, n < \omega$.*

This theorem derives strong facts.

Lemma 3.6. *If $\{x \in P_\kappa\lambda : 2^{x\cap\kappa} \leq |x|\}$ is A-subtle, then $\{x \in P_\kappa\lambda : o(x \cap \kappa) \geq 1\}$ is A-subtle.*

Proof. Note that κ is measurable if $P_\kappa 2^\kappa$ is Π^1_1-indescribable ([6],[7]). Let $X = \{x \in P_\kappa\lambda : P_{x\cap\kappa}2^{x\cap\kappa}$ is Π^1_1-indescribable$\}$. Then, X is A-subtle. For $x \in X$ $x \cap \kappa$ is measurable and $\{y \in P_{x\cap\kappa}2^{x\cap\kappa} : y \cap \kappa$ is measurable$\}$ is Π^1_1-indescribable. Hence $o(x \cap \kappa) \geq 1$. □

Note that $L[U] \models$"there exist $\kappa < \lambda$ such that $\{x \in P_\kappa\lambda : x \cap \kappa < |x|\} \in NS^+_{\kappa,\lambda}$".

Corollary 3.7. (1) $L[U] \models$ "*$\{x \in P_\kappa\lambda : x \cap \kappa < |x|\}$ is not A-subtle*".
(2) $L[U] \models$ "*$\neg\exists\kappa(\kappa$ is κ^+-Shelah)*".

Thus the existence of a cardinal κ such that $\{x \in P_\kappa\lambda : x \cap \kappa < |x|\}$ is A-subtle is rather strong in consistency strength. On the other hand subtlety is a Π^1_1 property of $P_\kappa\lambda$. The following proposition says the size of κ is not necessarily large.

Proposition 3.8. *The least cardinal κ such that κ is κ^+-subtle is not κ^+-Shelah.*

In the rest of this section we compare the almost ineffability and Shelah property using I_A, which reveals very useful in large cardinal hierarchy.
Carr [6] defined Shelah property as a $P_\kappa\lambda$ generalization of weak compactness. We show: *if $P_\kappa\lambda$ is subtle then there exist many $x \in P_\kappa\lambda$ such that $P_{x\cap\kappa}x$ is Shelah.*

Definition 3.9. *Let $X \subset P_\kappa\lambda$. We say X is Shelah if for any $\{f_x \in {}^x x : x \in P_\kappa\lambda\}$ there is $f : \lambda \to \lambda$ such that for every $y \in P_\kappa\lambda$ the set $\{x \in X \cap \hat{y} : f \restriction y = f_x \restriction y\} \in I^+_{\kappa,\lambda}$.*
We say X is almost ineffable (ineffable) if for any $\{f_x \in {}^x x : x \in P_\kappa\lambda\}$ there is $f : \lambda \to \lambda$ such that $\{x \in X : f \restriction x = f_x\} \in I^+_{\kappa,\lambda}$ $(NS^+_{\kappa,\lambda})$.
Let $NSh_{\kappa\lambda} = \{X \subset P_\kappa\lambda : X$ is not Shelah$\}$ and $NAIn_{\kappa\lambda} = \{X \subset P_\kappa\lambda : X$ is not almost ineffable$\}$.
We often say κ is λ-Shelah (almost λ-ineffable) if $P_\kappa\lambda$ is Shelah (almost ineffable).

Clearly X is Shelah if X is almost ineffable, and X is almost ineffable if X is ineffable. It is known that $NSh_{\kappa\lambda}$ and $NAIn_{\kappa\lambda}$ are strongly normal ideals if $cf(\lambda) \geq \kappa$. Moreover, Kamo [12] proved the following:

Fact 3.10. (Kamo) *If $X \subset P_\kappa\lambda$ is ineffable and $cf(\lambda) \geq \kappa$, then $\{x \in X : X \cap P_{x\cap\kappa}x$ is almost ineffable$\}$ is ineffable.*

The following follows immediately from definition and the remark after Definition 2.1 with strong normality of $NAIn_{\kappa\lambda}$.

Proposition 3.11. *If $cf(\lambda) \geq \kappa$ and $X \subset P_\kappa\lambda$ is almost ineffable, then X is A-subtle.*

Carr [7] proved the following:

Fact 3.12. *Let If $X \subset P_\kappa\lambda$ is Π^1_1-indescribable, then X is Shelah. The converse is also true if $cf(\lambda) \geq \kappa$.*

Corollary 3.13. *If $X \subset P_\kappa\lambda$ is subtle, then $Y = \{x \in X : X \cap P_{x\cap\kappa}x$ is not Shelah$\}$ is not A-subtle.*

Corollary 3.14. *Let $cf(\lambda) \geq \kappa$. If $X \subset P_\kappa\lambda$ is almost ineffable, then $\{x \in P_\kappa\lambda : X \cap P_{x\cap\kappa}x$ is Shelah$\}$ is almost ineffable. In particular, $\{x \in P_\kappa\lambda : x \cap \kappa$ is x-Shelah$\} \in NAIn^*_{\kappa,\lambda}$ if κ is almost λ-ineffable.*

This corollary tells that almost ineffability is much stronger hypothesis than Shelah property. For instance, suppose that κ is almost κ^+-ineffable. Then $\{x \in P_\kappa\lambda : o.t.(x) = (x \cap \kappa)^+\} \in NAIn^*_{\kappa,\lambda}$ hence $\{x \in P_\kappa\lambda : x \cap \kappa$ is $(x \cap \kappa)^+$-Shelah$\} \in NAIn^*_{\kappa,\lambda}$. Thus, below the least α that is almost α^+-ineffable, stationary many β which is β^+-Shelah exist.

4 Saturation of subtle ideals on $P_\kappa\lambda$

Now we turn to saturation of subtle ideals.

Proposition 4.1. *Let I be a normal λ saturated ideal on $P_\kappa\lambda$ with λ regular. Then, $\{x \in P_\kappa\lambda : o.t.(x)$ is regular$\} \in I^*$ hence $\{x \in P_\kappa\lambda : x \cap \kappa < |x|\} \in I^*$.*

Proof. Let G be P_I generic for V. By λ saturation the generic ultrapower $Ult(V,G)$ is well-founded. Let $j : V \to M \cong Ult(V,G)$ be an generic embedding. Since λ is regular in $V[G]$, $M \models$ "λ is regular" and $[\langle o.t.(x) | x \in P_\kappa\lambda\rangle]_G = o.t.(j``\lambda) = \lambda$. Clearly $\{x \in P_\kappa\lambda : x \cap \kappa \in \kappa \wedge x \setminus \kappa \neq \emptyset\} \in I^*$. □

Theorem 4.2. *Neither I_A nor I_M is λ saturated.*

Proof. First suppose that λ is regular. We know $X = \{x \in P_\kappa\lambda : x \cap \kappa = |x|$ and $\kappa \in x\}$ is A-subtle. For $x \in X$ $o.t.(x)$ is singular.
Second assume λ is singular and the subtle ideal on $P_\kappa\lambda$, say I, is λ saturated. In fact I is δ saturated for some regular $\delta < \lambda$. Then $I \restriction \delta$ is δ saturated and $\{x \in P_\kappa\delta : x \cap \kappa = |x|\} \notin I \restriction \delta$. Contradiction. □

Remark 4.3. *By Cummings' theorem I_A is not λ^+-saturated if $cf(\lambda) < \kappa$.*

By definition ineffability can be seen as a strengthening of Shelah property. One of the strongest version is complete ineffability defined as follows:

Definition 4.4. *An ideal I on $P_\kappa\lambda$ is (λ,λ)-distributive if for any $X \in I^+$ and $\{W_\alpha : \alpha < \lambda\}$ which is an I-partition of X with $|W_\alpha| \leq \lambda$ there exist $Y \in P(X) \cap I^+$ and $\{X_\alpha : \alpha < \lambda\}$ such that $X_\alpha \in W_\alpha$ and $Y \setminus X_\alpha \in I$ for every $\alpha < \lambda$.*
We say κ is completely λ ineffable if there is a normal (λ,λ)-distributive ideal on $P_\kappa\lambda$. When κ is completely λ ineffable, the minimal normal (λ,λ)-distributive ideal on $P_\kappa\lambda$, say I, is called completely ineffable ideal and $X \in I^+$ is said to be completely ineffable.

If I is a normal (λ,λ)-distributive ideal on $P_\kappa\lambda$ and $X \in I^+$, the following hold [11]:

1. For any $\{f_x \in {}^x x : x \in X\}$ there is $f : \lambda \to \lambda$ such that $\{x \in X : f_x = f \restriction x\} \in I^+$
2. I is strongly normal.

In the rest of this section we assume $cf(\lambda) \geq \kappa$ and I is a normal (λ,λ)-distributive ideal on $P_\kappa\lambda$. Hence κ is Mahlo, $\lambda^{<\kappa} = \lambda$, and I is strongly normal. Let $\mathbb{P}_I = (I, \subset)$, $G \subset \mathbb{P}_I$ be V generic, and $j : V \prec M \cong Ult(V,G)$ a generic elementary embedding defined in $V[G]$.

By normality of I $P(\lambda)^V \subset M$. Moreover $[\langle P_{x\cap\kappa}x \mid x \in P_\kappa\lambda\rangle]_G = j''P_\kappa\lambda \in M$, $j''P_\kappa\lambda = P_\kappa j''\lambda$, and $j(X) \cap j''P_\kappa\lambda = j''X \in M$ for every $X \in P(P_\kappa\lambda)^V$ by strong normality. Conversely we have:

Lemma 4.5. *For every $[f]_G \in {}^{j''\lambda}j''V \cap M$ there exists $g \in V$ such that $[f]_G = j(g) \restriction j''\lambda = j''g$.*

Proof. Suppose $f \in V$, $X \in I^+$ with $X \Vdash$"$[f]_G : j''\lambda \to j''V$". For $Y := \{s \in X : \mathrm{dom}(f(s)) = s\} \in G$, $\alpha < \lambda$, and $x \in V$, let $X_{\alpha,x} = \{s \in Y \cap \hat{\alpha} : f(s)(\alpha) = x\}$. Then $W_\alpha := \{X_{\alpha,x} : x \in V\} \cap I^+$ is a disjoint I-partition of Y and $|W_\alpha| \leq \lambda$. By (λ, λ)-distributivity there is $g : \lambda \to V$ such that for every $\alpha < \lambda$ $Y \setminus X_{\alpha,g(\alpha)} \in I$. Hence $Z := \Delta_\alpha X_{\alpha,g(\alpha)} \in I^+ \cap P(Y)$ and for every $s \in Z$ and $\alpha \in s$ $f(s)(\alpha) = g(\alpha)$, that is, $Z \Vdash$"$[f]_G = j(g) \restriction j''\lambda$". □

Remark 4.6. *By (λ, λ)-distributivity of I, λ remains a cardinal in $V[G]$ hence in M.*

Corollary 4.7. (1) *If $X \in M$, $X \subset j''V$, and $|X|^M \leq \lambda$, then $X = j''Y$ for some $Y \in V$.*
(2) *$P_\kappa j''\lambda = j''(P_\kappa\lambda)$ and $P(P_\kappa j''\lambda)^M = \{j''X : X \in P(P_\kappa\lambda)\}$.*

Proof. (1) Note that the collapsing map $\pi : j''\lambda \to \lambda$ is a bijection in M. Choose any surjection $f \in M$ from $j''\lambda$ to X. By the lemma $f = j(g) \restriction j''\lambda$ for some $g \in V$. Then $X = j(g)''(j''\lambda) = \{j(g)(j(\alpha)) : \alpha < \lambda\} = \{j(g(\alpha)) : \alpha < \lambda\}$. Hence $X = j''Y$ where $Y = g''\lambda$. Now (2) is clear. □

Next we present another characterization of completely ineffable subsets of $P_\kappa\lambda$.

Definition 4.8. *Let A be a set of ordinals. We inductively define $In_\alpha(\kappa, A) \subset P(P_\kappa A)$ as follows:*
(1) *$X \in In_0(\kappa, A)$ if $X \in NS^+_{\kappa,A}$, that is, for every $f : A \times A \to P_\kappa A$ there exists $x \in X$ such that $f''(x \times x) \subset P(x)$.*
(2) *$X \in In_{\alpha+1}(\kappa, A)$ if for every $f : P_\kappa A \to P_\kappa A$ such that $f(x) \subset x$ for any $x \in P_\kappa A$ there is $S \subset A$ with $\{x \in X : f(x) = x \cap S\} \in In_\alpha(\kappa, A)$.*
(3) *If α is a limit ordinal, $In_\alpha(\kappa, A) = \bigcap_{\beta<\alpha} In_\beta(\kappa, A)$.*

Thus $X \in In_1(\kappa, \lambda)$ if and only if $X \subset P_\kappa\lambda$ is ineffable. Clearly $In_\alpha(\kappa, A) \subset In_\beta(\kappa, A)$ if $\beta < \alpha$. Hence there is α such that $In_\alpha(\kappa, A) = In_{\alpha+1}(\kappa, A)$. If

$In_\alpha(\kappa, A) = In_{\alpha+1}(\kappa, A) \neq \emptyset$, $P(P_\kappa A) \setminus In_\alpha(\kappa, A)$ is the minimal normal (λ,λ)-distributive ideal, that is, completely ineffable ideal on $P_\kappa A$.

Definition 4.9. *We say I is precipitous if $\Vdash_{\mathbb{P}_I}$ "$Ult(V,G)$ is well-founded".*

Lemma 4.10. *Let $cf(\lambda) \geq \kappa$ and I be a normal (λ,λ)-distributive precipitous ideal on $P_\kappa\lambda$. For any $X \in P(P_\kappa\lambda)$ and α, $X \in In_\alpha(\kappa,\lambda)$ if and only if $M \models$ "$j''X \in In_\alpha(\kappa, j''\lambda)$".*

Proof. By induction on α. Assume first X is stationary, $f : j''\lambda \times j''\lambda \to P_\kappa j''\lambda$, and $f \in M$. Since $P_\kappa j''\lambda = j''(P_\kappa\lambda)$, $f = j(g) \restriction j''\lambda \times j''\lambda$ for some $g \in V$ such that $g : \lambda \times \lambda \to P_\kappa\lambda$. There is $x \in X$ such that $g''(x \times x) \subset P(x)$. Then $M \models$"$j(g)(a) \subset j(x)$ for all $a \in j(x) \times j(x)$". Since $j(x) = j''x \subset j''\lambda$, $M \models$"$f(a) \subset j(x)$ for all $a \in j(x) \times j(x)$". Hence $M \models$"$j''X \in NS^+_{\kappa,j''\lambda}$".
Assume conversely $M \models$"$j''X \in NS^+_{\kappa,j''\lambda}$", $f \in V$, and $f : \lambda \times \lambda \to P_\kappa\lambda$. Since $M \models$"$j(f) \restriction (j''\lambda \times j''\lambda) : j''\lambda \times j''\lambda \to P_\kappa j''\lambda$", there is $y \in j''X$ such that $j(f)''(y \times y) \subset P(y)$. For $x \in X$ with $j(x) = y$, $j(f)(j(a)) \subset j(x)$ for every $a \in x \times x$. Hence $f''(x \times x) \subset P(x)$, showing that X is stationary.
The case α is a limit ordinal is clear by our definition of $In_\alpha(\kappa,\lambda)$ and $In_\alpha(\kappa, j''\lambda)$. We prove for $\alpha = \beta + 1$. Suppose first $X \in In_\alpha(\kappa,\lambda)$, $M \models$"$f : P_\kappa j''\lambda \to P_\kappa j''\lambda$ and $f(j(x)) \subset j(x)$ for every $x \in P_\kappa\lambda$". We have $g \in V$ with $f = j(g) \restriction j''P_\kappa\lambda$. Since $g(x) \subset x$ for every $x \in P_\kappa\lambda$, there is $S \subset \lambda$ such that $Y := \{x \in X : g(x) = x \cap S\} \in In_\beta(\kappa,\lambda)$. By inductive hypothesis $j''Y \in In_\beta(\kappa, j''\lambda)$. For every $x \in Y$ $j(g)(j(x)) = j(g(x)) = j(x \cap S) = j''(x \cap S) = j(x) \cap j''S$ and $j''S \in M$. Now $f(y) = y \cap j''S$ for every $y \in j''Y$ hence $j''X \in In_\alpha(\kappa, j''\lambda)$
Suppose second $M \models$"$j''X \in In_\alpha(\kappa, j''\lambda)$", $f \in V$, and $f(x) \subset x$ for every $x \in P_\kappa\lambda$. Set $j(f) \restriction j''P_\kappa\lambda = g$. Since $M \models$"$g(x) \subset x$ for all $x \in j''P_\kappa\lambda$", there is $T \subset j''\lambda$ such that $S := \{y \in j''X : g(y) = y \cap T\} \in In_\beta(\kappa, j''\lambda)$. By the fact $S \in P(P_\kappa j''\lambda)$ we have $Y \in P(P_\kappa\lambda)^V$ with $S = j''Y$. For every $x \in Y$ $g(j(x)) = j(x) \cap T$. By the same reason for S, $T = j''T_1$ for some $T_1 \in V$. For every $x \in Y$ $j(f(x)) = g(j(x)) = j(x) \cap j''T_1 = j''(x \cap T_1) = j(x \cap T_1)$. So $f(x) = x \cap T_1$ for all $x \in Y$. The inductive hypothesis shows $Y \in In_\beta(\kappa,\lambda)$. Hence $X \in In_\alpha(\kappa,\lambda)$. □

Theorem 4.11. *Suppose $cf(\lambda) \geq \kappa$ and I is a normal (λ,λ)-distributive precipitous ideal on $P_\kappa\lambda$. Then $\{x : x \cap \kappa$ is completely o.t.(x)-ineffable$\} \in I^*$.*

Proof. Since $V \models$"κ is completely λ-ineffable", $In_\alpha(\kappa,\lambda) = In_{\alpha+1}(\kappa,\lambda) \neq \emptyset$ for some α. $In_\alpha(\kappa,\lambda) \neq \emptyset$ implies $In_\alpha(\kappa, j''\lambda) \neq \emptyset$. To show $In_\alpha(\kappa, j''\lambda) = In_{\alpha+1}(\kappa, j''\lambda)$, assume $X \in In_\alpha(\kappa, j''\lambda) - In_{\alpha+1}(\kappa, j''\lambda)$. Since $X \in P(P_\kappa j''\lambda)^M$, there is $Y \in P(P_\kappa\lambda)^V$ such that $X = j''Y$. Then $Y \in In_\alpha(\kappa,\lambda) = In_{\alpha+1}(\kappa,\lambda)$ hence $X = j''Y \in In_{\alpha+1}(\kappa, j''\lambda)$. Contradiction. □

Lemma 4.12. *Let κ be Mahlo, $X \subset P_\kappa\lambda$, and α an ordinal.*
(1) If $\{x \in P_\kappa\lambda : X \cap P_{x\cap\kappa}x \in In_\alpha(x\cap\kappa, x)\} \in In_\alpha(\kappa,\lambda)$, then $X \in In_\alpha(\kappa,\lambda)$.
(2) If $X \in In_\alpha(\kappa,\lambda)$, then $\{x \in P_\kappa\lambda : X \cap P_{x\cap\kappa}x \notin In_\alpha(x\cap\kappa, x)\} \in In_\alpha(\kappa,\lambda)$.
(3) If $X \subset P_\kappa\lambda$ is completely λ ineffable, then $\{x \in X : X \cap P_{x\cap\kappa}x$ is not completely ineffable $\}$ is completely ineffable.

Proof. (1) By induction on α. Suppose that $\{x \in P_\kappa\lambda : X \cap P_{x\cap\kappa}x \in NS^+_{x\cap\kappa,x}\} \in NS^+_{\kappa,\lambda}$ and $f : \lambda^2 \to P_\kappa\lambda$. There is x such that $X \cap P_{x\cap\kappa}x \in NS^+_{x\cap\kappa,x}$ and $f \restriction x \times x : x \times x \to P(x)$. Then we can find $y \in X \cap P_{x\cap\kappa}x$ such that $f''(y \times y) \subset P(y)$. Hence X is stationary.
Let α be a limit ordinal and $Y := \{x \in P_\kappa\lambda : X \cap P_{x\cap\kappa}x \in In_\alpha(x\cap\kappa, x)\} \in In_\alpha(\kappa,\lambda)$. For any $\beta < \alpha$ $Y \subset \{x \in P_\kappa\lambda : X \cap P_{x\cap\kappa}x \in In_\beta(x\cap\kappa, x)\} \in In_\alpha(\kappa,\lambda) \subset In_\beta(\kappa,\lambda)$. By induction hypothesis $X \in In_\beta(\kappa,\lambda)$. Thus $X \in In_\alpha(\kappa,\lambda)$.
Let $\alpha = \beta + 1$ and $f(x) \subset x$ for all $x \in P_\kappa\lambda$. For each $x \in Y$ there is $S_x \subset x$ such that $\{y \in X \cap P_{x\cap\kappa}x : f(y) = y \cap S_x\} \in In_\beta(x\cap\kappa, x)$. Since $Y \in In_{\beta+1}(\kappa,\lambda)$ we have $S \subset \lambda$ such that $Z := \{x \in Y : S_x = x \cap S\} \in In_\beta(\kappa,\lambda)$. For $x \in Z$ and $y \in X \cap P_{x\cap\kappa}x$ $f(y) = y \cap S_x = y \cap x \cap S = y \cap S$. So $\{x \in P_\kappa\lambda : \{y \in X : f(y) = y \cap S\} \cap P_{x\cap\kappa}x \in In_\beta(x\cap\kappa, x)\} \in In_\beta(\kappa,\lambda)$. By induction hypothesis $\{y \in X : f(y) = y \cap S\} \in In_\beta(\kappa,\lambda)$ hence $X \in In_{\beta+1}(\kappa,\lambda)$.
(2) We prove this by induction on κ. We may assume $A := \{x \in X : X \cap P_{x\cap\kappa}x \in In_\alpha(x\cap\kappa, x)\} \in In_\alpha(\kappa,\lambda)$. For any $x \in A$, by inductive hypothesis, $B_x := \{y \in P_{x\cap\kappa}x : X \cap P_{x\cap\kappa}x \cap P_{y\cap\kappa}y = X \cap P_{y\cap\kappa}y \notin In_\alpha(y\cap\kappa, y)\} \in In_\alpha(x\cap\kappa, x)$. By (1) and the fact $B_x = \{y \in P_\kappa\lambda : X \cap P_{y\cap\kappa}y \notin In_\alpha(y\cap\kappa, y)\} \cap P_{x\cap\kappa}x$, the conclusion holds.
(3) Let β be the least ordinal such that $In_{\beta+1}(\kappa,\lambda) = In_\beta(\kappa,\lambda)$ and $X \in In_\beta(\kappa,\lambda)$. We may assume $W := \{x \in X : X \cap P_{x\cap\kappa}x$ is completely ineffable $\} \in In_\beta(\kappa,\lambda)$. For $x \in W$ let β_x be the least ordinal such that $In_{\beta_x+1}(x\cap\kappa, x) = In_{\beta_x}(x\cap\kappa, x) \neq \emptyset$. By (2), for each $x \in W$ $T_x := \{y \in P_{x\cap\kappa}x : X \cap P_{x\cap\kappa}x \cap P_{y\cap\kappa}y = X \cap P_{y\cap\kappa}y \notin In_{\beta_x}(y\cap\kappa, y)\} \in In_{\beta_x}(x\cap\kappa, x)$. Set $\gamma = \max(\beta, \bigcup_{x\in W}\beta_x)$ and $T = \{x \in P_\kappa\lambda :$

$X \cap P_{x\cap\kappa}x \notin In_\gamma(x\cap\kappa, x)\}$. Then $T_x \subset T \cap P_{x\cap\kappa}x$ for every $x \in W$. Hence $T \in In_\gamma(\kappa,\lambda) = In_\beta(\kappa,\lambda)$ □

Now we conclude by the following corollary:

Corollary 4.13. *If $cf(\lambda) \geq \kappa$, then the completely ineffable ideal on $P_\kappa\lambda$ is not precipitous.*

5 Ineffability and almost ineffabilty

In this section we show $NAIn_{\kappa,\lambda} = NIn_{\kappa,\lambda}$ if $cf(\lambda) < \kappa$ and $\lambda^{<\kappa} = 2^\lambda$. Thus, two ideals are the same for "small" cofinality points if GCH holds.
Then we use it to prove that for λ with cofinality less than κ, "κ is λ-ineffable" does not always imply "κ is $\lambda^{<\kappa}$-ineffable". This contrasts to the fact "κ is $\lambda^{<\kappa}$-(super)compact if κ is λ-(super)compact.

Fact 5.1. (1) $WNS_{\kappa,\lambda} \subsetneq NAIn_{\kappa,\lambda} \subset NIn_{\kappa,\lambda}$.
(2) *If $P_\kappa\lambda \notin NAIn_{\kappa,\lambda}$ and $cf(\lambda) \geq \kappa$, then $\lambda^{<\kappa} = \lambda$.*
(3) *If $cf(\lambda) \geq \kappa$, then $\{x \in P_\kappa\lambda : P_{x\cap\kappa}x \in NAIn_{x\cap\kappa,x}\} \in NIn_{\kappa,\lambda}$ and $NAIn_{\kappa,\lambda} \subsetneq NIn_{\kappa,\lambda}$.*

We have known quite little when $cf(\lambda) < \kappa$ while the following conjecture has seemded reasonable comparing with supercompactness:

Conjecture 5.2. *Suppose that $cf(\lambda) < \kappa$ and κ is (almost) λ-ineffable. Then, $\lambda^{<\kappa} = \lambda^+$ and κ is λ^+-ineffable.*

Lemma 5.3. *If $\lambda^{<\kappa} = 2^\lambda$, then there exists $X \in WNS^*_{\kappa,\lambda}$ such that $I_{\kappa,\lambda} \restriction X = NS_{\kappa,\lambda} \restriction X$.*

Proof. Let ${}^{\lambda\times\lambda}\lambda = \{f_s : s \in P_\kappa\lambda\}$ be an enumeration and set $X = \{x \in P_\kappa\lambda : x \in \bigcap\{C_{f_s} : s \prec x\}\}$. For every $s \in P_\kappa\lambda$ $C_{f_s} \in NS^*_{\kappa,\lambda} \subset WNS^*_{\kappa,\lambda}$ hence $X \in WNS^*_{\kappa,\lambda}$. Let $Y \notin I_{\kappa,\lambda} \restriction X$. For every club $C \subset P_\kappa\lambda$ there exists $s \in P_\kappa\lambda$ such that $C_{f_s} \subset C$. We have $x \in Y \cap X$ with $s \prec x$. Then, $x \in C_{f_s} \cap Y \cap X \subset C \cap Y \cap X$ hence $Y \cap X \in NS^+_{\kappa,\lambda}$. □

Since $WNS_{\kappa,\lambda} \subset NAIn_{\kappa,\lambda}$ we have:

Theorem 5.4. *$NAIn_{\kappa,\lambda} = NIn_{\kappa,\lambda}$ if $\lambda^{<\kappa} = 2^\lambda$.*

By lemma 4.12 we have the following:

Lemma 5.5. *If κ is λ-ineffable, then $\{x \in P_\kappa\lambda : P_{x\cap\kappa}x \in NIn_{x\cap\kappa,x}\} \notin NIn_{\kappa,\lambda}$.*

Theorem 5.6. *Suppose that $\lambda^{<\kappa} = 2^\lambda$ and κ is λ^+-ineffable. Then, $\{x \in P_\kappa\lambda^+ : x \cap \kappa$ is $o.t.(x\cap\lambda)$-ineffable, not $o.t.(x)$-ineffable, and $o.t.(x) = o.t.(x\cap\lambda)^+\} \notin NIn_{\kappa,\lambda^+}$.*

Proof. We know $A = \{x \in P_\kappa\lambda^+ : x\cap\kappa$ is almost $o.t.(x)$-ineffable, $o.t.(x) = o.t.(x\cap\lambda)^+, cf(o.t.(x\cap\lambda)) < x\cap\kappa$, and $o.t.(x\cap\lambda)^{<x\cap\kappa} = 2^{o.t.(x\cap\lambda)}\} \in NIn^*_{\kappa,\lambda^+}$ ([12], [2]) . Every $x \in A$ is almost $o.t.(x\cap\lambda)$-ineffable hence $o.t.(x\cap\lambda)$-ineffable by the previous theorem. Now the conclusion follows by lemma 5.5. □

We conclude by the negation of $I^+_{\kappa,\lambda} \to (I^+_{\kappa,\lambda})^2$ with $\lambda^{<\kappa} = 2^\lambda$.

Definition 5.7. *For $X \subset P_\kappa\lambda$ let $[X]^2 = \{(x,y) \in X\times X : x \subsetneq y\}$. We say $I_{\kappa,\lambda}$ has the partition property if for every $X \notin I_{\kappa,\lambda}$ and $F : [X]^2 \to 2$ there exists $H \notin I_{\kappa,\lambda}$ such that $F \restriction [H]^2$ is constant.*

Theorem 5.8. *If $\lambda^{<\kappa} = 2^\lambda$, then $I_{\kappa,\lambda}$ does not have the partition property.*

Proof. Suppose otherwise and let $I = I_{\kappa,\lambda} \restriction X = NS_{\kappa,\lambda} \restriction X$ with X as in [?]. For every $X \in I^+$ and $F : [X]^2 \to 2$ there exists $H \notin NS_{\kappa,\lambda}$ such that $F \restriction [H]^2$ is constant. Hence $NIn_{\kappa,\lambda} \subset I$ ([15]) . However $I \subset WNS_{\kappa,\lambda} \subsetneq NIn_{\kappa,\lambda}$. Contradiction. □

Remark 5.9. *P. Matet proved that I_{κ,κ^+} does not have the partition property if $2^\kappa = \kappa^+$. While M. Shioya [17] constructed the model in which $I_{\kappa,\lambda}$ has the partition property with κ supercompact.*

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