数理解析研究所講究録 1306 巻 2003 年 109-117

無限個の制約条件を有する最適化問題について -ファジィ微分方程式への応用-

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Abstract In this study we give a new representation of fuzzy numbers with bounded supports and also we show that a fuzzy number means a bounded continuous curve in the two-dimensional metric space. Our aims of this research are to discuss optimization problems with objective functions and constraints both of which are $L-$fuzzy functions and to consider oil well equations which are represented by fuzzy differential equations $C_L'(t)+D_LC_L(t)=0$, where t is the time, $0\in\mathbf{R}$, $C_L(t)$ an $L-$fuzzy function and D_L a constant $L-$fuzzy number by applying the above criteria of $L-$optimization problems. Moreover we get an extension of the maxi-max theorem of optimization problems with an infinite number of constraints and a objective function which consists of infinite series. Finally we give remarks on a stochastic infinite horizon model of profit functions with a random variable.

Keywords: Fuzzy Numbers; Fuzzy Differential Equations; Attractive Set ; Couple Parametric Representation.

1 Set of fuzzy numbers

Let $I=[0,1]$. We define the following set of fuzzy numbers, where a fuzzy number x is characterized by a membership function μ_x as follows (cf. [2, 3]):

Definition 1 *Denote*

$\mathcal{F}_{\mathbf{b}}^{st}=\{\mu_x:\mathbf{R}\to I$ *satisfying* (i) $-$ (iv) *below*$\}$.

(i) *There exists a unique* $m\in\mathbf{R}$ *such that* $\mu_x(m)=$

(ii) *The support set* $supp(\mu_x)=cl(\{\xi\in\mathbf{R}:\mu(\xi)>0\})$ *is bounded in* $\mathbf{R}$;

(iii) *Let* $J=\{\xi\in\mathbf{R}:0<\mu_x(\xi)\}$. μ_x *is strictly quasi-convex on* J, *i.e.*, $\mu_x(\lambda\xi_1+(1-\lambda)\xi_2)>\min[\mu_x(\xi_1),\mu_x(\xi_2)]$ *for* $0<\lambda<1$ *and* $\xi_1,\xi_2\in J$ *such that* $\xi_1\neq\xi_2$;

(iv) μ_x *is upper semi-continuous on* $\mathbf{R}$.

In usual case a fuzzy number x satisfies quasi-convex on $\mathbf{R}$, *i.e.*,

$$\mu_x(\lambda\xi_1 + (1-\lambda)\xi_2) \geq \min[\mu_x(\xi_1), \mu_x(\xi_2)]$$

for $0 \leq \lambda \leq 1$ and $\xi_1, \xi_2 \in \mathbf{R}$. Condition (iii) plays an important role in proving properties of membership function μ_x in Theorem 1, where we show significant properties concerning the end-points of the α-cut set $L_\alpha(\mu_x) = \{\xi \in \mathbf{R} : \mu_x(\xi) \geq \alpha\}$.

In the similar way as [2, 3] we consider the following parametric representation of $\mu_x \in \mathcal{F}_{\mathrm{b}}^{st}$ such that

$$x_1(\alpha) = \min L_\alpha(\mu_x), \quad x_2(\alpha) = \max L_\alpha(\mu_x)$$

for $0 < \alpha \leq 1$ and that $x_1(0) = \min cl(supp(\mu_x)), x_2(0) = \max cl(supp(\mu_x))$. In what follows we denote a fuzzy numbers x by (x_1, x_2), *i.e.*, $x = (x_1, x_2)$.

By applying the above extension principle and the representation of fuzzy numbers we get the following results.

1) Addition. Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathcal{F}_{\mathrm{b}}^{st}$. We get the addition

$$\begin{aligned}\mu_{x+y}(\xi) &= \sup_{\xi=\xi_1+\xi_2} \min[\mu_x(\xi_1), \mu_y(\xi_2)] \\ &= \sup\{\alpha \in I : \xi = \xi_1 + \xi_2, \quad \xi_1 \in x_\alpha, \xi_2 \in y_\alpha\} \\ &= \sup_{\xi \in [x_1(\alpha)+y_1(\alpha), x_2(\alpha)+y_2(\alpha)]} \alpha,\end{aligned}$$

which means that $x + y = (x_1 + y_1, x_2 + y_2)$. Here $x_\alpha = L_\alpha(\mu_x)$ etc.

2) Subtraction. It follows that

$$\mu_{x-y}(\xi) = \sup\{\alpha \in I : \xi = \xi_1 - \xi_2, \quad \xi_1 \in x_\alpha, \xi_2 \in y_\alpha\}$$

means that $x - y = (x_1 - y_2, x_2 - y_1)$.

3) Product. It follows that

$$\mu_{xy}(\xi) = \sup\{\alpha \in I : \xi = \xi_1\xi_2, \quad \xi_1 \in x_\alpha, \xi_2 \in y_\alpha\}$$

means that the following relation.

$$xy = \begin{cases} (x_1y_1, x_2y_2) & (0 \leq x_1, 0 \leq y_1) \\ (x_2y_1, x_2y_2) & (0 \leq x_1, y_1 \leq 0 \leq y_2) \\ (x_2y_1, x_1y_2) & (0 \leq x_1, y_2 \leq 0) \\ (x_1y_2, x_2y_2) & (x_1 \leq 0 \leq x_2, 0 \leq y_1) \\ (\min\{x_2y_1, x_1y_2\}, \max\{x_1y_1, x_2y_2\}) & \\ \quad (x_1 \leq 0 \leq x_2, y_1 \leq 0 \leq y_2) & \\ (x_2y_1, x_1y_1) & (x_1 \leq 0 \leq x_2, y_2 \leq 0) \\ (x_1y_2, x_2y_1) & (x_2 \leq 0, 0 \leq y_1) \\ (x_1y_2, x_1y_1) & (x_2 \leq 0, y_1 \leq 0 \leq y_2) \\ (x_2y_2, x_1y_1) & (x_2 \leq 0, y_2 \leq 0) \end{cases}$$

By the above parametric representation of fuzzy numbers we get the following theorem concerning properties of end-points.

Theorem 1 *Denote* $x = (x_1, x_2) \in \mathcal{F}_{\mathrm{b}}^{st}$, *where* $x_1, x_2 : I \to \mathbf{R}$. *Then the following properties* (i)-(iii) *hold:*

(i) $x_i \in C(I), i = 1, 2$. *Here* $C(I)$ *is the set of all the continuous functions on* I*;*

(ii) *There exists a unique* $m \in \mathbf{R}$ *such that* $x_1(1) = x_2(1) = m$ *and* $x_1(\alpha) \leq m \leq x_2(\alpha)$ *for* $\alpha \in I$*;*

(iii) *One of the following statements* (a) *and* (b) *holds;*

(a) *Functions* x_1, x_2 *are non-decreasing, non-increasing on* I*, respectively, with* $x_1(\alpha) < x_2(\alpha)$ *for* $0 \leq \alpha < 1$*;*

(b) $x_1(\alpha) = x_2(\alpha) = m$ *for* $0 < \alpha \leq 1$.

Conversely, under the above conditions (i) -(iii), *if we denote*

$$\mu_x(\xi) = \sup\{\alpha \in I : x_1(\alpha) \leq \xi \leq x_2(\alpha)\} \quad (1.1)$$

then μ_x *is the membership function of* x, *i.e.*, $\mu_x \in \mathcal{F}_{\mathrm{b}}^{st}$.

Let a metric between $x = (x_1(\cdot), x_2(\cdot)), y = (y_1(\cdot), x_2(\cdot))$ be defined as follows.

$$d(x,y) = \sup_{\alpha \in I} \sqrt{|x_1(\alpha) - y_1(\alpha)|^2 + |x_2(\alpha) - y_2(\alpha)|^2}$$

Then we get following result immediately.

Theorem 2 $(\mathcal{F}_{\mathrm{b}}^{st}, d)$ *is complete metric space.*

2 Set of $L-$fuzzy numbers

Denote a shape function by $L : \mathbf{R} \to I$, where L is upper semi-continuous and satisfies the following properties (i) - (iv):

(i) $L(0) = \max_{\mathbf{R}} L(\xi) = 1$; (ii) $L(\xi)$ is strictly decreasing in $\xi \geq 0$;

(iii) $L(-\xi) = L(\xi)$ for $\xi \geq 0$; (iv) $\sup\{\xi \in \mathbf{R} : L(\xi) > 0\} = 1$.

In what follows we consider a set of $L-$fuzzy numbers $\mathcal{F}_L = \{\mu \in \mathcal{F}_{\mathrm{b}}^{st} : (a)$ or (b) hold.$\}$ Let $m \in \mathbf{R}$, $\ell \geq 0$. There exist two typical types (a) and (b) of $\mathcal{F}_L$.

(a) $\ell > 0$ and $\mu(\xi) = \begin{cases} L(\frac{m-\xi}{\ell}) & \text{for } \xi \leq m \\ L(\frac{\xi - m}{\ell}) & \text{for } \xi > m \end{cases}$

(b) $\ell = 0$ and $\mu(\xi) = \begin{cases} 1 & \text{for } \xi = m \\ 0 & \text{for } \xi \neq m \end{cases}$

In this section we introduce a total order relation λ-fuzzy max order $\preceq_\lambda$ over $\mathcal{F}_L$. Here $0 \leq \lambda \leq 1$ is given by decision makers. Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{F}_L$ with the center $x_1(1)$, the spread $\ell_x = x_1(1) - x_1(0) \geq 0$ and center $y_1(1)$, spread $\ell_y = y_1(1) - y_1(0) \geq 0$. We define that $x \preceq_\lambda y$ if and only if the following statements (i)-(iii) ([1]):

(i) $|\ell_y - \ell_x| \leq y_1(1) - x_1(1)$ for $y_1(1) \geq x_1(1)$;

(ii) $\lambda|\ell_y - \ell_x| \leq y_1(1) - x_1(1) < |\ell_y - \ell_x|$ for $y_1(1) > x_1(1)$ and $\ell_y \neq \ell_x$;

(iii) $|y_1(1) - x_1(1)| < \lambda(\ell_y - \ell_x)$ for $\ell_y - \ell_x > 0$.

Furukawa [1] gives the following theorem so that two any $L-$fuzzy numbers can be compared to each other.

Theorem F1 *For any* $x, y \in \mathcal{F}_L$ *it follows that one of the relations* $x \preceq_\lambda y$, *and* $y \preceq_\lambda x$ *hold. Thus* $\preceq_\lambda$ *is a total order relation over* $\mathcal{F}_L$.

The following theorem plays an important role in comparing two $L-$fuzzy numbers.

Theorem F2 *For* $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{F}_L$ *satisfying* $\ell_x = x_1(1) - x_1(0) \geq 0$, $\ell_y = y_1(1) - y_1(0) \geq 0$, *it follows that* $x \preceq_\lambda y$ *means that* (i) or (ii) *hold.*

(i) $\lambda \ell_x + x_1(1) < \lambda \ell_y + y_1(1)$ *for* $\ell_y > \ell_x$;

(ii) $\lambda \ell_x + x_1(1) \leq \lambda \ell_y + y_1(1)$ *for* $\ell_y \leq \ell_x$.

3 Fuzzy optimization problems

Let $0 \leq \lambda \leq 1$. In this section we show criteria concerning the following optimization problem

$$\text{minimize } f(z) \text{ subject to } z \in \mathcal{C}_\lambda^\delta. \qquad (P_\lambda^\delta)$$

where $\mathcal{C}_\lambda^\delta$ is a feasible set in $\mathcal{F}_L^n$ or $\mathcal{F}_L^\infty$ and $f : \mathcal{C}_\lambda^\delta \to \mathcal{F}_L$ is an objective function. We denote $z \in \mathcal{F}_L^\infty$ by

$$z = (z_1, z_2, z_3, \cdots) \text{ where } z_i \in \mathcal{F}_L \text{ for } i = 1, 2, \cdots.$$

In what follows we consider

$$\mathcal{C}_\lambda^\delta = \{z \in \mathcal{F}_L^\infty : g_j(z) \preceq_\lambda (0, \delta_j)_L, j = 1, 2, \cdots\}.$$

where $g_j : \mathcal{C}^{\delta}_{\lambda} \to \mathcal{F}_L$, $(0, \delta_j)_L \in \mathcal{F}_L$, and $\delta_j > 0$ are constants for $j = 1, 2, \cdots$. Let $\delta = (\delta_1, \delta_2, \cdots) \in \mathbf{R}^{\infty}$. If $z^* \in \mathcal{C}^{\delta}_{\lambda}$ satisfies $f(z^*) = \min\{f(z) : z \in \mathcal{C}^{\delta}_{\lambda}\}$ then z^* is called an optimal solution of (P^{δ}_{λ}).

In order to analyze (P^{δ}_{λ}) which is general case, we may consider an $\mathbf{R}$–valued optimization problem

$$\text{minimize } f(z) \text{ subject to } z \in \mathcal{C}^{\delta}_{\lambda} \cap \mathbf{R}. \qquad (P^0_0)$$

Then, letting $f^*_0 = \min_{z \in \mathcal{C}^{\delta}_{\lambda} \cap \mathbf{R}} f(z)$, we get $f^*_0 \in \mathbf{R}$ which gives the optimal value in $\mathbf{R}$ as follows.

Corollary 1 *Let $f^* \in \mathbf{R}$. Then there exists no L–fuzzy number $f \in \mathcal{F}_L \backslash \mathbf{R}$ such that $f = f^*$, i.e., $f \preceq_{\lambda} f^*$ and $f^* \preceq_{\lambda} f$.*

From the above corollary we get the following lemma immediately.

Lemma 1 *Denote $f^{\delta}_1 = \min\{f(z) : z \in \mathcal{C}^{\delta}_{\lambda}\}$, $f^{\delta}_2 = \min\{f(z^c) : z \in \mathcal{C}^{\delta}_{\lambda}\} \in \mathbf{R}$,*
$f^{\delta}_3 = \min\{f(z)^c : z \in \mathcal{C}^{\delta}_{\lambda}\} \in \mathbf{R}$,
$f^*_0 = \min\{f(z) : z \in \mathcal{C}^0_0\} \in \mathbf{R}$, where $z^c \in \mathbf{R}, f(z)^c \in \mathbf{R}$ are centers of $z, f(z)$, respectively, and $\mathcal{C}^0_0 = \mathcal{C}^{\delta}_{\lambda} \cap \mathbf{R}$.

If there exist $f^{\delta}_i, i = 1, 2, 3$ and f^*_0, then it follows that $f^{\delta}_1 \in \mathbf{R}$ and that

$$f^{\delta}_1 = f^{\delta}_2 = f^{\delta}_3 \le f^*_0.$$

If $\delta = 0$, then $f^0_1 = f^0_2 = f^0_3 = f^*_0$.

Remark 1 *It follows that*

$$\mathcal{C}^0_0 = \{z \in \mathbf{R}^{\infty} : g_j(z) \le 0, j = 1, 2, \cdots\} = \mathcal{C}^{\delta}_{\lambda} \cap \mathbf{R}.$$

When $\delta_j > 0$ for some integer j, there exists an example such that $f^{\delta}_1 = f^{\delta}_2 = f^{\delta}_3 < f^{\delta}_4$. See Exmaple

In case that there exists an optimal solution of L–fuzzy optimization problems, by the above lemma, the solution means a real number.

Theorem 3 *Denote $f^* = \min\{f(z) : z \in \mathcal{C}^{\delta}_{\lambda}\}$ and $f^*_0 = \min\{f(z) : z \in \mathcal{C}^{\delta}_{\lambda} \cap \mathbf{R}\}$. Suppose that (P^{δ}_{λ}) has at least one optimal solution in $\mathcal{C}^{\delta}_{\lambda}$. Then there exist f^*, f^*_0 in $\mathbf{R}$, which satisfy $f^* = f^*_0$.*

L–fuzzied numbers

In what follows we introduce an idea of L–fuzzied numbers generalized by $\mathcal{F}^{st}_{\mathbf{b}}$. Let $x \in \mathcal{F}_L$. The quadratic x^2 of an L–fuzzy number x isn't necessarily L–fuzzy number but fuzzy number in $\mathcal{F}^{st}_{\mathbf{b}}$ (see [6]). For $x = (x_1, x_2) \in \mathcal{F}_L$ and $\alpha \in I$, we have the following three cases:

- $x^2 = (x_1^2, x_2^2)$ if $x_1(\alpha) \ge 0$;
- $x^2 = (x_1 x_2, \max[x_1^2, x_2^2])$ if $x_1(\alpha) \le 0 \le x_2(\alpha)$;
- $x^2 = (x_2^2, x_1^2)$ if $x_2(\alpha) \le 0$.

In this study we consider the left portion of the membership function μ_{x^2} is more significant than the right portion of μ_{x^2}. Denote an operator $(\cdot)_L : \mathcal{F}^{st}_{\mathbf{b}} \to \mathcal{F}_L$ such that $(x)_L = (x_1(1), x_1(1) - x_1(0))_L$ for $x = (x_1, x_2) \in \mathcal{F}^{st}_{\mathbf{b}}$. We call that $(x)_L$ is an *L–fuzzized number.* Here the membership function of x is $\mu_x(\xi) = L(\frac{x_1(1)-\xi}{x_1(1)-x_1(0)})_+$ for $\xi \in \mathbf{R}$, $L : \mathbf{R} \to \mathbf{R}_+$ is a shape function and $\xi_+ = \max(\xi, 0)$ if $\xi \in \mathbf{R}$. For $x \in \mathcal{F}_L$ we get the L–fuzzied number

$$(x^2)_L = (x_1(1)^2, x_1(1)^2 - x_i(0)x_j(0)))_L,$$

where $i = 1, j = 2$ if $x_1(0)x_2(0) \le 0$, $i = j = 1$ if $x_1(0)x_2(0) > 0$ and $|x_1(0)| < |x_2(0)|$, $i = j = 2$ if $x_1(0)x_2(0) > 0$ and $|x_1(0)| \ge |x_2(0)|$.

Let a shape function be $L(\xi) = (1 - |\xi|)_+$. For an L−fuzzy number $x = (\xi_0, \ell)_L$ with $|\xi_0| \le \ell$, which has the membership function $\mu_x(\xi) = L(\frac{\xi_0 - \xi}{\ell})_+$ for $\xi \in \mathbf{R}$. Then we get the membership function

$$\mu_{x^2}(\xi) = \begin{cases} \left(1 - \frac{\sqrt{\xi_0^2 - \xi}}{\ell}\right)_+ & \text{for } \xi < \xi_0^2; \\ \left(1 - \frac{\xi_0 - \sqrt{\xi}}{\ell}\right)_+ & \text{for } \xi \ge \xi_0^2. \end{cases}$$

In this case we construct an L−fuzzy numbers $(x^2)_L$ with the same portion as the left one of μ_{x^2}. It follows that $(x^2)_L = (\xi_0^2, \ell^2)_L$. For $x \in \mathcal{F}_L$ and $k \in \mathbf{R}$ we have $(kx)_L = kx$.

In the following example we consider L−fuzzy optimization problem with a fuzzy objective function and fuzzy constraints.

Example 1 *Let $z = (u, v) \in \mathcal{F}_L^2$ and $\lambda \in I$. Fuzzy functions $F, g_j, j = 1, 2, 3$, are as follows (P_λ^δ):*

$$\begin{aligned} F(z) &= -u - v; \\ g_1(z) &= -u \preceq_\lambda (0, \delta_1)_L; \\ g_2(z) &= -v \preceq_\lambda (0, \delta_2)_L; \\ g_3(z) &= (u^2)_L + (v^2)_L \preceq_\lambda (1, \delta_3)_L. \end{aligned}$$

Here $(0,\delta_1)_L, (0,\delta_2)_L, (1,\delta_3)_L$ are L−fuzzy numbers and $(u^2)_L = (u_1(1)^2, \ell_{u^2})_L$, $(v^2)_L = (v_1(1)^2, \ell_{v^2})_L$ are L−fuzzized numbers.

The minimum of the above problem is attained at $u_1(1) = v_1(1) = (-\sqrt{\frac{1+\lambda\delta_3}{2}}, 0)_L$, which means that $\min_z f(z) = (-\sqrt{\frac{1+\lambda\delta_3}{2}}, 0)_L$ and $u^* = v^* = (-\sqrt{\frac{1+\lambda\delta_3}{2}}, 0)_L$. When $\lambda = 0$ and $\delta_j = 0, j = 1, 2, 3$, then the real type of optimization problem (P_0^0) gives $-\sqrt{2} \le f(z) \le 0$ in $\mathbf{R}$ and $u^* = v^* = 1/\sqrt{2} \in \mathbf{R}$.

This example shows that there exists a unique optimal solution of L−fuzzy number of fuzzy optimization problem (P_λ^δ) with a fuzzy coefficient, where (P_λ^δ) is an optimization problem with $\mathbf{R}$−valu coefficients if $\ell_z = 0$ and (P_λ^δ) is fuzzy type if $\ell_z \ne 0$, where ℓ_z is the spread of $z \in \mathcal{C}_\lambda^\delta$. Therefore the optimal solution to the real type (P_0^0) is the same as solution to the fuzzy type (P_λ^δ) concerning $\lambda = 0$ and $\ell_z = 0$.

4 Oil well equations with R-valued functions

In [8] they discuss exponential decay problems, *e.g.*, machine replacement and oil well extraction, etc. They analyze optimization problems for each oil well to determine its optimal replacement schedule. In order to give a mathematical model we introduce the following notations.

- $C(t)$: the quality remaining in the well at time t
- $D > 0$: rate of oil extraction
- P : unit profit of oil (sufficiently large)

As the oil reserves get depleted, the rate of extraction eventually decreases to uneconomic levels, making it worthwhile to abandon the well and drill a new one at a cost $f(\nu)$. Here ν is capacity of the well, f is continuously differentiable function. Assume that $V \in \mathbf{R}$ and that $0 \le \nu \le V$, $f(0) = 0$, $f'(\nu) \ge 0$.

Then they get the following rate of oil extraction $C'(t) = -DC(t)$ with $C(0) = \nu$. Then $C(t) = \nu e^{-Dt}$.

Moreover they discuss deterministic discounted models in case of horizon models with a continuous discount rate $r > 0$. Let $\mathcal{T} = \{t_i : i = 1, 2, \cdots\}$

be a sequence of drilling times such that $0 \le t_i < t_{i+1}$and $\mathcal{V} = \{\nu_i : i = 1, 2, \cdots\}$ a sequence of corresponding oil well capacity such that $\le \nu_i \le V$. They get a value of the net profit function

$$J(\mathcal{T}, \mathcal{V}) = \sum_{i=1}^{\infty}[\int_{t_i}^{t_{i+1}} P\nu_i e^{-D(t-t_i)} e^{-r(t-t_i)} dt - f(\nu_i)] e^{-rt_i}.$$

They consider maximizing problems of $J(\mathcal{T}, \mathcal{V})$ and show the following results.

Theorem STU The following statements 1) and 2) hold:

1) There exist optimal sequences $\mathcal{T}^* = \{t_i^* : i = 1, 2, \cdots\}$ and $\mathcal{V}^* = \{\nu_i^* : i = 1, 2, \cdots\}$ such that

$$\max_{\mathcal{T}, \mathcal{V}} J(\mathcal{T}, \mathcal{V}) = J(\mathcal{T}^*, \mathcal{V}^*).$$

2) It follows that $t_1^* = 0$, $t_2^* = t_{i+1}^* - t_i^*$, $\nu_1^* = \nu_i^*$ for $i = 1, 2, \cdots$.

Let $T = t_2^*$, $\nu = \nu_1^*$. Then

$$J = J(\mathcal{T}^*, \mathcal{V}^*) = \int_0^T P\nu e^{-(D+r)t} dt - f(\nu) + Je^{-rT},$$

$$\text{or } J = \frac{\nu}{1 - e^{-rT}} \Big[\frac{P(1 - e^{-(D+r)T})}{D + r} - \frac{f(\nu)}{\nu}\Big].$$

In [8] they assume that there eixits an optimal solution to the problem of maximizing J. They mention that a sufficiently large value of P will guarantee some drilling optimal but that the question of the existence is beyond the scope of the paper.

In the following section we show the maxi-max theorem [4] and we show an extnsion. Moreover we apply the extension theorem to the existence discussion for optimal solutions of J.

5 Maxi-max theorems

In [4] the author get the following theorem concerning the existence of dynamical programming.

Theorem I[IWAMOTO] *Let X be a set and $g : X \times \mathbf{R} \to \mathbf{R}$ such that $g(x, \cdot) : \mathbf{R} \to \mathbf{R}$ is non-decreasing for each $x \in X$. Denote a set-valued function by $\mathcal{Y}(\cdot) : X \to 2^{\mathbf{R}}$, where $2^{\mathbf{R}}$ is the power set of $\mathbf{R}$ and a gragh by $G(\mathcal{Y}) = \{(x, y) \in X \times \mathbf{R} : x \in X, y \in \mathcal{Y}(x)\}$.*

If a function $h : G(\mathcal{Y}) \to \mathbf{R}$ satisfy

$\exists \max_{x \in X} g(x, \max_{y \in \mathcal{Y}(x)} h(x, y)) = c$

then there exists $\max_{x \in X, y \in \mathcal{Y}(x)} g(x, h(x, y))$ *such that*

$c = \max_{x \in X, y \in \mathcal{Y}(x)} g(x, h(x, y))$.

In order to guarantee the existence of optimal sulitions of the maximizing problems J we discuss the following extension of the above theorem.

Theorem 4 *Let X and Y are sets. Denote $g : X \times Y \to \mathbf{R}$ such that $g(x, \cdot) : Y \to \mathbf{R}$ satisfies*

$\exists g(x, \max_{y \in \mathcal{Y}(x)} h(x, y))$ *and that*

$g(x, h(x, y)) \le g(x, \max_{y \in \mathcal{Y}(x)} h(x, y))$ *for $x \in X$. Denote a set-valued function by $\mathcal{Y}(\cdot) : X \to 2^{\mathbf{R}}$. Here $2^{\mathbf{R}}$ is the power set of $\mathbf{R}$ and a gragh by $G(\mathcal{Y}) = \{(x, y) \in X \times Y : x \in X, y \in \mathcal{Y}(x)\}$. Let a function $h : G(\mathcal{Y}) \to \mathbf{R}$ satisfy*

$\exists \max_{x \in X} g(x, \max_{y \in \mathcal{Y}(x)} h(x, y)) = c$. *Then we get*

$\exists \max_{x \in X, y \in \mathcal{Y}(x)} g(x, h(x, y))$ *such that*

$c = \max_{x \in X, y \in \mathcal{Y}(x)} g(x, h(x, y))$.

Let $\mathcal{J}(\mathcal{T}, \mathcal{V}) = g(\mathcal{T}, \mathcal{V}) = \sum_{i=1}^{\infty} h(\mathcal{T}, \mathcal{V})$, and

$$h(\mathcal{T}, \mathcal{V}) = [\frac{P(1 - e^{-(D+r)(t_{i+1}-t_i)})\nu_i}{D + r} - f(\nu_i)] e^{-rt_i}.$$

Denote $X = \{\mathcal{T} = (t_1, t_2 \cdots)\}$, and $Y = \{\mathcal{V} = (\nu_1, \nu_2 \cdots)\}$. From the above we get

$$\max_{\mathcal{V} \in Y} h(\mathcal{T}, \mathcal{V}^*) = [\frac{P(1 - e^{-(D+r)(t_{i+1}-t_i)})\nu_i^*}{D + r} - f(\nu_i^*)] e^{-rt_i}.$$

Here $\mathcal{V}^* = \{\nu_i^*\}, \nu_i^* = \min(V, \bar{\nu_i})$ and $f'(\bar{\nu_i}) = \frac{P(1 - e^{-(D+r)(t_{i+1}-t_i)})}{D+r}$.

Assumption. $t_1^* = 0$ and $\exists T > 0$ such that $\Delta t_i \leq \exists T$.

Denote $\mathcal{T}^* = \{t_i^*\}$ such that $T = t_{i+1}^* - t_i^* = t_2^*$. $\nu_i^* = \min(V, \bar{\nu_i})$ and $f'(\bar{\nu}_i) = \frac{P(1-e^{-(D+r)T})}{D+r}$ so $\nu_i^* = \nu_{i+1}^*$ for $i \geq 1$.

Then, by the extension of the maxi-max theorem, we have

$$\begin{aligned} \max_{\mathcal{T}\in X} g(\mathcal{T}, \max_{\mathcal{V}\in Y} \mathcal{V}) &= g(\mathcal{T}^*, \mathcal{V}^*) \\ &= \max_{\mathcal{T}\in X, \mathcal{V}\in Y} g(\mathcal{T}, \mathcal{V}). \end{aligned}$$

6 Fuzzy functions

Consider a function $x(t) : \mathbf{R} \rightarrow \mathcal{F}_{\mathbf{b}}^{st}$. Then $x(t)$ is said to be a fuzzy function. In [5] we find the following definition of fuzzy functions.

$$\begin{aligned} x(t,\cdot) &= \{(x_1(t,\alpha), x_2(t,\alpha))^T \in \mathbf{R}^2 : \alpha \in I\} \\ &= (x_1(t,\cdot), x_2(t,\cdot)) \end{aligned}$$

for $t \in [t_1, t_2]$. Denote $x(t) = (x_1(t), x_2(t))$.

An L-fuzzy function $x(t) = (x_1(t), x_2(t)) : \mathbf{R} \rightarrow \mathcal{F}_{\mathbf{b}}^{st}$ is H-differentiable at t in the sense of Hukuhara if there exists an $\eta \in \mathcal{F}_{\mathbf{b}}^{st}$ such that (i) and (ii) hold as $h \rightarrow +0$.

(i) $x(t+h) = x(t) + h\eta + o(h)$; (ii) $x(t) = x(t-h) + h\eta + o(h)$.

Here $o(h) = (o_1(h), o_2(h)) \in C[0,\varepsilon] \times C[0,\varepsilon]$ with $\varepsilon > 0$, which means that

$$\lim_{|h|\rightarrow 0} \frac{d(o(h), 0)}{|h|} = 0.$$

Then $x(t) = (x_1(t), x_2(t))$ is H-differentiable at t if and only if $x_1(t,\alpha), x_2(t,\alpha)$ are differentiable in t for each $\alpha \in I$ such that $\eta = (\frac{\partial x_1}{\partial t}, \frac{\partial x_2}{\partial t}) \in \mathcal{F}_{\mathbf{b}}^{st}$.

In [5] the author discuss the integration of fuzzy function $x(t)$.

Definition 2 *An L-fuzzy function* $x(t,\cdot) = (x_1(t,\cdot), x_2(t,\cdot))$ *is called integrable over* $[t_1, t_2]$ *if* $x_1(t,\alpha)$ *and* $x_2(t,\alpha)$ *are integrable over* $[t_1, t_2]$ *for* $\alpha \in I$. *Define*

$$\begin{aligned} &\int_{t_1}^{t_2} x(s,\cdot)ds \\ &= \{(\int_{t_1}^{t_2} x_1(s,\alpha)ds, \int_{t_1}^{t_2} x_2(s,\alpha)ds)^T \in \mathbf{R}^2 : \\ &\qquad \alpha \in I\}. \end{aligned}$$

7 Oil well equations with L-fuzzy functions

In the same way as analyzing oil well equations of $\mathbf{R}-$valued functions we consider the following problem with $0 \leq \lambda \leq 1$. We consider the rate of oil extraction D_L as a constant $L-$fuzzy number $D_L = (D_1, D_2) \in \mathcal{F}_L$ such that $0 \preceq_\lambda D_L$, where $D_1(\alpha)$ is the left end-point of the $\alpha-$cut set and $D_1(\alpha) > 0$ for $\alpha \in I$. Then we assume that the oil quality and unit profit of oil are L-fuzzy function and $L-$fuzzy number, respectively.

- $C_L(t) = (C_1(t), C_2(t)) \in \mathcal{F}_L$: $L-$fuzzy function which means the quality remaining in the well at time t
- $P_L = (P_1, P_2) \in \mathcal{F}_L$: unit profit of oil with $0 \preceq_\lambda P_L$

In this section we consider the following notations, $\nu \in \mathbf{R}$ is a capacity of the well, $f(\nu)$ is renewal cost which is continuously differentiable with $f(0) = 0$ and $f'(\nu) \geq 0$ and V is the upper bound such that $0 \leq \nu \leq V$.

Then we get an initial value problem of $L-$fuzzy differential equation $\frac{dC_L}{dt}(t) + D_L C_L(t) = 0$ with

$C_L(0) = \nu$, where $0 \in \mathbf{R}$. It follows that as long as $C_1(t) \geq 0$

$$C_1'(t) + D_1C_1(t) = 0$$
$$C_2'(t) + D_2C_2(t) = 0$$

with $C_1(0) = C_2(0) = \nu$. Therefore

$$C_1(t,\alpha) = \nu e^{-D_1(\alpha)t},\ C_2(t,\alpha) = \nu e^{-D_2(\alpha)t}$$

for $t \geq 0, \alpha \in I$. We can solve some cases of the above fuzzy differential equations (see [6]). Without using the above information about the $L-$fuzzy function $C_L(t)$ we can find optimal solutions for maximizing problems of values of the net profit function with $L-$fuzzy value concerning $\mathcal{T} = \{t_i \in \mathbf{R} : i = 1,2,\cdots\}$ such that $0 \leq t_i < t_{i+1}$ and $\mathcal{V} = \{\nu_i \in \mathbf{R} : i = 1,2,\cdots\}$ such that $0 \leq \nu \leq V$ as follows.

$$\mathcal{J}(\mathcal{T},\mathcal{V}) = \sum_{i=1}^{\infty}[\int_{t_i}^{t_{i+1}} P_L\nu_iC_L(t-t_i)e^{-r(t-t_i)}dt - f(\nu_i)]e^{-rt_i}.$$

It can be seen that $\mathcal{J}(\mathcal{T},\mathcal{V}) = (J_1, J_2) \in \mathcal{F}_L$, where

$$J_1 = \sum_{i=1}^{\infty}[\int_{t_i}^{t_{i+1}} P_1\nu_ie^{-D_1t}e^{-r(t-t_i)}dt - f(\nu_i)]e^{-rt_i},$$
$$J_2 = \sum_{i=1}^{\infty}[\int_{t_i}^{t_{i+1}} P_2\nu_ie^{-D_2t}e^{-r(t-t_i)}dt - f(\nu_i)]e^{-rt_i}.$$

Here $J_1(\mathcal{T},\mathcal{V},\alpha), J_2(\mathcal{T},\mathcal{V},\alpha)$ are $\mathbf{R}-$valued functions defined on $X \times Y \times I$. Denote

$$X = \{\mathcal{T} = (t_1,t_2,\cdots) : t_i \in \mathbf{R},\ 0 \leq t_i < t_{i+1}\},$$
$$Y = \{\mathcal{V} = (\nu_1,\nu_2,\cdots) : \nu_i \in \mathbf{R},\ 0 \leq \nu \leq V\}.$$

Consider the maximizing problem of $\mathcal{J}(\mathcal{T},\mathcal{V})$. We get the following theorem by applying Theorem 3 and STU.

Theorem 5 *The following statements 1) and 2) hold.*

1) *There exists at least one pair of optimal sequences $\mathcal{T}^* = \{t_i^* : i = 1,2,\cdots\}$ and $\mathcal{V}^* = \{\nu_i : i = 1,2,\cdots\}$ such that*

$$\max_{\mathcal{T},\mathcal{V}\in X\times Y} \mathcal{J}(\mathcal{T},\mathcal{V}) = \mathcal{J}(\mathcal{T}^*,\mathcal{V}^*) \in \mathbf{R}.$$

2) *It follows that $t_1^* = 0,\ t_2^* = t_{i+1}^* - t_i^*, \nu_1^* = \nu_i^* \in \mathbf{R}$ for $i = 1,2,\cdots$. Let $T = t_2^*,\ \nu = \nu_1^*$. Then*

$$\mathcal{J}(\mathcal{T}^*,\mathcal{V}^*) = \frac{\nu}{1-e^{-rT}}\Big[\frac{P(1-e^{-(D+r)T})}{D+r} - \frac{f(\nu)}{\nu}\Big].$$

Here $P, D \in \mathbf{R}$ are respective centers of $L-$fuzzy numbers P_L, D_L.

8 Concluding remarks

In [8] they analyze stochastic infinite horizon models. In the case where a continuous discount rate $r > 0$ they consider the expected infinite horizon profit functions $J(C_i)$ at times t_i such that

$$J(C_i) = J(C_{i+1})e^{-r(t_{i+1}-t_i)} + \int_\nu\int_{t_i}^{t_{i+1}} Pe^{-(D+r)(t-t_i)}dt\nu\phi(\nu)d\nu - Q.$$

Here $C_{i+1} = \nu e^{-D(t_{i+1}-t_i)}$ and the capacity ν of the oil well is a random variable, instead of a decision one. Let $\phi(\nu)$ be the probability density function and $\nu > C_i$. They assume that a fixed cost Q is incurred per drilling.

From now on we treat the capacity ν as a random variable and the rate of oil extraction D as a fuzzy number as well as we we analyze fuzzy functions $J(C_i),\ i = 1,2,\cdots$, with a random variable ν.

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