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Cardinal invariants associated with some combinatorial statements

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Abstract

T. Bartoszyński [1] characterized the uniformity $\mathbf{non}(\mathcal{M})$ of the meager ideal on the real line as the smallest size of a family $X \subset \omega^\omega$ such that $\forall y \in \omega^\omega\ \exists x \in X\ \exists^\infty n < \omega\ y(n) = x(n)$. By replacing ω^ω by certain restricted subsets, we can get weaker combinatorial statements and define cardinal invariants. In this talk, we study these cardinal invariants.

0 Introduction

We use standard notion and notations in set theory (see e.g. [3]). Set

$$\mathcal{F} = \{ f \in (\omega \setminus \{0\})^\omega \mid f \text{ is non-decreasing and } \lim_{n<\omega} f(n) = \omega \}.$$

For each $f \in \mathcal{F}$, define the cardinal θ_f by

$$\theta_f = \min\{ |X| \mid X \subset \prod_{n<\omega} f(n) \text{ and } \forall y \in \prod_{n<\omega} f(n)\ \exists^\infty n < \omega\ y(n) = x(n) \}.$$

By the Bartoszyński's characterization of $\mathbf{non}(\mathcal{M})$, it holds that $\theta_f \leq \mathbf{non}(\mathcal{M})$ for all $f \in \mathcal{F}$. Also, it is easy to see that $\theta_{f_1} \leq \theta_{f_2}$ if $f_1,\ f_2 \in \mathcal{F}$ and $f_1 \leq^* f_2$. In the next section, we show that, in a certain generic model which is obtained by adjoining random reals, $\theta_{f_1} < \theta_{f_2}$ holds for some $f_1,\ f_2 \in \mathcal{F}$. Put $\theta = \min\{ \theta_f \mid f \in \mathcal{F} \}$. Let me introduce another cardinal invariant θ^* which is associated with a weaker combinatorial statement. For this, we need some definitions. Set

$$\mathcal{H} = \{ h \in \omega^\omega \mid h \text{ is strictly increasing and } \lim_{n<\omega} h(n+1) - h(n) = \omega \}.$$

For each $h \in \mathcal{H}$ and $n < \omega$, a^h_n denotes the interval $[h(n), h(n+1))$ of ω. Define θ^* by

$$\theta^* = \min\{ |W| \mid W \subset 2^\omega \times \mathcal{H} \text{ and } \forall y \in 2^\omega\ \exists (x,h) \in W\ \exists^\infty n < \omega\ y \restriction a^h_n = x \restriction a^h_n \}.$$

It is easy to check that $\omega_1 \leq \theta^* \leq \theta$. Furtheremore, we have:

Theorem 0.1 *Assume that* $\mathrm{cof}([\mathbf{d}]^\omega, \subset) = \mathbf{d}$. *Then, it holds that* $\theta^* \leq \mathbf{d}$.

Proof Take a sufficiently large regular cardinal ρ. By using the assumption, take an elementary substructure M of $H(\rho)$ such that

$$M \cap \omega^\omega \text{ is a dominating family and } |M| = \mathbf{d} \text{ and } M \cap [M]^\omega \text{ is } \subset\text{-cofinal in } [M]^\omega.$$

Since $M \cap \omega^\omega$ is a dominating family, it holds that

(*) $\forall h \in \mathcal{H}\, \exists h' \in M \cap \mathcal{H}\, \forall^\infty n < \omega\, \exists m < \omega \;\; a^{h'}_m \subset a^h_n$.

We show that $W = M \cap (2^\omega \times \mathcal{H})$ satisfy the definition of θ^*. To get a contradiction, assume that there is $y \in 2^\omega$ such that

$$\forall^\infty n < \omega \;\; y \restriction a^h_n \neq x \restriction a^h_n, \text{ for all } (x,h) \in W.$$

Put $X = 2^\omega \cap M$. The next claim is easily verfied by using (*).

Claim 0.2 $\forall x \in X\, \exists k < \omega\, \forall^\infty m < \omega \;\; y \restriction [m, m+k) \neq x \restriction [m, m+k)$. △

By Claim 0.2, define $\varphi : X \to \omega$ by

$$\varphi(x) = \text{ the largest } k < \omega \text{ such that } \exists^\infty m < \omega \;\; x \restriction [m, m+k) \subset y.$$

It is easy to check that $\sup\{\, \varphi(x) \mid x \in X \,\} = \omega$. By this, since $[M]^\omega \cap M$ is $\subset$-cofinal in $[M]^\omega$, we can take $A = \{\, a_i \mid i < \omega \,\} \in M$ such that $\sup\{\, \varphi(a_i) \mid i < \omega \,\} = \omega$. Take $\psi : \omega \times \omega \to \omega$ such that, for each $(i,n) \in \omega \times \omega$,

$$i + n + \varphi(a_i) \leq \psi(i,n) \text{ and } \exists m \in [n, \psi(i,n) - \varphi(a_i)) \;\; a_i \restriction [m, m + \varphi(a_i)) \subset y.$$

Without loss of generality, we may assume that $\psi \in M$. Define $\langle\, k_i \mid i < \omega \,\rangle \in M$ by

$$\begin{cases} k_0 &= 0 \\ k_{i+1} &= \psi(i, k_i), \text{ for } i < \omega \end{cases}$$

and set $x = \bigcup_{i<\omega} a_i \restriction [k_i, k_{i+1}) \in X$. Then, it holds that

$$\forall k < \omega\, \exists m < \omega \; x \restriction [m, m+k) \subset y.$$

But this contradicts Claim 0.2 □

Let $\mathbf{C}_\omega$ be the notion of forcing which adds a Cohen real. Then, it holds that

$$\Vdash_{\mathbf{C}_\omega} \forall y \in 2^\omega\, \exists x \in 2^\omega \cap \mathbf{V}\, \exists^\infty n < \omega \;\; x \restriction [n^2, n^2 + n) = y \restriction [n^2, n^2 + n).$$

So, $\theta^* < \mathbf{d}$ holds in a certain Cohen generic model.

It is known that the assumption $\mathrm{cof}([\mathbf{d}]^\omega, \subset) = \mathbf{d}$ is followed from the non-existence of $0^\#$. So, it seems to prove Theorem 0.1 without this assumption. But I failed to find a proof.

Question 0.1 *Is $\theta^* \leq \mathbf{d}$ proved in* ZFC*?*

In sections 2, 3, 4, we show that the cardinals ω_1, θ, θ^* can be separated for certain generic models.

1 Generic extensions by random reals

For each infinite cardinal κ, we denote by $\mathbf{B}(\kappa)$ the measure algebra which adds a random function from κ to 2 and by $\mu_\kappa : \mathbf{B}(\kappa) \to [0,1]$ the measure function. In this section, we prove the following theorem.

Theorem 1.1 *Assume* CH. *Let $\kappa > \omega_1$ be a regular cardinal. Then, there are f_1, $f_2 \in \mathcal{F}$ such that*

$\Vdash_{\mathbf{B}(\kappa)} \theta_{f_1} = \omega_1$ *and* $\theta_{f_2} = \kappa$.

Set $f_2 = \langle 2^n \mid n < \omega \rangle \in \mathcal{F}$. The next well-known lemma guarantees that this f_2 is as required in Theorem 1.1.

Lemma 1.2 (Forklore) $\Vdash_{\mathbf{B}(\omega)} \exists y \in \prod_{n<\omega} f_2(n) \, \forall x \in \prod_{n<\omega} f_2(n) \cap \mathbf{V} \, \forall^\infty n < \omega \; x(n) \neq y(n)$.

Proof Define $k_n < \omega$ (for $n < \omega$) by

$k_0 = 0$ and $k_{n+1} = k_n + n$ for $n < \omega$.

For each $n < \omega$, put $I_n = [k_n, k_{n+1})$ and take a bijections from ${}^{I_n}2$ to $f_2(n)$. Using these bijections, we identify $\prod_{n<\omega} f_2(n)$ with $\prod_{n<\omega} {}^{I_n}2$. Let $\dot{g}$ be the canonical $\mathbf{B}(\omega)$-name of generic real. Define $\dot{y}$ by

$\Vdash \dot{y} = \langle \dot{g} \restriction I_n \mid n < \omega \rangle$.

It holds that, for each $n < \omega$ and $s : I_n \to 2$,

$\mu_\omega([\![s = \dot{g} \restriction I_n]\!]) = 2^{-|I_n|} = 2^{-n}$.

So, $\mu_\omega([\![\exists^\infty n < \omega \; x \restriction I_n = \dot{y}(n)]\!]) = 0$ for all $x \in 2^\omega$. This implies that

$\Vdash \forall^\infty n < \omega \; x(n) \neq \dot{y}(n)$, for all $x \in \prod_{n<\omega} {}^{I_n}2$. □

Lemma 1.3 *Let $0 < K$, $M < \omega$. Suppose that $\{ b_i^m \mid i < K$ and $m < M \} \subset \mathbf{B}(\omega)$ and $b \in \mathbf{B}(\omega)$ satisfy*

$b = \sum_{i<K} b_i^m$, *for all* $m < M$.

Then, there is a function $\varphi : M \to K$ such that

$$\mu_\omega\Big(\sum_{m<M} b_{\varphi(m)}^m\Big) \geq \mu_\omega(b) - \Big(\frac{K-1}{K}\Big)^M \mu_\omega(b).$$

Proof By induction on $M \in [1, \omega)$. The case $M = 1$ is clear. Let $M = M_0 + 1 > 1$. Using the induction hypothesis, take $\varphi_0 : M_0 \to K$ such that

$$\mu_\omega(\sum_{m<M_0} b^m_{\varphi_0(m)}) \geq \mu_\omega(b) - (\frac{K-1}{K})^{M_0}\mu_\omega(b).$$

Put $c = \sum_{m<M_0} b^m_{\varphi_0(m)}$. Since $b - c = \sum_{i<K}(b^{M_0}_i - c)$, there exists $j < K$ such that $\mu_\omega(b^{M_0}_j) \geq \frac{1}{K}\mu_\omega(b-c)$. Then, $\varphi = \varphi_0{}^\frown\langle j \rangle$ is as required. □

For each $n < \omega$, let

$$M_n = \min\{ M < \omega \mid (\frac{n}{n+1})^M < 2^{-n} \}.$$

Define $f_1 \in \mathcal{F}$ by

$$|\{ k < \omega \mid f_1(k) = n+1 \}| = M_n, \text{ for all } n < \omega.$$

The next lemma implies that f_1 satisfies the condition in Theorem 1.1.

Lemma 1.4 $\Vdash_{\mathbf{B}(\omega)} \forall y \in \prod_{k<\omega} f_1(k)\, \exists x \in \prod_{k<\omega} f_1(k) \cap \mathbf{V}\, \exists^\infty k < \omega\; x(k) = y(k).$

Proof For each $n < \omega$, put $J_n = \{ k < \omega \mid f_1(k) = n+1 \}$. To show this lemma, let $\dot{y}$ be a $\mathbf{B}(\omega)$-name such that $\Vdash\ \dot{y} \in \prod_{k<\omega} f_1(k)$. For each $n < \omega$, using Lemma 1.3, take $s_n \in \prod_{k\in J_n} f_1(k)$ such that

$$\mu_\omega(\sum_{k\in J_n} [\![s_n(k) = \dot{y}(k)]\!]) \geq 1 - (\frac{n}{n+1})^{M_n}.$$

Put $x = \bigcup_{n<\omega} s_n$. It is easy to check that

$$\mu_\omega([\![\forall^\infty n < \omega\, \exists k \in J_n\; x(k) = \dot{y}(k)]\!]) = 0.$$

So, it holds that $\Vdash\ \exists^\infty k < \omega\; x(k) = \dot{y}(k)$. □

2 A forcing notion with the ccc which lifts up θ^*

Define the forcing notion $(Q, \leq)$ by

$$Q \subset 2^{<\omega} \times [2^\omega \times \mathcal{H}]^{<\omega}$$

and, for any $(s,u) \in 2^{<\omega} \times [2^\omega \times \mathcal{H}]^{<\omega}$,

$$(s,u) \in Q$$

if and only if, for all $(x,h) \in u$ and all $k < \omega$,

if $a^h_k \setminus \mathrm{dom}(s) \neq \phi$ then $|a^h_k \setminus \mathrm{dom}(s)| \geq |u|$ or $\exists i \in a^h_k \cap \mathrm{dom}(s)\; x(i) \neq s(i)$,

and, for any (s,u), $(s',u') \in Q$,

$$(s',u') \leq (s,u)$$

if and only if

$s' \supset s$ and $u' \supset u$ and, for all $(x,h) \in u$ and all $k < \omega$, if $a^h_k \cap (\mathrm{dom}(s') \setminus \mathrm{dom}(s)) \neq \phi$ then $|a^h_k \setminus \mathrm{dom}(s')| \geq |u'|$ or $\exists i \in a^h_k \cap \mathrm{dom}(s')\; x(i) \neq s'(i)$.

We show that a finite support iteration by the above forcing notion lifts up the value θ^*. For this, we need several lemmas.

Lemma 2.1 *Let $n < \omega$. Then, for every $(s,u) \in Q$, there is $s' \in 2^{<\omega}$ such that $(s',u) \in Q$ and $(s',u) \leq (s,u)$ and $n \in \mathrm{dom}(s)$.*

Proof For each $j < \omega$, define $\varphi_j : \mathcal{H} \to \omega$ by

$$\varphi_j(h) = \text{the unique } k < \omega \text{ such that } j \in a^h_k.$$

For each $t \in 2^{<\omega}$, define $\psi_t : 2^\omega \times \mathcal{H} \to \omega$ by

$$\psi_t(x,h) = \begin{cases} |a^h_{\varphi_{\mathrm{dom}(t)}(h)} \setminus \mathrm{dom}(t)|, & \text{if } t \restriction a^h_{\varphi_{\mathrm{dom}(t)}(h)} \subset x, \\ |a^h_{\varphi_{\mathrm{dom}(t)}(h)+1}|, & \text{otherwise.} \end{cases}$$

To show this lemma, let $n < \omega$ and $(s,u) \in Q$. Put $m = \mathrm{dom}(s)$. Take $M < \omega$ such that

$$n,\ m < M \text{ and } |a^h_{\varphi_M(h)} \setminus M| \geq |u|, \text{ for all } (x,h) \in u$$

By induction on $j \in [m, M]$, take $s_j : j \to 2$ as follows:

Put $s_m = s$. Suppose that $j \in [m, M)$ and s_j has been defined. Let l_j be the smallest element of $\{ \psi_{s_j}(x,h) \mid (x,h) \in u \}$. Take $(x_j, h_j) \in u$ such that $\psi_{s_j}(x_j, h_j) = l_j$. Set $s_{j+1} = s_j{}^\frown\langle 1 - x_j(j) \rangle$.

Claim 2.2 $|\{ (x,h) \in u \mid \psi_{s_j}(x,h) < l \}| < l$, *for all $0 < l < \omega$ and $j \in [m, M]$*

$\because$) By induction on $j \in [m, M]$. The case $j = m$ is followed from the fact $(s,u) \in Q$. The case $j = j_0 + 1 > m$ is followed from the fact $\psi_{s_j}(x_{j_0}, h_{j_0}) \geq |u|$. △

By Claim 2.2, it holds that $l_j > 0$, for all $j \in [m, M)$. So, it holds that $(s_M, u) \in Q$ and $(s_M, u) \leq (s,u)$. □

Lemma 2.3 *For each $(x,h) \in 2^\omega \times \mathcal{H}$,*

$\{ (s,u) \in Q \mid (x,h) \in u \}$ is dense in Q.

Proof Let $(x,h) \in 2^\omega \times \mathcal{H}$ and $(s,u) \in u$. Take $M < \omega$ such that

(1) $|s| \leq M$,

(2) if $a^{h'}_k \setminus M \neq \phi$ then $|a^{h'}_k \setminus M| > |u|$, for all $k < \omega$ and $(x', h') \in u$.

(3) if $a^h_k \setminus M \neq \phi$ then $|a^h_k \setminus M| > |u|$, for all $k < \omega$.

Using Lemma 2.1, take $(t,u) \leq (s,u)$ such that $\mathrm{dom}(t) = M$. Then, it holds that $(t, u \cup \{ (x,h) \}) \in Q$ and $(t, u \cup \{ (x,h) \}) \leq (s,u)$. □

Lemma 2.4 *Q satisfies the countable chain condition.*

Proof Let W be an uncountable subset of Q. Using Lemma 2.1, replace W by certain stronger conditions if necessary, we may assume that, for all $(s,u) \in W$,

$$\text{for all } (x,h) \in u \text{ and } k < \omega, \text{ if } a_k^h \setminus k \neq \phi \text{ then } |a_k^h \setminus k| \geq 2|u|.$$

Take $s_0 \in 2^{<\omega}$ and $m < \omega$ such that $W' = \{(s,u) \in W \mid s = s_0 \text{ and } |u| = m\}$ is uncountable. Then, every elements in W' are compatible. □

Let $\dot{G}$ be the canonical generic Q-name. Define $\dot{g}$ by

$$\Vdash_Q \dot{g} = \bigcup\{s \mid (s,u) \in \dot{G}, \text{ for some } u\}.$$

Lemma 2.5 $\Vdash_Q \dot{g} \in 2^\omega$ *and* $\forall x \in 2^\omega \cap \mathbf{V}\ \forall h \in \mathcal{H} \cap \mathbf{V}\ \forall^\infty n < \omega\ \dot{g} \restriction a_n^h \neq x \restriction a_n^h$.

Proof This is directly followed from Lemmas 2.1 and 2.3. □

Let κ be a regular uncountalbe cardinal and P the κ-stage finite support iteration by the above forcing Q. Then, by the above argments, it holds that $\theta^* = \kappa$ in the generic model $\mathbf{V}^P$. Since P is finite support, it adds cofinally many Cohen reals. So, in $\mathbf{V}^P$, the covering number $\mathbf{cov}(\mathcal{M})$ of the meager ideal on the real line lifts up to κ. Futhermore, the next lemma shows that the unbounding number $\mathbf{b}$ of ω^ω lifts up to κ, too.

Lemma 2.6 *There is a Q-name $\dot{d}$ such that*

$$\Vdash_Q \dot{d} \in \omega^\omega \textit{ dominates } \omega^\omega \cap \mathbf{V}.$$

Proof For each set X, denote by $\mathbf{0}_X$ the constantly zero function from X to 2.

Claim 2.7 *For any $n < \omega$,*

$$\{(s,u) \in Q \mid \exists m < \omega\ (\mathbf{0}_{[m,m+n)} \subset s)\} \textit{ is dense in } Q.$$

∵) Let $n < \omega$ and $(s,u) \in Q$. Take $(t,u) \leq (s,u)$ such that, for all $(x,h) \in u$ and $k < \omega$,

$$\text{if } a_k^h \setminus \mathrm{dom}(t) \neq \phi \text{ then } |a_k^h \setminus \mathrm{dom}(t)| \geq |u| + n.$$

Define $t' : |t| + n \to 2$ by $t \subset t'$ and $t'(|t| + j) = 0$, for $j < n$. It is easy to check that $(t',u) \in Q$ and $(t',u) \leq (s,u)$. △

By Claim 2.7, it holds that

$$\Vdash_Q \forall n < \omega\ \exists m < \omega\ \dot{g} \restriction [m, m+n) = \mathbf{0}_{[m,m+n)}.$$

So, in $\mathbf{V}^Q$, define $\dot{d} \in \omega^\omega$ by

$$\dot{d}(n) = \text{the smallest } m < \omega \text{ such that } n \leq m \text{ and } \dot{g} \restriction [m, m+2n) = \mathbf{0}_{[m,m+2n)}.$$

To show $\dot{d}$ is a required one, let $f \in \omega^\omega$ and $(s,u) \in Q$. Without loss of generality, we may assume that f is strictly increasing. Take $h \in \mathcal{H}$ such that

$|a^h_k| \leq |a^h_{k+1}|$, for all $k < \omega$ and $|\{ k < \omega \mid |a^h_k| = n \}| \geq f(n) + 1$, for all $n < \omega$. By Lemma 2.3, take $(t, v) \leq (s, u)$ such that $(\mathbf{0}_\omega, h) \in v$. Let k_0 be the smallest $k < \omega$ such that $|t| \geq h(k)$ and set $n_0 = |a^h_{k_0}| + |t|$. The next claim completes the proof of the lemma.

Claim 2.8 $(t, v) \Vdash_Q \forall n > n_0 \ f(n) < \dot{d}(n)$.

$\because$) To get a contradiction, assume that there are $(t', v') \leq (t, v)$ and $n > n_0$ such that $(t', v') \Vdash_Q \dot{d}(n) \leq f(n)$. Replace (t', v') by a stronger condition if necessary, we may assume that (t', v') decides the value of $\dot{d}(n)$. Let $m < \omega$ be such that $(t', v') \Vdash_Q \dot{d}(n) = m$. Without loss of generality, we may assume that $m + 2n \subset \mathrm{dom}(t')$. Let k be the unique $k < \omega$ such that $m \in a^h_k$. By the choise of h, it holds that $|a^h_k|$, $|a^h_{k+1}| \leq n$. $\therefore$ $a^h_{k+1} \subset [m, m + 2n)$. Since $(t', v') \Vdash_Q \dot{g} \restriction [m, m + 2n) = \mathbf{0}_{[m,m+2n)}$, it holds that $t' \restriction a^h_{k+1} = \mathbf{0}_{a^h_{k+1}}$. This contradicts the facts that $(t', v') \leq (t, v)$ and $(\mathbf{0}_\omega, h) \in v$ and $\mathrm{dom}(t) \cap [m, m + 2n) = \phi$. □

In section 4, we give a genric model in which holds $\theta^* = \omega_2$ and $\mathbf{cov}(\mathcal{M}) = \omega_1$. But I do not known whether there is a model which satisfies $\mathbf{b} < \theta^*$.

Question 2.1 *Is* $\mathbf{b} < \theta^*$ *consistent with* ZFC*?*

3 A forcing notion which lifts up θ

In this section, we give a forcing notion which gives a generic model of $\theta^* = \omega_1$ and $\theta = \omega_2$. The forcing notion which we give here is constructed by the ω_2-stage countable support iteration. We begin with the definition of a forcing notion $\mathbf{BT}_f$ for $f \in \mathcal{F}$ which will be used each stage in the iteration.

Let $f \in \mathcal{F}$. For each $n < \omega$, denote $\prod_{m<n} f(m)$ by S^f_n. Put $S^f = \bigcup_{n<\omega} S^f_n$. Note that $(S^f, \subset)$ is a tree. Define the forcing notion $(\mathbf{BT}_f, \leq)$ by

$q \in \mathbf{BT}_f$

if and only if

(1) q is a subtree of S^f.

(2) there is a function $f' \in \mathcal{F}$ such that $|\mathrm{succ}_q(s)| \geq f'(|s|)$ for every $s \in q$.

$q' \leq q$ if and only if $q' \subset q$.

For each $q \in \mathbf{BT}_f$, define $\pi_q \in \omega^\omega$ by

$$\pi_q(n) = \max\{ k < \omega \mid \forall n' \geq n \, \forall s \in q \cap S^f_{n'} \ |\mathrm{succ}_q(s)| \geq k \}.$$

Note that $\pi_q \in \mathcal{F}$ for all $q \in \mathbf{BT}_f$. For each $k < \omega$, define the ordering $\leq_k$ on $\mathbf{BT}_f$ by

$$q' \leq_k q \text{ if and only if } q' \leq q \text{ and } \pi_q \restriction m_k = \pi_{q'} \restriction m_k,$$

where m_k denotes the smallest $m < \omega$ such that $\pi_q(m) > k$.

In [2], Bartoszyński, Judah and Shelah have used similar but more complicated forcing notions $\mathbf{Q}_{f,g}$. The proof of the next lemma is similar to, but quite easier than the proof of Claim 2.6 in [2].

Lemma 3.1 *Let $\dot{e}$ be a $\mathbf{BT}_f$-name such that $\Vdash \dot{e} \in \mathbf{V}$. Then, for each $k < \omega$ and $q \in \mathbf{BT}_f$, there are $q' \leq_k q$ and a finite set E such that $q' \Vdash \dot{e} \in E$.*

Proof Let $\dot{e}$, $k < \omega$, $q \in \mathbf{BT}_f$ be as in the lemma. For each $s \in q$, denote by $q[s]$ the condition $\{ t \in q \mid s \subset t \text{ or } t \subset s \}$. Take $M < \omega$ such that $\pi_q(M) \geq 2k$. Set

$$T = \{ s \in q \mid |s| \geq M \text{ and } \exists q' \leq_k q[s] \exists E \ (E \text{ is finite and } q' \Vdash \dot{e} \in E \ ) \}.$$

Note that, whenever $s \in q \setminus T$ and $|s| \geq M$, $|\mathrm{succ}_q(s) \cap T| < k$.

Claim 3.2 $q \cap S^f_M \subset T$.

$\because$) To get a contradiction, assume that $s \in q \cap S^f_M \setminus T$. Let $U = \{ t \in q \setminus T \mid s \subset t \}$. Then, it holds that

$$\forall t \in U \ (\ |\{ u \in U \mid t \subset u \text{ and } |u| = |t| + 1 \}| > \pi_q(|u|) - k \ ).$$

This implies that $r = \{ s \restriction j \mid j < |s| \} \cup U \in \mathbf{BT}_f$ and $r \leq_k q[s]$. Take $r' \leq r$ such that r' decides $\dot{e}$. Take $t \in r'$ such that $\pi_{r'}(|t|) \geq k$. Since $r'[t] \leq_k q[t]$, we have that $t \in T$. This contradicts that $U \cap T = \phi$. △

By Claim 3.2, for each $s \in q \cap S^f_M$, take $q_s \leq_k q[s]$ and a finite set E_s such that $q_s \Vdash \dot{e} \in E_s$. Then $q' = \bigcup_{s \in q \cap S^f_M} q_s$ and $E = \bigcup_{s \in q \cap S^f_M} E_s$ satisfy this lemma. □

Corollary 3.3 $(\mathbf{BT}_f, (\leq_k)_{k<\omega})$ *satisfies Axiom A and* $\mathbf{BT}_f$ *is ω^ω-bounding.* □

Let $\dot{G}$ be the canonical generic $\mathbf{BT}_f$-name. Define $\mathbf{BT}_f$-name $\dot{g}$ by

$$\Vdash \dot{g} = \bigcup (\bigcap \dot{G}) \in \prod_{n<\omega} f(n).$$

Then, it is easy to check that

$$\Vdash \forall x \in \prod_{n<\omega} f(n) \cap \mathbf{V} \ \forall^\infty n < \omega \ \ \dot{g}(n) \neq x(n).$$

Now we can describe how to construct a model which satisfies $\theta = \omega_2$ and $\theta^* = \omega_1$. Start with a ground model with CH. Let $\{ f_\alpha \mid \alpha < \omega_2 \} \subset \mathcal{F}$ be such that

$$\{ \alpha < \omega_2 \mid f_\alpha = f \} \text{ is cofinal in } \omega_2 \text{ for each } f \in \mathcal{F}.$$

Define the ω_2-stage countable support iteration P_α (for $\alpha \le \omega_2$), $\dot{Q}_\alpha$ (for $\alpha < \omega_2$) by

$$\Vdash_\alpha \dot{Q}_\alpha = \mathbf{BT}_{f_\alpha}.$$

Let $P = P_{\omega_2}$. Then, by the above argments, it holds that, in $\mathbf{V}^P$, $\theta = \omega_2$ and $\mathbf{d} = \omega_1$. Since $\mathrm{cof}([\omega_1]^\omega, \subset) = \omega_1$ does always hold, it holds that, in $\mathbf{V}^P$, $\theta^* \le \mathbf{d} = \omega_1$.

4 A generic model of $\theta = \omega_2$ and $\mathbf{cov}(\mathcal{M}) = \omega_1$

In the previous section, we show that $\mathbf{BT}_f$ does not lift up θ^*. But, if we first add a dominating real then we get a certain function $f \in \mathcal{F}$ such that $\mathbf{BT}_f$ lifts up θ^*. In this section, we show that θ^* can be separeted from $\mathbf{cov}(\mathcal{M})$ by using it.

Lemma 4.1 *Let* $\mathbf{V}$, $\mathbf{W}$ *be transitive models of* ZFC *such that* $\mathbf{V} \subset \mathbf{W}$. *Assume that* $d \in \mathbf{W} \cap \omega^\omega$ *dominates* $\mathbf{V} \cap \omega^\omega$. *In* $\mathbf{W}$, *define* $h \in \mathcal{H}$ *by*

$$|a^h_k| \le |a^h_{k+1}|, \text{ for all } k < \omega \text{ and } |\{ k < \omega \mid |a^h_k| = n \}| = d(n) + 1, \text{ for all } n < \omega.$$

Then, it holds that $\forall^\infty m < \omega \, \exists k < \omega \; a^h_k \subset a^{h'}_m$ *for all* $h' \in \mathbf{V} \cap \mathcal{H}$.

Proof Let $h' \in \mathbf{V} \cap \mathcal{H}$. In $\mathbf{V}$, define f_0, $f_1 \in \omega^\omega$ by

$f_0(n) =$ the smallest $m < \omega$ such that $\forall m' \ge m \;\; |a^{h'}_{m'}| \ge 2n$, and

$f_1(n) = \max \, a^{h'}_{f_0(n+1)}$.

Since d dominates f_0, f_1, there is $n_0 < \omega$ such that $\forall n \ge n_0 \;\; f_0(n), f_1(n) < d(n)$. Put $k_0 = f_0(n_0)$. To show that $\forall k \ge k_0 \, \exists j < \omega \; a^h_j \subset a^{h'}_k$, let $k \ge k_0$. Take $n < \omega$ such that $f_0(n) \le k < f_0(n+1)$. Then, it holds that $|a^{h'}_k| \ge 2n$ and $\max a^{h'}_k < \max a^{h'}_{f_0(n+1)} = f_1(n) \le d(n)$. Since $[0, d(n))$ is covered by $\{ a^h_j \mid j < d(n) \}$ and $|a^h_j| \le n$ for all $j < d(n)$, there is $j < d(n)$ such that $a^h_j \subset a^{h'}_k$. □

Lemma 4.2 *Let* $\mathbf{V}$, $\mathbf{W}$, d *and* h *be as in Lemma 4.1. Working in* $\mathbf{W}$. *Define* $f \in \mathcal{F}$ *by*

$$f(k) = 2^{|a^h_k|}, \text{ for all } k < \omega.$$

Then, there is a $\mathbf{BT}_f$*-name* $\dot{y}$ *such that*

$$\Vdash \dot{y} \in 2^\omega \text{ and } \Vdash \forall^\infty k < \omega \; \dot{y} \restriction a^{h'}_k \ne x \restriction a^{h'}_k, \text{ for all } x \in 2^\omega \cap \mathbf{V} \text{ and } h' \in \mathcal{H} \cap \mathbf{V}.$$

Proof Working in $\mathbf{W}$. Considering bijections from $f(k)$ to ${}^{a^h_k}2$ for $k < \omega$, we may identfy $\prod_{k<\omega} f(k)$ with $\prod_{k<\omega} {}^{a^h_k}2$. Let $\dot{G}$ be the canonical generic $\mathbf{BT}_f$-name. Define $\mathbf{BT}_f$-names $\dot{g}$ and $\dot{y}$ by

$$\Vdash \dot{g} = \bigcup(\bigcap \dot{G}) \text{ and } \dot{y} = \bigcup_{k<\omega} \dot{g}(k).$$

Note that $\Vdash \dot{g} \in \prod_{k<\omega} {}^{a_k^h}2$ and $\dot{y} \in 2^\omega$. It is easy to check that

$$\Vdash \forall x \in 2^\omega \cap \mathbf{W}\, \forall^\infty k < \omega \;\; \dot{y} \restriction a_k^h \neq x \restriction a_k^h.$$

To show $\dot{y}$ is as required, let $x \in \mathbf{V} \cap 2^\omega$ and $h' \in \mathbf{V} \cap \mathcal{H}$. Since it holdds that $x \in \mathbf{W}$ and $\forall^\infty m < \omega\, \exists\, k < \omega\;\, a_k^h \subset a_m^{h'}$, we have that

$$\Vdash \forall^\infty m < \omega \;\; \dot{y} \restriction a_m^{h'} \neq x \restriction a_m^{h'}. \qquad \square$$

Corollary 4.3 *Assume that* CH *holds. There are a forcing notion R and R-name $\dot{y}$ such that*

(1) *R is proper and does not add a Cohen real and $|R| = \omega_1$.*

(2) $\Vdash_R \; \dot{y} \in 2^\omega$ *and* $\forall x \in 2^\omega \cap \mathbf{V}\, \forall h \in \mathcal{H} \cap \mathbf{V}\, \forall^\infty k < \omega\, \dot{y} \restriction a_k^h \neq x \restriction a_k^h$. $\square$

Using Corollary 4.3, we can construct a generic model which satisfies $\mathbf{cov}(\mathcal{M}) = \omega_1$ and $\theta^* = \omega_2$. Start with a ground model with CH. Take an ω_2-stage countable support iteration by the forcing notion as in Corollary 4.3. Since the iteration does not add a Cohen real, $\mathbf{cov}(\mathcal{M})$ remains ω_1. On the other hand, since functions $\dot{y} \in 2^\omega$ which satisfy (2) in the corollary is added cofinally, θ^* must be lifted up.

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