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# Eigenvalues of elliptic operators on singularly perturbed domains

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**Abstract.** In this note we discuss eigenvalues of elliptic operators in divergence form on domains which have a thin tubular hole. We derive an approximate formula for the eigenvalues as the hole degenerates into a one-dimensional manifold.

## 1 Introduction

This is a joint work with Shuichi Jimbo (Hokkaido University).

Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. We deal with eigenvalue problems of elliptic operators:

$$L\Phi + \mu\Phi = 0 \text{ in } D, \quad \Phi = 0 \text{ on } \partial D \tag{1.1}$$

where $D$ means one of some subdomains of $\Omega$ and $L$ is a uniformly elliptic operator in divergence form:

$$L\Phi = \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial \Phi}{\partial x_j}\right), \quad a_{ij} \in C^2(\Omega), \quad a_{ij} = a_{ji}.$$

Let $Y$ be an embedding of $S^1$ into $\Omega$ and $\Omega(\zeta)$ a subdomain of $\Omega$ with a thin tubular hole defined as

$$\Omega(\zeta) = \Omega \setminus \overline{Y(\zeta)}$$

where $Y(\zeta)$ is a tubular neighborhood:

$$Y(\zeta) = \{x \in \mathbb{R}^3 : \mathrm{dist}(x, Y) < \zeta\} \Subset \Omega \quad \text{for } \zeta \in (0, \zeta^*).$$

The domain $\Omega(\zeta)$ is a singularly perturbed domain of $\Omega$ and $\zeta$ is a perturbation parameter. We consider eigenvalues $\mu_n(\zeta)$ of (1.1) for $D = \Omega(\zeta)$. It is well-known that each eigenvalue $\mu_n(\zeta)$ converges to an eigenvalue of (1.1) for $D = \Omega$ since the codimension of $Y$ is 2 (see Lemma 2.1). The purpose of this paper is to find a precise asymptotic expression of $\mu_n(\zeta)$ at $\zeta = 0$, that is, we will show that $\mu_n(\zeta)$ satisfies

$$\mu_n(\zeta) = \mu_n + \mu_n^{(1)}(\log(1/\zeta))^{-1} + o((\log(1/\zeta))^{-1}) \quad \text{as } \zeta \to 0$$

and $\mu_n^{(1)}$ is an eigenvalue of a certain matrix.

Many authors have studied the continuous dependence of eigenvalues of operators under singular variations of domains. Rauch and Taylor [12] showed that the spectrum of the Laplacian on a bounded domain does not change after imposed Dirichlet B. C. on a compact subset of Newtonian capacity zero. Ozawa [11] studied the asymptotic behavior of eigenvalues of the Laplacian subject to the Dirichlet B. C. on domains with a small hole when the hole degenerates into a point. He gave an asymptotic expression of the eigenvalues. Chavel and Feldman [3] derive an approximate formula of the eigenvalues of the Laplacian on compact Riemannian manifolds with a neighborhood of a closed submanifold removed. The codimension of the submanifold is greater than or equal to 2. If the codimension of the submanifold is 2, the approximate formula is given in the form

$$\mu_n(\zeta) - \mu_n = \frac{2\pi}{\log(1/\zeta)} \int_Y \Phi_n^2\, dV_x + o\left(\frac{1}{\log(1/\zeta)}\right) \quad \text{as } \zeta \to 0$$

where $\{\Phi_n\}_{n=1}^{\infty}$ is an arranged complete system of eigenfunctions on the manifold and $Y$ means the submanifold. See also Flucher [7] and references therein.

To state our main result, we use the following notation. Let $\mu_n$ ($n = 1, 2, \dots$) be the eigenvalues of (1.1) for $D = \Omega$ arranged in increasing order with counting multiplicity and $\{\Phi_n\}_{n=1}^{\infty}$ a complete system of corresponding $L^2$-orthonormalized eigenfunctions which is realvalued. Let $n(k)$ ($k = 1, 2, \dots$) be natural numbers defined by

$$n(1) = 1, \quad n(k+1) = \min\{n \in \mathbb{N} : \mu_n > \mu_{n(k)}\}$$

and $m(k)$ the multiplicity, that is, $m(k) = n(k+1) - n(k)$. Let $\mu_n(\zeta)$ ($n = 1, 2, \dots$) be the eigenvalues of (1.1) for $D = \Omega(\zeta)$ arranged in increasing order with counting multiplicity. Let $N(x)$ be the normal space at $x \in Y$ and $P_x$ the projection from $\mathbb{R}^3$ into $N(x)$ and $I_x$ the inclusion from $N(x)$ into $\mathbb{R}^3$. We define the function $\beta$ on $Y$ as

$$\beta(x) = \sqrt{\det(P_x\, \boldsymbol{A}_x\, I_x)} \qquad (x \in Y)$$

where $\boldsymbol{A}_x$ means a linear mapping on $\mathbb{R}^3$ defined as the coefficient matrix $(a_{ij}(x))_{1 \le i,j \le 3}$ and $P_x\, \boldsymbol{A}_x\, I_x$ is a composite mapping from $N(x)$ into $N(x)$.

**Theorem 1.1.** Assume the above. Then for all $n \in \mathbb{N}$ there exists

$$\mu_n^{(1)} = \lim_{\zeta \to 0} (\mu_n(\zeta) - \mu_n) \log(1/\zeta)$$

and for each $k$ the limits $\mu_n^{(1)}$ $(n(k) \leq n < n(k+1))$ are eigenvalues of an $m(k)$ square symmetric matrix

$$\boldsymbol{M}_k = \left( 2\pi \int_Y \Phi_{n(k)+i-1}(x)\, \Phi_{n(k)+j-1}(x)\, \beta(x)\, dl_x \right)_{1 \leq i,j \leq m(k)}.$$

Here $dl_x$ means the standard line element.

We remark that the eigenvalues of the matrix $\boldsymbol{M}_k$ are independent of choices of systems of eigenfunctions.

# 2 Notation and Preparation

To prove Theorem 1.1, we use the following notation. Let $s$ be an arc length parameter of $Y$ and $l > 0$ the length of $Y$. Let $p \in C^\infty(\mathbb{R}; \mathbb{R}^3)$ such that

$$Y = \{x \in \mathbb{R}^3 : x = p(s),\ 0 \leq s < l\}, \quad p(s+l) = p(s)$$

and $q_i \in C^\infty(\mathbb{R}; \mathbb{R}^3)$ for $i = 1, 2, 3$ such that $q_3(s) = p'(s)$, $q_i(s+l) = q_i(s)$ and that the series $\{q_1(s), q_2(s), q_3(s)\}$ is an orthonormal basis of $\mathbb{R}^3$ with $\det({}^t q_1(s), {}^t q_2(s), {}^t q_3(s)) = 1$. Let $T(r)$ be a cylindrical domain

$$T(r) = \left\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : {x_1}^2 + {x_1}^2 < r^2\right\}$$

and $T(r; s)$ a cylindrical domain with a height $s$

$$T(r; s) = T(r) \cap \{0 \leq x_3 < s\}.$$

We define a mapping $Q : T(r) \to \Omega$ as

$$Q(y) = y_1\, q_1(y_3) + y_2\, q_2(y_3) + p(y_3), \quad y = (y_1, y_2, y_3) \in T(r).$$

We remark $\Omega(\zeta) = \Omega \setminus \overline{Q(T(\zeta))}$ for small $\zeta > 0$.

Let $\widetilde{\boldsymbol{A}}_{y_3}$ be the symmetric matrix

$$\widetilde{\boldsymbol{A}}_{y_3} = \begin{pmatrix} q_1(y_3) \\ q_2(y_3) \\ q_3(y_3) \end{pmatrix} \boldsymbol{A}_{Q(0,0,y_3)} {}^t\!\begin{pmatrix} q_1(y_3) \\ q_2(y_3) \\ q_3(y_3) \end{pmatrix}$$

and $\boldsymbol{B}_{y_3}$ a matrix such that

$$\boldsymbol{B}_{y_3} = \begin{pmatrix} b_{11}(y_3) & b_{12}(y_3) & 0 \\ b_{21}(y_3) & b_{22}(y_3) & 0 \\ b_{31}(y_3) & b_{32}(y_3) & b_{33}(y_3) \end{pmatrix}, \quad \boldsymbol{B}_{y_3}\widetilde{\boldsymbol{A}}_{y_3}{}^{\mathrm{t}}\boldsymbol{B}_{y_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\det(\boldsymbol{B}_{y_3}) > 0, \quad b_{ij} \in C^\infty(\mathbb{R}), \quad b_{ij}(y_3 + l) = b_{ij}(y_3), \quad b_{33}(y_3) > 0.$$

We note that the existence of $\boldsymbol{B}_{y_3}$ is shown by the following simple argument. Since the matrix $\widetilde{\boldsymbol{A}}_{y_3}$ is a positive definite symmetric matrix, there exists an orthogonal matrix $\boldsymbol{P}$ such that

$$\boldsymbol{P}\widetilde{\boldsymbol{A}}_{y_3}{}^{\mathrm{t}}\boldsymbol{P} = \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{pmatrix}.$$

Let

$$\tilde{\boldsymbol{P}} = ({}^{\mathrm{t}}\tilde{p}_1 \; {}^{\mathrm{t}}\tilde{p}_2 \; {}^{\mathrm{t}}\tilde{p}_3) = \begin{pmatrix} 1/\sqrt{\Lambda_1} & 0 & 0 \\ 0 & 1/\sqrt{\Lambda_2} & 0 \\ 0 & 0 & 1/\sqrt{\Lambda_3} \end{pmatrix} \boldsymbol{P}$$

and $\{\hat{p}_1, \hat{p}_2, \hat{p}_3\}$ an orthonormal system such that

$$\hat{p}_3 = |\tilde{p}_3|^{-1}\tilde{p}_3 \quad \text{and} \quad \det({}^{\mathrm{t}}\hat{p}_1 \; {}^{\mathrm{t}}\hat{p}_2 \; {}^{\mathrm{t}}\hat{p}_3) = 1.$$

Then

$$\boldsymbol{B}_{y_3} = \begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \end{pmatrix} \tilde{\boldsymbol{P}}.$$

We define a mapping $z = (z_1, z_2, z_3) = B(y)$ as

$$\begin{aligned} z_1 &= b_{11}(y_3)\, y_1 + b_{12}(y_3)\, y_2, \\ z_2 &= b_{21}(y_3)\, y_1 + b_{22}(y_3)\, y_2, \\ z_3 &= b_{31}(y_3)\, y_1 + b_{32}(y_3)\, y_2 + \int_0^{y_3} b_{33}(s)\, ds. \end{aligned}$$

We note that the inverse mapping $B^{-1} : B(T(r)) \to T(r)$ exists for some $r > 0$ since the Jacobian $\det(\partial z/\partial y) = \det(\boldsymbol{B}_{y_3}) > 0$ on the $y_3$-axis.

Let $G = Q \circ B^{-1} : B(T(\zeta^*)) \to Q(T(\zeta^*))$ and $\Omega_1(\zeta) = \Omega \setminus \overline{G(T(\zeta; l'))}$ for small $\zeta > 0$ where $\zeta^*$ is a positive constant such that $G|_{B(T(\zeta^*; l'))}$ is a one-to-one mapping and

$$l' = \int_0^l b_{33}(s)\, ds.$$

It is obvious that there exist constants $c > 1$ and $\zeta_0 > 0$ such that

$$T(\zeta/c) \subset B(T(\zeta)) \subset T(c\zeta) \subset B(T(\zeta^*)) \quad \text{for } \zeta \in (0, \zeta_0)$$

and hence

$$\Omega(\zeta^*) \subset \Omega_1(c\zeta) \subset \Omega(\zeta) \subset \Omega_1(\zeta/c) \quad \text{for } \zeta \in (0, \zeta_0). \tag{2.1}$$

Let $\omega_n(\zeta)$ $(n = 1, 2, \dots)$ be the eigenvalues of (1.1) for $D = \Omega_1(\zeta)$ arranged in increasing order with counting multiplicity and $\{\Psi_{n,\zeta}\}_{n=1}^{\infty}$ a complete system of corresponding $L^2$-orthonormalized eigenfunctions which is realvalued. Then we have the following.

**Lemma 2.1.** The eigenvalues $\mu_n$, $\mu_n(\zeta)$ and $\omega_n(\zeta)$ satisfy

$$\mu_n \le \omega_n(\zeta/c) \le \mu_n(\zeta) \le \omega_n(c\zeta) \quad \text{for } 0 < \zeta < \zeta_0, \tag{2.2}$$

$$\lim_{\zeta \to 0} \mu_n(\zeta) = \lim_{\zeta \to 0} \omega_n(\zeta) = \mu_n. \tag{2.3}$$

*Proof.* By a standard mini-max principle,

$$\mu_n = \sup_{\substack{K \subset L^2(\Omega) \\ \dim K \le n-1}} \inf_{\substack{\Phi \in H_0^1(\Omega) \\ \Phi \perp K}} \int_\Omega \nabla\Phi\, \boldsymbol{A}_x\, {}^t\nabla\Phi \, dx \Big/ \int_\Omega |\Phi|^2 \, dx \tag{2.4}$$

where $\Phi \perp K$ means

$$\int_\Omega \Phi\, \Psi \, dx = 0 \quad \text{for all } \Psi \in K.$$

The eigenvalues $\mu_n(\zeta)$ and $\omega_n(\zeta)$ are given by (2.4) when $\Omega$ is replaced with $\Omega(\zeta)$ and $\Omega_1(\zeta)$ respectively. Since (2.1), we have (2.2).

Let

$$h_\zeta(x) = \begin{cases} 1 & x \in \Omega \setminus Q(T(\zeta_0)), \\ \dfrac{\log(r/\zeta)}{\log(\zeta_0/\zeta)} & x \in Q(T(\zeta_0)) \setminus \overline{Q(T(\zeta))}, \\ 0 & x \in \overline{Q(T(\zeta))}, \end{cases}$$

where $r = \operatorname{dist}(x, Y) = \sqrt{{y_1}^2 + {y_2}^2}$ for $y = Q^{-1}(x)$. Then we have

$$h_\zeta \Phi_n \in H_0^1(\Omega(\zeta)),$$

$$\int_{T(\zeta_0;l) \setminus T(\zeta;l)} |\nabla_y h_\zeta|^2 dy = \mathcal{O}\left((\log(1/\zeta))^{-1}\right), \tag{2.5}$$

$$\int_{T(\zeta_0;l) \setminus T(\zeta;l)} |1 - h_\zeta|^2 dy = \mathcal{O}\left((\log(1/\zeta))^{-2}\right) \tag{2.6}$$

and hence

$$\int_{\Omega(\zeta)} h_\zeta{}^2\, \Phi_n\, \Phi_m\, dx = \delta_{nm} + \mathcal{O}\left((\log(1/\zeta))^{-1}\right),$$

$$\int_{\Omega(\zeta)} \nabla_x(h_\zeta\Phi_n)\, \boldsymbol{A}_x\, \nabla_x(h_\zeta\Phi_m)\, dx = \mu_n\delta_{nm} + \mathcal{O}\left((\log(1/\zeta))^{-1}\right),$$

where $\delta_{nm}$ is Kronecker's delta symbol. Thus

$$\mu_n(\zeta) \leq \mu_n + \mathcal{O}\left((\log(1/\zeta))^{-1}\right) \quad \text{for each } n \in \mathbb{N}$$

and we obtain (2.3). □

Next, we will observe the behavior of the eigenfunctions $\Psi_{n,\zeta}$ as $\zeta \to 0$. It is well known that each eigenfunction converges to an eigenfunction of (1.1) for $D = \Omega$ by choosing a subsequence.

**Lemma 2.2.** For any positive sequence which converges to zero, there exist a subsequence $\{\zeta_i\}_{i=1}^{\infty}$ and a complete system of orthonormalized eigenfunctions $\{\Psi_n\}_{n=1}^{\infty}$ of (1.1) for $D = \Omega$ such that $\Psi_{n,\zeta_i}$ converges to $\Psi_n$ in $H_0^1(\Omega)$ as $i \to \infty$ where $\Psi_{n,\zeta_i}$ is extended to vanish on $G(T(\zeta_i))$.

*Proof.* By the estimate in the proof of Lemma 2.1, there exists a constant $c_1 = c_1(n) > 0$ such that $\|\Psi_{n,\zeta}\|_{H^1(\Omega)} \leq c_1$ for all $\zeta > 0$. By the Rellich theorem, there exist a sequence $\{\zeta(1,i)\}_{i=1}^{\infty}$ with $\zeta(1,i) \to 0$ as $i \to \infty$ and $\Psi_1 \in H_0^1(\Omega)$ such that

$$\Psi_{1,\zeta(1,i)} \rightharpoonup \Psi_1 \text{ weakly in } H_0^1(\Omega) \text{ as } i \to \infty,$$
$$\Psi_{1,\zeta(1,i)} \to \Psi_1 \text{ strongly in } L^2(\Omega) \text{ as } i \to \infty.$$

Since (2.5), (2.6) and

$$\int_\Omega \nabla\Psi_{1,\zeta}\boldsymbol{A}_x{}^{\mathrm{t}}\nabla(\varphi h_{c\zeta}) - \omega_1(\zeta)\, \Psi_{1,\zeta}\, (\varphi\, h_{c\zeta})\, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega),$$

we have

$$\int_\Omega \nabla\Psi_1\, \boldsymbol{A}_x{}^{\mathrm{t}}\nabla\varphi - \mu_1\, \Psi_1\, \varphi\, dx = 0 \text{ for } \varphi \in C_0^\infty(\Omega), \qquad \int_\Omega |\Psi_1|^2\, dx = 1$$

where $c$ is the constant defined in (2.1). This means $\Psi_1$ is an eigenfunction associated with $\mu_1$. Inductively, by choosing subsequences $\{\zeta(n,i)\}_{i=1}^{\infty} \subset \{\zeta(n-1,i)\}_{i=1}^{\infty}$ for $n \geq 2$, we have

$$\int_\Omega \nabla\Psi_n\, \boldsymbol{A}_x{}^{\mathrm{t}}\nabla\varphi - \mu_n\, \Psi_n\, \varphi(x)\, dx = 0 \text{ for } \varphi \in C_0^\infty(\Omega),$$

$$\int_\Omega \Psi_n \Psi_m \, dx = \delta_{nm} \text{ for } 1 \le m \le n.$$

Let $\zeta_i = \zeta(i,i)$. Then we have for all $n \in \mathbb{N}$

$$\Psi_{n,\zeta_i} \to \Psi_n \text{ strongly in } L^2(\Omega) \text{ as } i \to \infty,$$
$$\Psi_{n,\zeta_i} \rightharpoonup \Psi_n \text{ weakly in } H_0^1(\Omega) \text{ as } i \to \infty.$$

Since $L$ is uniformly elliptic, there exists a constant $c_2 > 0$ such that

$$\begin{aligned} c_2 \int_\Omega |\nabla(\Psi_{n,\zeta} - \Psi_n)|^2 dx &\le \int_\Omega \nabla(\Psi_{n,\zeta} - \Psi_n) \boldsymbol{A}_x {}^{\mathrm{t}}\nabla(\Psi_{n,\zeta} - \Psi_n) \, dx \\ &= \omega_n(\zeta) \int_\Omega |\Psi_{n,\zeta}|^2 dx - 2\mu_n \int_\Omega \Psi_{n,\zeta} \Psi_n \, dx \\ &\quad + \mu_n \int_\Omega |\Psi_n|^2 dx. \end{aligned}$$

Therefore we obtain Lemma 2.2. □

If there exists a limit of $(\omega_n(\zeta) - \mu_n)\log(1/\zeta)$ as $\zeta \to 0$, there exists a limit of $(\mu_n(\zeta) - \mu_n)\log(1/\zeta)$ as $\zeta \to 0$ and those limits are equal by Lemma 2.1. Therefore we consider $\omega_n(\zeta)$ instead of $\mu_n(\zeta)$. To obtain an accurate approximation of $\omega_n(\zeta)$, we construct an approximate function of the eigenfunction $\Psi_{n,\zeta}$. For that purpose, we introduce a solution to a certain elliptic boundary value problem on a tubular neighborhood of $Y$. We define $\alpha(s)$ $(s \in \mathbb{R})$ as

$$\alpha(s) = \sqrt{\det\left(\boldsymbol{A}_{G(0,0,s)}\right)}$$

and for $\Psi \in C^2(\Omega)$, let $U = U(\Psi, \zeta) = U(\Psi, \zeta; z)$ be a unique solution to

$$\begin{cases} \operatorname{div}(\alpha(z_3)\nabla U) = 0 & \text{in } T(\zeta_0) \setminus \overline{T(\zeta)}, \\ U(z_1, z_2, z_3 + l') = U(z_1, z_2, z_3) & \text{in } T(\zeta_0) \setminus \overline{T(\zeta)}, \\ U = 0 & \text{on } \partial T(\zeta_0), \\ U = \Psi \circ G & \text{on } \partial T(\zeta). \end{cases} \tag{2.7}$$

By the separation of variables method, we have

$$U(\Psi, \zeta; z) = \sum_{\eta=-\infty}^{\infty} \sum_{\xi=0}^{\infty} F_\zeta(\Psi; \eta, \xi) \frac{R_{\eta\xi}(r)}{R_{\eta\xi}(\zeta)} \Theta_\eta(\theta) Z_\xi(s) \tag{2.8}$$

where $z = (z_1, z_2, z_3) = (r\cos\theta, r\sin\theta, s)$ and $\Theta_\eta(\theta) = e^{i\eta\theta}/\sqrt{2\pi}$ $(\eta = 0, \pm1, \pm2, \dots)$. $Z_\xi$ is an eigenfunction of

$$\frac{1}{\alpha(s)} \frac{d}{ds}\left(\alpha(s) \frac{dZ_\xi}{ds}\right) + \lambda_\xi Z_\xi = 0 \text{ in } \mathbb{R}, \quad Z_\xi(s + l') = Z_\xi(s) \text{ in } \mathbb{R}$$

$$\int_0^{l'} Z_\xi \, Z_{\xi'} \, \alpha \, ds = \delta_{\xi\,\xi'}$$

and $\{Z_\xi\}$ is a complete system of $L^2((0,l'),\alpha\, ds)$. Here $\lambda_0 = 0$ and we arrange the eigenvalues $\lambda_\xi$ ($\xi = 0,1,2,\dots$) in increasing order counting multiplicity. $R_{\eta\xi}$ is a function defined as

$$R_{\eta\xi}(r) = S_{\eta\xi}(r) \int_r^{\zeta_0} \frac{1}{t\, S_{\eta\xi}(t)^2} dt \quad (0 < r \le \zeta_0) \tag{2.9}$$

for

$$S_{\eta 0}(r) = r^{|\eta|}, \qquad S_{\eta\xi}(r) = \sum_{k=0}^{\infty} \frac{|\eta|!\, r^{|\eta|}}{k!\,(k+|\eta|)!} \left(\frac{\lambda_\xi r^2}{4}\right)^k \quad (\xi \ne 0),$$

which is a solution of

$$\begin{cases} \dfrac{d^2 R_{\eta\xi}}{dr^2} + \dfrac{1}{r}\dfrac{dR_{\eta\xi}}{dr} - \left(\dfrac{\eta^2}{r^2} + \lambda_\xi\right) R_{\eta\xi} = 0 \quad (0 < r < \zeta_0), \\ R_{\eta\xi}(r) > 0 \quad (0 < r < \zeta_0), \qquad R_{\eta\xi}(\zeta_0) = 0. \end{cases} \tag{2.10}$$

$F_r(\Psi;\eta,\xi)$ is the Fourier coefficient

$$F_r(\Psi;\eta,\xi) = \int_0^{l'} \int_{-\pi}^{\pi} \tilde{\Psi}(r,\theta,s)\, \Theta_\eta(-\theta)\, Z_\xi(s)\, \alpha(s)\, d\theta\, ds \tag{2.11}$$

where $\tilde{\Psi}(r,\theta,s) = \Psi \circ G(r\cos\theta, r\sin\theta, s)$.

First we state several well known results without proof.

**Lemma 2.3.** There exists a constant $c_1 > 1$ such that $\xi^2/c_1 \le \lambda_\xi \le c_1\xi^2$.

**Lemma 2.4.** There exists a constant $c_2 > 0$ such that

$$\sup_{s\in\mathbf{R}} |Z_\xi(s)| \le c_2 \quad \text{and} \quad \sup_{s\in\mathbf{R}} |Z'_\xi(s)| \le c_2\sqrt{\lambda_\xi}.$$

By standard arguments of self adjoint operators, a system of eigenfunctions

$$\{\Theta_\eta(\theta) Z_\xi(s) : \eta = 0, \pm1, \pm2, \dots, \xi = 0,1,2,\dots\} \tag{2.12}$$

is a complete orthonormal system of $L^2((-\pi,\pi)\times(0,l'),\ \alpha\, d\theta\, ds)$ and we have the following Parseval equality and Bessel inequalities.

**Lemma 2.5.** Assume the above. Then we have

$$\sum_{\eta,\xi} |F_r(\Psi;\eta,\xi)|^2 = \int_0^{l'} \int_{-\pi}^{\pi} |\tilde{\Psi}(r,\theta,s)|^2 \alpha(s)\, d\theta\, ds, \tag{2.13}$$

$$\sum_{\eta,\xi} \lambda_\xi |F_r(\Psi;\eta,\xi)|^2 \le \int_0^{l'} \int_{-\pi}^{\pi} |\partial_s \tilde{\Psi}(r,\theta,s)|^2 \alpha(s)\, d\theta\, ds, \tag{2.14}$$

$$\sum_{\eta,\xi} \eta^2 |F_r(\Psi;\eta,\xi)|^2 \le \int_0^{l'} \int_{-\pi}^{\pi} |\partial_\theta \tilde{\Psi}(r,\theta,s)|^2 \alpha(s)\, d\theta\, ds, \tag{2.15}$$

$$\sum_{\eta,\xi} \lambda_\xi \eta^2 |F_r(\Psi;\eta,\xi)|^2 \le \int_0^{l'} \int_{-\pi}^{\pi} |\partial_s \partial_\theta \tilde{\Psi}(r,\theta,s)|^2 \alpha(s)\, d\theta\, ds$$

and

$$\sum_{\eta,\xi} \lambda_\xi^2 |F_r(\Psi;\eta,\xi)|^2 \le \int_0^{l'} \int_{-\pi}^{\pi} \left| \frac{\partial_s(\alpha(s)\, \partial_s \tilde{\Psi}(r,\theta,s))}{\alpha(s)} \right|^2 \alpha(s)\, d\theta\, ds.$$

Next we prove several estimates of $R_{\eta\xi}$ needed later.

**Lemma 2.6.** Let $R_{\eta\xi}$ be the solution of (2.10) defined as (2.9). Then

$$\log(\zeta_0/\zeta) \frac{R_{\eta\xi}(r)}{R_{\eta\xi}(\zeta)} \le \log(\zeta_0/r) \quad (0 < \zeta \le r \le \zeta_0), \tag{2.16}$$

$$R_{0\xi}(r) \le \log(\zeta_0/r) \quad (0 < r \le \zeta_0), \tag{2.17}$$

$$\sup_{0<r<\zeta_0} |\log(\zeta_0/r) - R_{0\xi}(r)| \le 2\zeta_0 \sqrt{\lambda_\xi}. \tag{2.18}$$

*Proof.* Let

$$w(r) = \log(\zeta_0/\zeta) \frac{R_{\eta\xi}(r)}{R_{\eta\xi}(\zeta)} - \log(\zeta_0/r).$$

Then $w(\zeta) = w(\zeta_0) = 0$ and

$$w'' + \frac{1}{r} w' = \left( \frac{\eta^2}{r^2} + \lambda_\xi \right) \log(\zeta_0/\zeta) \frac{R_{\eta\xi}(r)}{R_{\eta\xi}(\zeta)} > 0 \quad (\zeta < r < \zeta_0).$$

By the maximum principle, we obtain (2.16).

Since

$$\frac{d}{dt} \left( \frac{\log(\zeta_0/t)}{S_{0\xi}(t)} \right) = -\frac{S_{0\xi}(t) + t \log(\zeta_0/t)\, S'_{0\xi}(t)}{t\, S_{0\xi}(t)^2},$$

we have

$$\log(\zeta_0/r) - R_{0\xi}(r) = S_{0\xi}(r) \int_r^{\zeta_0} \frac{S_{0\xi}(t) - 1 + t\,\log(\zeta_0/t)\,S'_{0\xi}(t)}{t\,S_{0\xi}(t)^2}\,dt \tag{2.19}$$

and hence we obtain (2.17).

It is clear that

$$r^2 S'_{0\xi}(r)^2 = \lambda_\xi \left( r^2 S_{0\xi}(r)^2 - 2\int_0^r t S_{0\xi}(t)^2 dt \right)$$

so that $S'_{0\xi}(r) \le \sqrt{\lambda_\xi}\, S_{0\xi}(r)$ and hence $S_{0\xi}(t) - 1 \le \sqrt{\lambda_\xi}\, t\, S_{0\xi}(t)$. By (2.19), we have

$$\begin{aligned}
|\log(\zeta_0/r) - R_{0\xi}(r)| &\le S_{0\xi}(r) \int_r^{\zeta_0} \frac{\sqrt{\lambda_\xi}\, t\, S_{0\xi}(t)(1 + \log(\zeta_0/t))}{t\,S_{0\xi}(t)^2}\,dt \\
&\le \sqrt{\lambda_\xi} \int_r^{\zeta_0} (1 + \log(\zeta_0/t))\,dt
\end{aligned}$$

and hence we obtain (2.18). □

**Lemma 2.7.** Let $R_{\eta\xi}$ be the solution of (2.10) defined as (2.9). Then

$$\left( \log(\zeta_0/\zeta) \frac{r R'_{\eta\xi}(r)}{R_{\eta\xi}(\zeta)} \right)^2 \le 1 + 2\eta^2 \left(\log(\zeta_0/r)\right)^2 + \lambda_\xi\, {\zeta_0}^2 \tag{2.20}$$

$$(0 < \zeta \le r \le \zeta_0),$$

$$\left( r R'_{0\xi}(r) \right)^2 \le 1 + \lambda_\xi {\zeta_0}^2 \quad (0 < r \le \zeta_0), \tag{2.21}$$

$$\left| r\, R'_{0\xi}(r) + 1 \right| \le \frac{8 + {\zeta_0}^2 \lambda_\xi}{2\,\log(\zeta_0/r)} \quad (0 < r < \zeta_0). \tag{2.22}$$

*Proof.* It is clear $(r^2 R'_{\eta\xi}(r)^2)' = (\eta^2 + \lambda r^2)(R_{\eta\xi}(r)^2)'$ and $(r R'_{\eta\xi}(r))' = (\eta^2/r + \lambda r) R_{\eta\xi}(r)$ so that

$$\begin{aligned}
&\left( r R'_{\eta\xi}(r) \right)^2 + \frac{R_{\eta\xi}(r)}{\log(\zeta_0/r)}\, r R'_{\eta\xi}(r) \\
&= (\eta^2 + \lambda_\xi r^2)\, R_{\eta\xi}(r)^2 + \frac{2\lambda_\xi}{\log(\zeta_0/r)} \int_r^{\zeta_0} t\,\log(\zeta_0/t)\, R_{\eta\xi}(t)^2\,dt \\
&\quad - \frac{2}{\log(\zeta_0/r)} \int_r^{\zeta_0} \left( \frac{\eta^2}{t} + \lambda_\xi t \right) R_{\eta\xi}(t)^2\,dt.
\end{aligned}$$

Since

$$\left|\frac{R_{\eta\xi}(r)}{\log(\zeta_0/r)}\, r R'_{\eta\xi}(r)\right| \le \frac{1}{2}\left(\frac{R_{\eta\xi}(r)}{\log(\zeta_0/r)}\right)^2 + \frac{1}{2}\left(r R'_{\eta\xi}(r)\right)^2,$$

we have

$$\begin{aligned}(r R'_{\eta\xi}(r))^2 \le \left(\frac{R_{\eta\xi}(r)}{\log(\zeta_0/r)}\right)^2 &+ 2(\eta^2 + \lambda_\xi r^2)\, R_{\eta\xi}(r)^2 \\ &+ \frac{4\,\lambda_\xi}{\log(\zeta_0/r)} \int_r^{\zeta_0} t \log(\zeta_0/t)\, R_{\eta\xi}(t)^2\, dt.\end{aligned}$$

By (2.16), we have

$$\begin{aligned}\left(\log(\zeta_0/\zeta)\frac{r R'_{\eta\xi}(r)}{R_{\eta\xi}(\zeta)}\right)^2 \le 1 &+ 2\left(\eta^2 + \lambda_\xi r^2\right)\left(\log(\zeta_0/r)\right)^2 \\ &+ 4\,\lambda_\xi \int_r^{\zeta_0} t\,(\log(\zeta_0/t))^2 dt\end{aligned}$$

and we obtain (2.20). Similarly we obtain (2.21).

It is clear that

$$\tau\, R'_{0\xi}(\tau) - r\, R'_{0\xi}(r) = \lambda_\xi \int_r^\tau t\, R_{0\xi}(t)\, dt$$

so that

$$r\, R'_{0\xi}(r) + 1 = 1 - \frac{R_{0\xi}(r)}{\log(\zeta_0/r)} - \frac{\lambda_\xi}{\log(\zeta_0/r)} \int_r^{\zeta_0} \log(\zeta_0/t)\, t\, R_{0\xi}(t)\, dr.$$

Hence we obtain (2.22) by (2.17) and (2.18). □

It is clear that finite sums of the right hand side of (2.8) satisfy (2.7) except the boundary condition on $\partial T(\zeta)$. Since the above Lemmas and standard convergence theorems of solutions of elliptic partial differential equations, $U(\Psi, \zeta)$ is the unique solution to (2.7).

Next we consider the limit of $\log(\zeta_0/\zeta)\, U(\Psi, \zeta)$ as $\zeta \to 0$.

**Lemma 2.8.** Let

$$V(\Psi) = V(\Psi; z) = \sum_{\xi=0}^{\infty} \frac{F_0(\Psi; 0, \xi)}{\sqrt{2\pi}}\, R_{0\xi}(r)\, Z_\xi(s), \tag{2.23}$$

$$z = (r\cos\theta, r\sin\theta, s) \in T(\zeta_0) \setminus \{z_1 = z_2 = 0\}.$$

Then there exists a constant $c_3 > 0$ such that

$$|V(\Psi; z)| \le c_3 \log(\zeta_0/r) \quad (0 < r \le \zeta_0) \tag{2.24}$$

and

$$\partial_s V(\Psi; z) = \sum_{\xi=0}^{\infty} \frac{F_0(\Psi; 0, \xi)}{\sqrt{2\pi}} R_{0\xi}(r)\, Z'_\xi(s), \tag{2.25}$$

$$\partial_r V(\Psi; z) = \sum_{\xi=0}^{\infty} \frac{F_0(\Psi; 0, \xi)}{\sqrt{2\pi}} R'_{0\xi}(r)\, Z_\xi(s), \tag{2.26}$$

$$\begin{cases} \operatorname{div}(\alpha(z_3)\nabla V(\Psi; z)) = 0 & \text{in } T(\zeta_0) \setminus \{z_1 = z_2 = 0\}, \\ V(\Psi; z_1, z_2, z_3 + l') = V(\Psi; z_1, z_2, z_3) & \text{in } T(\zeta_0) \setminus \{z_1 = z_2 = 0\}, \\ V(\Psi) = 0 & \text{on } \partial T(\zeta_0), \end{cases} \tag{2.27}$$

and

$$V(\Psi),\ r\nabla_z V(\Psi) \in L^2(T(\zeta_0; l'), \alpha(z_3)\, dz).$$

*Proof.* By Lemmas 2.3 to 2.7, we obtain the conclusion. □

**Lemma 2.9.** Let $V(\Psi, \zeta) = V(\Psi, \zeta; z) = \log(\zeta_0/\zeta)\, U(\Psi, \zeta; z)$ which is extended to vanish on $T(\zeta)$. Then

$$V(\Psi, \zeta) \to V(\Psi) \text{ as } \zeta \to 0 \quad \text{in } L^2(T(\zeta_0; l'),\ \alpha(z_3)\, dz), \tag{2.28}$$

$$r\nabla V(\Psi, \zeta) \to r\nabla V(\Psi) \text{ as } \zeta \to 0 \quad \text{in } L^2(T(\zeta_0; l'),\ \alpha(z_3)\, dz). \tag{2.29}$$

For each $0 < r \le \zeta_0$,

$$\partial_r V(\Psi, \zeta; r\cos\theta, r\sin\theta, s) \to \partial_r V(\Psi; r\cos\theta, r\sin\theta, s) \text{ as } \zeta \to 0 \tag{2.30}$$

in $L^2((-\pi, \pi) \times (0, l'),\ \alpha(s)\, d\theta\, ds)$.

*Proof.* Clearly, we have

$$\begin{aligned} V(\Psi, \zeta) - V(\Psi) &= \sum_{\xi=0}^{\infty} I(\zeta, \xi) \log(\zeta_0/\zeta) \frac{R_{0,\xi}(r)}{R_{0,\xi}(\zeta)} \Theta_0(\theta)\, Z_\xi(s) \\ &\quad + \sum_{\substack{\eta=-\infty \\ \eta \ne 0}}^{\infty} \sum_{\xi=0}^{\infty} F_\zeta(\Psi; \eta, \xi) \log(\zeta_0/\zeta) \frac{R_{\eta,\xi}(r)}{R_{\eta,\xi}(\zeta)} \Theta_\eta(\theta)\, Z_\xi(s) \end{aligned}$$

where $I(\zeta,\xi) = I_1(\zeta,\xi) + I_2(\zeta,\xi)$ and

$$I_1(\zeta,\xi) = F_\zeta(\Psi;0,\xi) - F_0(\Psi;0,\xi),$$

$$I_2(\zeta,\xi) = F_0(\Psi;0,\xi)\frac{\log(\zeta_0/\zeta) - R_{0,\xi}(\zeta)}{\log(\zeta_0/\zeta)},$$

so that

$$\begin{aligned}
&\|V(\Psi,\zeta) - V(\Psi)\|^2_{L^2(T(\zeta_0;l')\setminus T(\zeta;l'),\, \alpha(z_3)\, dz)} \\
&\le \left(\sum_\xi |I(\zeta,\xi)|^2 + \sum_{\substack{\eta,\xi \\ \eta\neq 0}} |F_\zeta(\Psi;\eta,\xi)|^2\right)\frac{\zeta_0^2}{4}.
\end{aligned}$$

By the orthogonality of the eigenfunctions, we have

$$\sum_\xi |I_1(\zeta,\xi)|^2 \le \int_0^{l'}\int_{-\pi}^{\pi} |\tilde\Psi(\zeta,\theta,s) - \tilde\Psi(0,\theta,s)|^2\,\alpha(s)\,d\theta\,ds,$$

by (2.18) and (2.14), we have

$$\sum_\xi |I_2(\zeta,\xi)|^2 \le \frac{4\zeta_0^2}{\log(\zeta_0/\zeta)}\int_0^{l'}\int_{-\pi}^{\pi} |\partial_s\tilde\Psi(0,\theta,s)|^2\,\alpha(s)\,d\theta\,ds,$$

and by (2.15), we have

$$\begin{aligned}
\sum_{\substack{\eta,\xi \\ \eta\neq 0}} |F_\zeta(\Psi;\eta,\xi)|^2 &\le \sum_{\substack{\eta,\xi \\ \eta\neq 0}} |F_\zeta(\Psi;\eta,\xi)|^2\eta^2 \\
&\le \zeta^2\int_0^{l'}\int_{-\pi}^{\pi} |\partial_{z_1}\tilde\Psi|^2 + |\partial_{z_2}\tilde\Psi|^2\,\alpha(s)\,d\theta\,ds.
\end{aligned}$$

Therefore we obtain (2.28). Similarly we obtain (2.29) and (2.30) by Lemmas 2.5 and 2.6. □

# 3 Proof of Theorem 1.1

Now we prove Theorem 1.1 by using the above lemmas. Let

$$\psi_{m,i}(x) = \begin{cases} 0 & x \in G\left(T(\zeta_i;l')\right), \\ \Psi_m(x) - U(\Psi_m,\zeta_i)\circ G^{-1}(x) & x \in G\left(\overline{T(\zeta_0;l')}\setminus T(\zeta_i;l')\right), \\ \Psi_m(x) & x \in \Omega\setminus G\left(\overline{T(\zeta_0;l')}\right) \end{cases}$$

where $\Psi_m$ is the eigenfunction of (1.1) for $D = \Omega$ and $\{\zeta_i\}_{i=1}^{\infty}$ are the su
quence given in Lemma 2.2. $U(\Psi_m, \zeta)$ is the solution of (2.7) for $\Psi =$
Clearly we have

$$\int_{\Omega_1(\zeta_i)} \nabla\Psi_{n,\zeta_i} \boldsymbol{A}_x {}^{\mathrm{t}}\nabla\psi_{m,i} - \omega_n(\zeta_i)\, \Psi_{n,\zeta_i}\, \psi_{m,i}\, dx = 0.$$

By a simple calculation, we have

$$\int_{\Omega_1(\zeta_i)} \nabla\Psi_{n,\zeta_i} \boldsymbol{A}_x {}^{\mathrm{t}}\nabla\Psi_m\, dx = \mu_m \int_{\Omega_1(\zeta_i)} \Psi_{n,\zeta_i}\, \Psi_m\, dx$$

and

$$\begin{aligned}
&\int_{G\left(T(\zeta_0;l')\setminus\overline{T(\zeta_i;l')}\right)} \nabla\Psi_{n,\zeta_i} \boldsymbol{A}_x {}^{\mathrm{t}}\nabla U(\Psi_m, \zeta_i)\, dx \\
&= \int_{T(\zeta_0;l')\setminus\overline{T(\zeta_i;l')}} \nabla_z\Psi_{n,\zeta_i} \boldsymbol{J}(z) {}^{\mathrm{t}}\nabla_z U(\Psi_m, \zeta_i)\, dz \\
&= \int_{\partial T(\zeta_0)\cap\{0<z_3<l'\}} \Psi_{n,\zeta_i}\, \partial_r U(\Psi_m, \zeta_i)\, \alpha(z_3)\, dS_z.
\end{aligned}$$

where

$$\boldsymbol{J}(z) = \left|\det \frac{\partial z}{\partial x}\right|^{-1} \frac{\partial z}{\partial x} \boldsymbol{A}_{G(z)} {}^{\mathrm{t}}\frac{\partial z}{\partial x}.$$

Let $\boldsymbol{J}'(s) = \boldsymbol{J}(0, 0, s)$. By the definition of $G$, we have

$$\boldsymbol{J}'(s) = \alpha(s) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\tilde{U}(\Psi_m, \zeta_i) = \log(\zeta_0/\zeta_i) U(\Psi_n, \zeta_i)$. By the above, we have

$$\log(\zeta_0/\zeta_i)\, (\omega_n(\zeta_i) - \mu_m) \int_{\Omega_1(\zeta_i)} \Psi_{n,\zeta_i}\, \Psi_m\, dx = I_1(\zeta_i) + I_2(\zeta_i) + I_3(\zeta_i)$$

where

$$\begin{aligned}
I_1(\zeta_i) &= \int_{T(\zeta_0;l')\setminus\overline{T(\zeta_i;l')}} \nabla_z\Psi_{n,\zeta_i}\, (\boldsymbol{J}'(z_3) - \boldsymbol{J}(z)) {}^{\mathrm{t}}\nabla_z\tilde{U}(\Psi_m, \zeta_i)\, dz, \\
I_2(\zeta_i) &= -\int_{\partial T(\zeta_0)\cap\{0<z_3<l'\}} \Psi_{n,\zeta_i}\, \partial_r\tilde{U}(\Psi_m, \zeta_i)\, \alpha(z_3)\, dS_z, \\
I_3(\zeta_i) &= \omega_n(\zeta_i) \int_{T(\zeta_0;l')\setminus\overline{T(\zeta_i;l')}} \Psi_{n,\zeta_i}\, \tilde{U}(\Psi_m, \zeta_i) \left|\det \frac{\partial z}{\partial x}\right|^{-1} dz.
\end{aligned}$$

By Lemma 2.2 and (2.29) and that

$$\sup\{r^{-1}|\boldsymbol{J}'(z_3)-\boldsymbol{J}(z)| : z\in T(\zeta_0)\setminus\{z_1=z_2=0\}\}<\infty,$$

we have

$$\lim_{i\to\infty} I_1(\zeta_i)=\int_{T(\zeta_0;l')\setminus\{z_1=z_2=0\}} \nabla_z\Psi_n\,(\boldsymbol{J}'(z_3)-\boldsymbol{J}(z))\,{}^{\mathrm{t}}\nabla_z V(\Psi_m)\,dz.$$

By Lemma 2.2 and the trace theorem and (2.30), we have

$$\lim_{i\to\infty} I_2(\zeta_i)=-\int_{\partial T(\zeta_0)\cap\{0<z_3<l'\}} \Psi_n\,\partial_r V(\Psi_m)\,\alpha(z_3)\,dS_z.$$

By (2.3) and Lemma 2.2 and (2.28), we have

$$\lim_{i\to\infty} I_3(\zeta_i)=\mu_n\int_{T(\zeta_0;l')\setminus\{z_1=z_2=0\}} \Psi_n\,V(\Psi_m)\left|\det\frac{\partial z}{\partial x}\right|^{-1}dz.$$

Combining the above, there exists a limit of $(\omega_n(\zeta_i)-\mu_n)\log(\zeta_0/\zeta_i)$ as $i\to\infty$. Let $\mu_n^{(1)}$ denote the limit. For $m,n$ with $\mu_n=\mu_m$, we have

$$\begin{aligned}\mu_n^{(1)}\delta_{nm}=&\int_{T(\zeta_0;l')\setminus\{z_1=z_2=0\}} \nabla_z\Psi_n\,(\boldsymbol{J}'(z_3)-\boldsymbol{J}(z))\,{}^{\mathrm{t}}\nabla_z V(\Psi_m)\,dz\\ &-\int_{\partial T(\zeta_0)\cap\{0<z_3<l'\}} \Psi_n\,\partial_r V(\Psi_m)\,\alpha(z_3)\,dS_z\\ &+\mu_n\int_{T(\zeta_0;l')\setminus\{z_1=z_2=0\}} \Psi_n\,V(\Psi_m)\left|\det\frac{\partial z}{\partial x}\right|^{-1}dz.\end{aligned}$$

Let $\delta\in(0,\zeta_0)$. By the divergence theorem and (2.27), we have

$$\begin{aligned}&\int_{T(\zeta_0;l')\setminus T(\delta;l')} \nabla_z\Psi_n\boldsymbol{J}'(z_3)\,{}^{\mathrm{t}}\nabla_z V(\Psi_m)\,dz,\\ &=-\int_{\partial T(\delta)\cap\{0<z_3<l'\}} \Psi_n\partial_r V(\Psi_m)\,\alpha(z_3)\,dS_z\\ &\quad+\int_{\partial T(\zeta_0)\cap\{0<z_3<l'\}} \Psi_n\,\partial_r V(\Psi_m)\,\alpha(z_3)\,dS_z.\end{aligned}$$

By (1.1), we have

$$\int_{T(\zeta_0;l')\setminus T(\delta;l')} \nabla_z\Psi_n\boldsymbol{J}(z)\,{}^{\mathrm{t}}\nabla_z V(\Psi_m)\,dz$$

$$= \int_{\partial T(\delta)\cap\{0<z_3<l'\}} \nabla_z \Psi_n \boldsymbol{J}(z)\, {}^{\mathrm{t}}\nu(z)\, V(\Phi_m)\, dS_z$$
$$+ \mu_n \int_{T(\zeta_0;l')\setminus T(\delta;l')} \Psi_n\, V(\Phi_m) \left|\det \frac{\partial z}{\partial x}\right|^{-1} dz.$$

By (2.24),

$$\int_{\partial T(\delta)\cap\{0<z_3<l'\}} \nabla_z \Psi_n \boldsymbol{J}(z)\, {}^{\mathrm{t}}\nu(z)\, V(\Phi_m)\, dS_z = \mathcal{O}(\delta \log(1/\delta)) \text{ as } \delta \to 0.$$

By (2.11), (2.22), (2.23) and (2.14), we have

$$\begin{aligned}
\mu_n^{(1)} \delta_{nm} &= -\lim_{\delta\to 0} \int_{\partial T(\delta)\cap\{0<z_3<l'\}} \Psi_n \partial_r V(\Psi_m)\, \alpha(z_3)\, dS_z \\
&= -\lim_{\delta\to 0} \sum_{\xi=0}^{\infty} \delta\, R'_{0\xi}(\delta)\, F_0(\Psi_m; 0, \xi)\, F_\delta(\Psi_n; 0, \xi) \\
&= \sum_{\xi=0}^{\infty} F_0(\Psi_m; 0, \xi)\, F_0(\Psi_n; 0, \xi).
\end{aligned}$$

By the completeness of the system (2.12), we have

$$\begin{aligned}
\mu_n^{(1)}\, \delta_{nm} &= \int_0^{l'} \int_{-\pi}^{\pi} \tilde{\Psi}_m(0,\theta,s)\, \tilde{\Psi}_n(0,\theta,s)\, \alpha(s)\, d\theta\, ds \\
&= 2\pi \int_0^{l'} \Psi_n \circ G(0,0,z_3)\, \Psi_m \circ G(0,0,z_3)\, \alpha(z_3)\, dz_3.
\end{aligned}$$

Since ${}^{\mathrm{t}}\boldsymbol{B}_{y_3}\, \boldsymbol{B}_{y_3} = \widetilde{\boldsymbol{A}}_{y_3}^{-1}$, we have

$$\begin{aligned}
(b_{33}(y_3))^2 &= \frac{1}{\det(\widetilde{\boldsymbol{A}}_{y_3})} \det \begin{pmatrix} q_1(y_3)\boldsymbol{A}_x {}^{\mathrm{t}}q_1(y_3) & q_1(y_3)\boldsymbol{A}_x {}^{\mathrm{t}}q_2(y_3) \\ q_2(y_3)\boldsymbol{A}_x {}^{\mathrm{t}}q_1(y_3) & q_2(y_3)\boldsymbol{A}_x {}^{\mathrm{t}}q_2(y_3) \end{pmatrix} \\
&= \frac{\det(P_x \boldsymbol{A}_x I_x)}{\det(\boldsymbol{A}_x)} \quad \text{for } x = Q(0,0,y_3).
\end{aligned}$$

Since

$$z_3 = \int_0^{y_3} b_{33}(s)\, ds \quad \text{on the } z_3\text{-axis},$$

we have

$$\int_0^{l'} \Psi_n \circ G(0,0,z_3)\, \Psi_m \circ G(0,0,z_3)\, \alpha(z_3)\, dz_3 = \int_Y \Psi_n\, \Psi_m\, \beta\, dl_x.$$

Accordingly, we have

$$\mu_n^{(1)}\delta_{nm} = 2\pi \int_Y \Psi_n \Psi_m \beta\, dl_x \quad \text{for } n, m \text{ with } \mu_n = \mu_m.$$

Since

$$\Psi_n = \sum_{j=n(k)}^{n(k+1)-1} (\Psi_n, \Phi_j)_{L^2(\Omega)} \Phi_j \quad \text{for } n = n(k), \ldots, n(k+1)-1,$$

we have

$$\begin{pmatrix} \mu_{n(k)}^{(1)} & & \boldsymbol{O} \\ & \ddots & \\ \boldsymbol{O} & & \mu_{n(k+1)-1}^{(1)} \end{pmatrix} = \boldsymbol{P} \boldsymbol{M}_k {}^{\mathrm{t}}\boldsymbol{P}$$

where

$$\boldsymbol{P} = \left( (\Psi_{n(k)+i-1}, \Phi_{n(k)+j-1})_{L^2(\Omega)} \right)_{1 \le i,j \le m(k)}.$$

These eigenvalues are independent of choices of sequences $\{\zeta_i\}$, we obtain Theorem 1.1. □

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