Analysis of the singular sets of the Landau-Lifshitz system *

Liu Xiangao

Institute of Mathematics, Fudan University, Shanghai 200433, China
email: xgliuk@online.sh.cn

1   Introduction

- **Physical model**

In 1935, L.D. Landau and E.M. Lifshitz derived the following equation, the so called Landau-Lifshitz system, which describes evolution of spin fields in continuum ferrimagnetism (see [LL]).

$$\partial_t u = -\alpha_1 u \times (u \times F_{eff}) + \alpha_2 u \times F_{eff},$$

where $u = (u^1, u^2, u^3) : R^m \times R_+ \to S^2 \subset R^3$ is the spin field; "$\times$" denotes the vector cross product in R^3; $\alpha_1 > 0$ is a Gilbert damping constant, α_2 is a exchange constant, and F_{eff} is the effective field containing contributions from exchange interaction crystalline anisotropy, magneto-static self energy, external magnetic field, etc (see [LN]).

In particular, taking $F_{eff} = \Delta u$, corresponding to the pure isotropic case and without external magnetic fields, Landau-Lifshitz equation reads

$$\partial_t u = -\alpha_1 u \times (u \times \Delta u) + \alpha_2 u \times \Delta u.$$

When $\alpha_1 = 0$, the system is called the Heisenberg system.

- **The equivalent equation**

Using the following formula $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$, and the fact that $|u| = 1$ implies $u \Delta u = -|\nabla u|^2$, we have the following equivalent equation

$$\begin{cases} \lambda_1 \partial_t u - \lambda_2 u \times \partial_t u = \Delta u + |\nabla u|^2 u & (x,t) \in R^m \times R_+ \\ u(x,0) = u_0(x), & x \in R^m. \end{cases} \tag{1.1}$$

where $\lambda_1 = \frac{\alpha_1}{\alpha_1^2 + \alpha_2^2}$, $\lambda_2 = \frac{\alpha_2}{\alpha_1^2 + \alpha_2^2}$, and $|u_0(x)| = 1$,

• **Well known results**

- **1-Dimension** A lot work contributed to the study of solutions of L-L system has been made by some physicists and mathematicians such as H.C.Fogedby [Fo], M.Laksmanan, K. Nakamura [LN], K.Nakamura, T.Sasada [NS], L.A.Takhtalian [T], J.Tjon, J.Wright [TW], Y.Zhou, B.Guo S.Tan [ZGT] and so on.

- **2-Dimension:** Singular points in the finite time are finite, because of, well-known, the conformal in-variation of the energy in the 2-dimension.

- **High-Dimension** The global existence of weak solutions for the equation has been established by F.Alouges A.Soyeur [AS](1992) and B.Guo, M.Hong [GA](1993).

• **The problems concerned and their difficulties**

- **What is the regularity to (1.1)?** This is a basic problem to any nonlinear equations considered in the space of the generalized functions such as Sobolev space. Furthermore we hope to know the behavior of the solution at a singular point, that is, one ask what happen at a singular point?

- **The characteristics of the equation (1.1)**

  * The landau-Lifshitz equation is a parabolic type equation with the natural increasing term $|\nabla u|^2$.

  * In appearance, (1.1) is similar to the heat flow of harmonic map into sphere (if $\alpha_2 = 0$), however there is an anti-symmetric term $u \times \partial_t u$. In other words, the Landau-Lifshitz system is a more general system, which contains in particularly the equations of harmonic map into sphere and its heat flow.

- **The difficulties in getting regularity** The natural increasing term and the anti-symmetric term are both difficult to regularity (no existence). The classical methods can't be used for the first one, and the second one breaks down the monotone property which is a main tool to deal with the harmonic maps and its heat flows as known. We can define the **stationary condition** in an analogous way as in harmonic maps by R.Schoen, the so called the variation of domain.

  In the case of harmonic map and its heat flow, we have the following fact:

  The stationary condition $\implies$ monotonicity.

  But In our case, the stationary condition does not imply the monotonicity , we will see it in the following. Maybe this is the crucial reason as which up to now one could not to get the regularity of Landau-Lifshitz system.

  It is well known that the weak harmonic map, without monotone property, may be almost discontinuous in three dimension (see T. Riviera).

- **Our main results**

  - **Part one:** We prove the stationary weak solution is smooth except $\mathcal{H}^m_\rho$-zero set, i.e.,

    $$\mathcal{H}^m_\rho(\text{Sing}(u)) = 0.$$

  The following results largely depend on the regularity got in the first part.

  - **Part two:** Let u_k be a sequence of the stationary weak solutions of (1.1) with the initial data u_{k0} and $\int_{R^m} |\nabla u_{k0}|^2 \leq \Lambda$. For fixed t let Σ^t be the **blow up set** of the sequence, then we have

    1. Σ^t is rectifiable, i.e., almost C^1 smooth.
    2. Σ^t moves by quasi-mean curvature if u_k are the strong stationary weak solutions of (1.1) and $\lambda_1 \geq 2|\lambda_2|$.

  - **Part three:** Let (x_0, t_0) be a singular point of u, by scaling we obtain the two blow up formulas.

2 Stationary weak solutions

In this section, we introduce the notions of the stationary weak solutions of Landau-Lifshitz system, and show some generalized monotonicity inequalities.

DEFINITION 2.1 $u(x,t) \in W^{1,2}(R^m \times R_+, S^2)$ *is called a* **stationary** *weak solution of (1.1), if it is a weak solution of (1.1) and satisfies the following two assumptions:*

$$\int_{R^m} 2(\lambda_1 u_t - \lambda_2 u \times u_t)\zeta \cdot \nabla u - |\nabla u|^2 div\zeta + 2\partial_j u \partial_k u \partial_j \zeta^k = 0, \qquad (2.1)$$

$$(\int_{R^m \times t_2} - \int_{R^m \times t_1})|\nabla u|^2 \vartheta dx \qquad (2.2)$$

$$\leq -\int_{t_1}^{t_2} \int_{R^m} [2(\lambda_1 u_t^2 \vartheta + \nabla u \nabla \vartheta u_t) - |\nabla u|^2 \vartheta_t] dx dt,$$

where $t_2 > t_1 > 0$, the functions ζ, ϑ are smooth, and $\vartheta \geq 0$

u is called a **strong stationary** *weak solution if the equality in (2.2) holds.*

REMARK 2.1 *If a weak solution u of (1.1) satisfies the stability hypothesis defined by the requirement that, similar to heat flow in [Fe],*

$$\int_0^\infty \int_{R^m} (\lambda_1 u_t - \lambda_2 u \times u_t)\partial_\tau \hat{u}^\tau|_{\tau=0}$$

$$+\partial_\tau^+ \int_0^\infty \int_{R^m} |\nabla \hat{u}^\tau|^2 dx dt|_{\tau=0} \leq 0 \qquad (2.3)$$

holds for each family $\hat{u}^\tau$ of the domain variation defined by $\hat{u}^\tau = u(F_\tau(x,t))$, where $F_\tau = (x + \tau\tilde{\zeta}, t + \tau\tilde{\vartheta})$, $\tilde{\zeta}, \tilde{\vartheta}$ are smooth functions and $\tilde{\vartheta} \geq 0$. Then the assumptions (2.1) and (2.2) hold (see Proposition 7 and 8 in [Fe]).

Certainly smooth solution is strong stationary.

Notations:

- **Point Sets and Fundamental Solution**

$$z = (x,t) \in R^m \times R, \quad z_0 = (x_0, t_0);$$

$$B_r(x_0) = \{x \in R^m : \ |x_0 - x| < r\}, \ S_r(t_0) = \{(x,t) : \ t = t_0 - r^2\},$$

$$P_r(z_0) = \{(x,t) \in R^m \times R : |x - x_0| < r, |t - t_0| < r^2\},$$

$$T_r(z_0) = \{(x,t) \in R^m \times R : t_0 - 4r^2 < t < t_0 - r^2\}.$$

$$G_{z_0} = \frac{1}{[4\pi(t_0 - t)]^{m/2}} \exp(-\frac{|x - x_0|^2 \lambda_1}{4(t_0 - t)}), \quad t < t_0,$$

is the standard fundamental solution of the backward heat equation $\partial_t G - \lambda_1^{-1} \Delta G = 0$;

- **Functionals**

$$\Phi_{z_0}(R, u) = R^2 \int_{S_R(t_0)} |\nabla u|^2 G_{z_0} dx; \quad \Psi_{z_0}(R, u) = \int_{T_R(t_0)} |\nabla u|^2 G_{z_0} dxdt;$$

$$\Theta_{z_0}(R_1, R_2, u) = \frac{\lambda_2^2}{\lambda_1} \int_{t_0 - R_2^2}^{t_0 - R_1^2} \int_{R^m} u_t^2 (t_0 - t) G_{z_0} dxdt;$$

$$E(r, u, z) = r^{2-m} \int_{B_r(z)} |\nabla u|^2 dy; \quad I(r, u, z) = r^{2-m} \int_{B_r(x)} |\partial_t u|^2 dy;$$

$$E_p(r, u, z) = r^{-m} \int_{P_r(z)} |\nabla u|^2 dydt; \quad \mathcal{E}(r, u, z) = E_p(r, u, z) + r^\beta.$$

The stationary condition

⇓

- **Energy inequality**

$$\int_0^T \int_{R^m} 2\lambda_1 u_t^2 dxdt + \int_{R^m} |\nabla u|^2 (x, T) dx$$

$$\leq \int_{R^m} |\nabla u_0|^2 dx = E_0. \tag{2.4}$$

- **Generalized monotone inequality I**

$$\int_{t_0 - R_2^2}^{t_0 - R_1^2} \int_{R^m} \lambda_1 [\frac{r u_r}{\sqrt{2(t_0 - t)}}$$

$$-\sqrt{2(t_0 - t)}(u_t - \frac{\lambda_2}{2\lambda_1} u \times u_t)]^2 G_{z_0} dxdt$$

$$\leq \ \Phi_{z_0}(R_2, u) - \Phi_{z_0}(R_1, u) + \Theta_{z_0}(R_1, R_2, u). \tag{2.5}$$

The equality holds if and only if u is strong stationary weak solution, i.e. the equality in (2.2) (or (2.3))holds.

- **Generalized monotone inequality II**

Suppose that $I(r, u, z) \leq \frac{K}{\lambda_1^2 + \lambda_2^2} r^{-\alpha}$ for $0 < r \leq r_0$, where $K > 0, 0 < \alpha < 1/2$. Then $e^r (E(r, u, z) + \frac{K}{1-\alpha} r^{1-\alpha})$ is non-deceasing with respect to r, precisely

$$\frac{d(e^r (E(r, u, z) + \frac{K}{1-\alpha} r^{1-\alpha}))}{dr}$$

$$\geq 2 e^r r^{2-m} \int_{\partial B_r} |\partial_r u|^2 d\sigma + \frac{K e^r}{1-\alpha} r^{1-\alpha}. \tag{2.6}$$

REMARK 2.2 *In the heat flow case,* $\Theta_{z_0}(R_1, R_2, u) = 0$ *since* $\alpha_2 = 0$. *So* $\Phi_{z_0}(R, u)$ *is increasing with respect to R.*

Our idea is to hope that $\Theta_{z_0}(R, u)$ **is small whenever** R **is small**, which enables us to define the following set Σ_β. For any fixed $1/2 > \alpha, \beta > 0$, and fixed constant c_0, we define

$$\Sigma_\beta = \left\{ z \in R^m \times R_+ : \frac{\limsup_{R \to 0} R^{-(m-2+\beta)}}{\int_{t-R^2}^{t+R^2} \int_{B_{R\sqrt{|\log R|}}(x)} u_t^2(x, t) dx dt} = \infty \right\}$$

$$S_\alpha(u, t) = \left\{ x \in R^m : \limsup_{r \to 0} r^\alpha I(r, z, u) \geq c_0 > 0 \right\}.$$

The following lemma is true (see [Liu] [LT]).

LEMMA 2.1

$$\mathcal{H}_\varrho^{m-2+\beta+\epsilon}(\Sigma_\beta) = 0,$$

where $\mathcal{H}_\varrho$ *denotes the parabolic metric Hausdorff measure, and* $\epsilon > 0$ *is any positive constant.*

If $\int_{R^m} |\partial_t u|^2 < \infty$, *then for any* $0 < \alpha < 1/2$, *we have*

$$\mathcal{H}^{m-2-\alpha}(S_\alpha(u, t)) < \infty.$$

The following lemma enables our idea to become possible, which is the key part of getting regularity.

LEMMA 2.2 *Let* $z_0 \notin \Sigma_\beta$, *i.e.* *there exist constants* $C_0(z_0)$ *and* $R_0(z_0) > 0$, *such that whenever* $R_0 \geq R > 0$,

$$\frac{1}{R^{m-2+\beta}} \int_{t_0-R^2}^{t_0+R^2} \int_{B_{\sqrt{|\log R|}R}(x_0)} u_t^2(x, t) dx \leq C_0.$$

Then for any $0 < R < R_1 \leq R_0$, *there exists a constant* C_1, *depending only on* C_0, E_0, m, β *and* $\alpha_i, i = 1, 2$, *such that*

$$\Theta_{z_0}(R, R_1, u) \leq C_1 R_1^\beta.$$

REMARK 2.3 *If $z_0 \notin \Sigma_\beta$, then for large $\sqrt{t_0} \geq R_1 \geq R_0$ the same inequality holds from energy estimate (2.4).*

From the lemma above we have the following estimate for the normalized energy

PROPOSITION 2.1 *Let $z_0 \notin \Sigma_\beta$, corresponding C_0, $R_0 > 0$, and let $R_2 < 1/3$, and $\mathcal{E}(R_2, u, z_0) \leq \epsilon_1$, where ϵ_1 small enough (e.g. $\epsilon_1 \leq \exp(-(80R_0^2)^{-1}\lambda_1)$), then there exists a constant C_3, depending on C_0, E_0, m, and β and $\alpha_i, i = 1, 2$, such that for any $0 < R < R_2$, the following inequality*

$$\mathcal{E}(R, u, z_0) \leq C_3 |\log \mathcal{E}(R_2, u, z_0)|^{m/2} \mathcal{E}(R_2, u, z_0).$$

holds.

REMARK 2.4 *Let $z_0 \notin \Sigma_\beta$. Then*

$$\mathcal{E}(R, u, z) \leq C_2(R_0)$$

3 Partial regularity of the stationary solutions

We have

THEOREM 3.1 *Let $u \in W^{1,2}(R^m \times R_+, S^2)$ be a global stationary weak solution of the Landau-Lifshitz system (1.1) with $E(u_0) < \infty$, where $E(u_0) = \int_{R^m} |\nabla u_0|^2 dV$. Then singular set of u, $Sing(u)$, is a closed set with $\mathcal{H}_\rho^m(Sing(u)) = 0$. More precisely, $Sing(u) \subset \Sigma_\beta \cup \{z : \limsup_{r \to 0} \mathcal{E}(r, u, z) \geq \epsilon_0\}$ and $\mathcal{H}_\rho^{m-2+\beta+\alpha}(\Sigma_\beta) = 0, \mathcal{H}_\rho^m(\{z : \limsup_{r \to 0} \mathcal{E}(r, u, z) \geq \epsilon_0\}) = 0$ for any $1/2 > \beta, \alpha > 0$, where $\mathcal{H}_\rho$ denotes the parabolic metric Hausdorff measure.*

As the usual blow-up argument, the key part to the proof of Theorem 3.1 is the following small energy decay lemma.

LEMMA 3.1 *There exist constants $0 < \epsilon_0, \tau < 1$ such that if $\mathcal{E}(r, u, z) \leq \epsilon_0 \leq \epsilon_1 \leq 1/2$, then*

$$\mathcal{E}(\tau r, u, z) \leq \frac{1}{2}\mathcal{E}(r, u, z) \tag{3.1}$$

for any $z \in R^m \times R_+$ and $z \notin \Sigma_\beta$, and $0 < r < \sqrt{t}$ small enough.

The compact Lemma 3.1 can be proved by using Proposition 2.1 and famous compensated compactness principle [CLMS] and Hélein's trick.

4 Analysis of the blow up sets

4.1 Problems

Let u_k be a sequence of stationary weak solutions of (1.1) with initial data $u_k(x,0)$ and $\int_{R^m} |\nabla u_k(x,0)|^2 \leq \Lambda$. By the energy inequality we have

$$\int_0^T \int_{R^m} 2\lambda_1 \partial_t u_k^2 \, dx \, dt + \int_{R^m} |\nabla u_k|^2(x,T) \, dx$$
$$\leq \int_{R^m} |\nabla u_k(x,0)|^2 dx = E_{k0} \leq \Lambda. \tag{4.1}$$

Therefore we may assume that $u_k \to u$ weakly in $W^{1,2}(R^m \times R_+, S^2)$. We set

$$\Sigma_{\epsilon_0}^t = \cap_{r>0} \left\{ x \in R^m \,|\, \liminf_{k \to \infty} \mathcal{E}(r, u_k, z) \geq \epsilon_0 \right\},$$

$$\Sigma_\beta^t = \left\{ x \in R^m \,|\, \begin{array}{l} \limsup_{r \to 0} \liminf_{k \to \infty} \frac{1}{r^{m-2+\beta}} \\ \int_{t-r^2}^{t+r^2} \int_{B_{r|\log r|^{1/2}}(x)} |\partial_t u_k|^2 = \infty \end{array} \right\},$$

where $\epsilon_0 > 0$ defined in Theorem 3.1 and $1/2 > \beta > 0$ are the fixed constants. We call the set $\Sigma^t = \Sigma_{\epsilon_0}^t \cup \Sigma_\beta^t$ the **blow up set** for the sequence u_k at t and $\Sigma = \cup_{0 < t < \infty} \Sigma^t \times \{t\}$ the **total blow up set** for the sequence u_k.

The purpose in this part is to analyze the blow up set Σ^t. In view of the geometric measure theory we first hope to know that

- is Σ^t rectifiable?

- Furthermore, one ask how to move Σ^t with respect to t?

- What happens at a singular point?

4.2 Rectifiable

Define
$$\mathcal{T}_\infty = \left\{ t \in R_+ \,|\, \liminf_{k \to \infty} \int_{R^m} |\partial_t u_k|^2 = \infty \right\}.$$

Then we have
$$\mathcal{H}^1(\mathcal{T}_\infty) = 0;$$
$$\mathcal{H}^{m-2}(\Sigma_{\epsilon_0}^t) < \infty;$$

If $t \notin \mathcal{T}_\infty$ then for any $\epsilon > 0$, $\mathcal{H}^{m-4+\beta+\epsilon}(\Sigma_\beta^t) = 0$.

REMARK 4.1 *Here we can not expect that $\mathcal{H}^{m-2}(\Sigma_{\epsilon_0}^t) = 0$, as we do not know if the sequence $\{|\nabla u_k|^2\}$ is of the equivalent continuous in the sense of integration, or equivalently, strong convergence.*

The small energy regularity and (4.1) imply we may assume that

$$|\nabla u_k|^2(\cdot, t)dxdt \rightharpoonup |\nabla u|^2(\cdot, t)dxdt + \nu_t dt,$$

$$|\partial_t u_k|^2(\cdot, t)dxdt \rightharpoonup |\partial_t u|^2(\cdot, t)dxdt + \mu,$$

in the sense of measure as $k \to \infty$, where ν_t is a nonnegative Radon measure in R^m supported in Σ^t, μ is a nonnegative Radon measure in $R^m \times R_+$ supported in Σ.

We have from the following monotonicity (2.6).

LEMMA 4.1 *If* $\liminf_{k\to\infty} I(r, u_k, z) \le \frac{r^{-\alpha}}{\lambda_1^2 + \lambda_2^2}$ *for* $0 < r \le r_0$, *then* $e^r(E(r, u, z) + r^{2-m}\nu_t(B_r(x)) + \frac{r^{1-\alpha}}{1-\alpha})$ *is non-deceasing. Precisely*

$$\frac{d}{dr}(e^r(E(r, u, z) + r^{2-m}\nu_t(B_r(x)) + \frac{r^{1-\alpha}}{1-\alpha})) \ge \frac{e^r r^{1-\alpha}}{1-\alpha}.$$

Now we present our main theorem in this section.

THEOREM 4.1 *Let* u_k *be a strong stationary weak solution of (1.1) with the initial energy* $E_{k0} \le \Lambda$. *Then for almost every* $t \in R_+$, ν_t *is* $\mathcal{H}^{m-2}$-*rectifiable, therefore* Σ^t *is* $\mathcal{H}^{m-2}$ *-rectifiable.*

We obvious have the following properties for $\theta(x, t)$.

LEMMA 4.2 *For almost every* t, $\nu_t = \theta(x, t)\mathcal{H}^{m-2}\lfloor\Sigma^t$, *and* $\theta(x, t)$ *is upper semicontinuous in* $\Sigma^t/S_\alpha(t)$ *with* $C(\epsilon_0) \le \theta(x, t) \le C(\Lambda)$ *for* $\mathcal{H}^{m-2}$-*a.e.x* $\in \Sigma^t$, *and therefore* $\theta(x, t)$ *is* $\mathcal{H}^{m-2}$ *approximate continuous for* $\mathcal{H}^{m-2}$ *a.e.* $x \in \Sigma^t$.

4.3 Quasi-mean curvature flow

We know that the blow up sets of the heat flow for harmonic maps move by the mean curvature (see [LT]). In this section we will calculate the curvature of Σ^t at the point $t \notin \mathcal{T}_\infty$ and verify Σ^t moves by the quasi-mean curvature. The difference from the heat flows is that the blow up set Σ^t in Landau-Lifshitz case moves by the quasi-mean curvature defined in the following, no the mean curvature except $\alpha_2 = 0$.

Using the monotonicity inequality (2.6) we can obtain the following important lemmas.

LEMMA 4.3 *let* $T \in T\Sigma^t$, *where* $T\Sigma^t$ *is the tangent bundle on* Σ^t. *If* $t \notin \mathcal{T}_\infty$, *then we have*

$$\lim_{\epsilon \to 0} \liminf_{k \to \infty} \int_{B_\epsilon(\Sigma^t)} |\nabla_T u_k|^2 = 0.$$

Here and in the sequel we denote by

$$B_\epsilon(X) = \{x \in R^m | dist(x, X) < \epsilon\}.$$

LEMMA 4.4

$$\lim_{\epsilon \to 0} \lim_{k \to \infty} \int_{B_\epsilon(\Sigma^t)} \partial_j u_k \partial_i u_k \partial_j \zeta^i = \frac{1}{2} \int_{\Sigma^t} div_{(\Sigma^t)^\perp}(\zeta) \, \nu_t.$$

We also define the induced Radon measures $\overrightarrow{w_t}$, $\overrightarrow{w_t^*}$, and $\overrightarrow{w_t^{*\perp}}$ on Σ^t by the following respectively,

$$\lim_{k \to \infty} 2 \int_{R^m} (\lambda_1 \partial_t u_k - \lambda_2 u_k \times \partial_t u_k) \nabla u_k \zeta$$

$$= 2 \int_{R^m} (\lambda_1 u_t - \lambda_2 u \times u_t) \zeta \cdot \nabla u + 2 \int_{\Sigma^t} \overrightarrow{w_t} \zeta$$

$$\lim_{k \to \infty} \int_{R^m} (\zeta \cdot \nabla u_k) \partial_t u_k$$

$$= \int_{R^m} (\zeta \cdot \nabla u) u_t + \int_{\Sigma^t} \zeta \overrightarrow{w_t^*}$$

$$\lim_{k \to \infty} \int_{R^m} (u_k \times \partial_t u_k)(\zeta \cdot \nabla u_k)$$

$$= \int_{R^m} (u \times \partial_t u)(\zeta \cdot \nabla u) + \int_{R^m} \zeta \cdot \overrightarrow{w_t^{*\perp}}$$

where ζ is a smooth vector field with compact support in R^m, and $\overrightarrow{w_t} = \overrightarrow{w_t^1} - \overrightarrow{w_t^2}$ and every component $\overrightarrow{w_t^i}$, $i = 1, 2$ is a nonnegative Radon measure on Σ^t. Analogously for $\overrightarrow{w_t^*}$ and $\overrightarrow{w_t^{*\perp}}$. It is obvious that

$$\overrightarrow{w_t} = \lambda_1 \overrightarrow{w_t^*} - \lambda_2 \overrightarrow{w_t^{*\perp}}.$$

Then we obtain from the definition of the stationary weak solutions and the two lemmas above

THEOREM 4.2 *Let ζ be a smooth vector field with compact support in R^m. If $t \notin \mathcal{T}_\infty$, then*

$$\int_{\Sigma^t} div_{\Sigma^t}(\zeta)\nu_t + \int_{R^m} |\nabla u|^2 div\zeta - 2\partial_j u \partial_k u \partial_j \zeta^k$$

$$= 2 \int_{R^m} (\lambda_1 u_t - \lambda_2 u \times u_t)\zeta \cdot \nabla u + 2 \int_{\Sigma^t} \overrightarrow{w_t}\zeta.$$

We also have the following theorem.

THEOREM 4.3 *Suppose that u_k is a stationary weak solution of (1.1). Then for any $\vartheta \in C_0^\infty(R^m, R_+)$, we have*

$$\int_{R^m} \vartheta \nu_t - \int_{R^m} \vartheta \nu_{t_0} + \left(\int_{R^m \times t} - \int_{R^m \times t_0} \right) |\nabla u|^2 \vartheta$$

$$\leq -2 \int_{t_0}^t \int_{\Sigma^t} (\lambda_1 \vartheta \mu + \nabla \vartheta \overrightarrow{w_t^*})$$

$$-2 \int_{t_0}^t \int_{R^m} (\lambda_1 u_t^2 \vartheta + \nabla u \nabla \vartheta u_t).$$

And if u_k is strong stationary weak solution of (1.1). Then the equality above holds.

We are in the position to introduce the curvature of Σ^t. Suppose that u_k are the strong stationary weak solutions, and the limiting map u is a strong stationary weak solution. We then have from Theorem 4.2 that, for any $t \notin \mathcal{T}_\infty$,

$$\int_{\Sigma^t} div_{\Sigma^t}(\zeta)\nu_t = 2\int_{\Sigma^t} \overrightarrow{w_t}\zeta = -\int_{\Sigma^t} \overrightarrow{H_t}\zeta\nu_t$$

where $\overrightarrow{H_t} = -2\frac{d\overrightarrow{w_t}}{d\nu_t}$. Clearly for $\mathcal{H}^{m-2}$-a.e. $x \in \Sigma^t$

$$\overrightarrow{H_t}(x) = -2\lim_{r\to 0}\lim_{k\to\infty}\frac{\int_{B_r(x)}(\lambda_1\partial_t u_k - \lambda_2 u_k \times \partial_t u_k)\nabla u_k}{\int_{B_r(x)}|\nabla u_k|^2}.$$

Applying Lemma 4.3 we see that

$$\overrightarrow{H_t}(x) \in S^\perp(x).$$

Where $S = S(x) = T_x\nu_t$. Thus we obtain the first variation of varifold V_{ν_t} :

$$\delta V_{\nu_t} = -\nu_t \lfloor \overrightarrow{H_t}.$$

And $\overrightarrow{H_t}$ is the **generalized mean curvature** of Σ^t (see 4.3 in [A]). We define the total variation $\|\delta V_{\nu_t}\|$ of δV_{ν_t} by the requirement that

$$\|\delta V_{\nu_t}\|(U) = \sup\{\delta V_{\nu_t}(\zeta) : \zeta \in T\Sigma^t, spt\zeta \subset U \text{ and } |\zeta| \leq 1\}.$$

Then by the representation theorems (see 2.5 in [F])

$$\overrightarrow{H_t} = -\frac{d\|\delta V_{\nu_t}\|}{d\nu_t}\eta_t, \tag{4.2}$$

where $\eta_t(x) \in S^{m-1}$. We also define the varifolds $\delta^* V_{\nu_t}$ and $\delta^{*\perp}V_{\nu_t}$ respectively by

$$\delta^* V_{\nu_t}(\zeta) = \int_{\Sigma^t} 2\zeta\overrightarrow{w_t^*}\nu_t,$$

$$\delta^{*\perp}V_{\nu_t}(\zeta) = \int_{\Sigma^t} 2\zeta\overrightarrow{w_t^{*\perp}}\nu_t.$$

Then we have

$$\|\delta^* V_{\nu_t}\| = \|\delta^{*\perp}V_{\nu_t}\|, \tag{4.3}$$

$$|\overrightarrow{H_t^*}| = |\overrightarrow{H_t^{*\perp}}|, \tag{4.4}$$

$$\overrightarrow{H_t} = \lambda_1\overrightarrow{H_t^*} - \lambda_2\overrightarrow{H_t^{*\perp}}, \tag{4.5}$$

$$|\overrightarrow{H_t}| \leq (\lambda_1 + |\lambda_2|)|\overrightarrow{H_t^*}|. \tag{4.6}$$

Evidently $\overrightarrow{H_t^*} = -2\frac{d\overrightarrow{w_t^*}}{d\nu_t}$ and $\overrightarrow{H_t^{*\perp}} = -2\frac{d\overrightarrow{w_t^{*\perp}}}{d\nu_t}$, and for $\mathcal{H}^{m-2}$-a.e. $x \in \Sigma^t$

$$\overrightarrow{H_t^*}(x) = -2\lim_{r\to 0}\lim_{k\to\infty}\frac{\int_{B_r(x)}\partial_t u_k\nabla u_k}{\int_{B_r(x)}|\nabla u_k|^2}.$$

$$\overrightarrow{H_t^{*\perp}}(x) = -2 \lim_{r \to 0} \lim_{k \to \infty} \frac{\int_{B_r(x)}(u_k \times \partial_t u_k)\nabla u_k}{\int_{B_r(x)} |\nabla u_k|^2}.$$

Applying Lemma 4.3 again we have

$$\overrightarrow{H_t^*}(x), \overrightarrow{H_t^{*\perp}}(x) \in S^\perp(x).$$

$\overrightarrow{H_t^*}$ is called the **quasi-mean curvature** of Σ^t.

Now we introduce the Brakke's quantity $\mathcal{B}^*(\nu_t, \vartheta)$.

DEFINITION 4.1 *Let* $\vartheta \in C_0^2(R^m, R_+)$. *Define*

$$\mathcal{B}^*(\nu_t, \vartheta) = -\int_{R^m} \frac{\lambda_1}{2}\vartheta|\overrightarrow{H_t^*}|^2 \nu_t + \int_{R^m} \nabla\vartheta \cdot S^\perp \cdot \overrightarrow{H_t^*}\nu_t,$$

where $S = S(x) \equiv T_x \nu_t$ *for* H^{n-2} *a.e.* $x \in \{\vartheta > 0\}$.

DEFINITION 4.2 *We say* $\{\nu_t\}_{t \geq 0}$ *a generalized Brakke's motion provided that for a.e.* $t \geq 0$ *and all* $\vartheta \in C_0^2(R^m, R_+)$,

$$d_t^+ \nu_t(\vartheta) \leq \mathcal{B}^*(\nu_t, \vartheta),$$

where we denote by

$$d_t^+ f = \limsup_{s \to t} \frac{f(s) - f(t)}{s - t}.$$

We also say the measure family $\{\nu_t\}_{t \geq 0}$ *(or surface* Σ^t *equivalently) moving by the quasi-mean curvature in the case.*

We have

THEOREM 4.4 *Suppose that* u_k *are the strong stationary weak solutions of (1.1) and the limiting map* u *is also a strong stationary weak solution of (1.1), then the blow up measure* $\{\nu_t\}$ *is a generalized Brakke's motion.*

The following theorem asserts that the singular set Σ^t of Landau-Lifshitz system is a quasi-mean curvature flow.

THEOREM 4.5 *Suppose that blow up set* Σ^t *of Landau-Lifshitz system (1.1) with* $\lambda_1 \geq 2|\lambda_2|$ *is a smooth family of sub-manifolds in* R^m *and assume that it is a generalized Brakke's flow in the sense of Theorem 4.4. Then* Σ^t *is a quasi-mean curvature flow.*

Proof. We write $\Sigma^t = F(\cdot, t)(M_0^{m-2})$, which is a parametric representation of Σ^t. Suppose that

$$\partial_t F(x, t) = \vec{Y}(x, t)$$

with $\vec{Y} \in T\Sigma^{t\perp}$. By the first variation of a varifold with respect to integrands (see 4.9 in [A] or 2.4 in [I]), we have

$$\delta V_{\nu_t}(\vartheta)(\vec{Y}) = d_t^+ \int_{\Sigma^t} \vartheta \nu_t = \int_{\Sigma^t} (-\vartheta \vec{H_t} + \nabla \vartheta \cdot S^\perp) \cdot \vec{Y} \nu_t, \qquad (4.7)$$

where $\vec{H_t}$ defined in (4.2) is the mean curvature of Σ^t. On the other hand, by the Theorem 4.4, we have

$$d_t^+ \int_{\Sigma^t} \vartheta \nu_t \leq \int_{\Sigma^t} (-\frac{\lambda_1}{2}\vartheta|\vec{H_t^*}|^2 + \nabla \vartheta \cdot S^\perp \cdot \vec{H_t^*})\nu_t. \qquad (4.8)$$

Therefore we obtain that from (4.7) (4.8) and (4.5)

$$\int_{\Sigma^t} \vartheta[\lambda_1\vec{H_t^*} \cdot (\vec{Y} - \frac{1}{2}\vec{H_t^*}) - \lambda_2\vec{H_t^{*\perp}} \cdot \vec{Y}]\nu_t \geq \int_{\Sigma^t} \nabla \vartheta \cdot (\vec{H_t^*} - \vec{Y})\nu_t.$$

Now since $\vartheta > 0$ is arbitrary and $|\vec{H_t^*} \cdot \vec{H_t^{*\perp}}| \leq |\vec{H_t^*}|^2$ and $\lambda_1 \geq 2|\lambda_2|$, we have to have $\vec{H_t^*} = \vec{Y}$, which is the desired result.

4.4   Blow up analysis at a singular point

Let $z_0 = (x_0, t_0) \notin \Sigma_\beta$ be a singular point of u such that

$$\lim_{r \to 0} \mathcal{E}(r, u, z_0) \geq \epsilon_0$$

by Theorem 3.1. Set $u_k(z) = u(x_0 + r_k x, t_0 + r_k^2 t)$ where $x \in R^m$ and $t \in R_-$, then u_k satisfies (1.1) and by scaling , for any $z \in R^m \times R_-$,

$$\mathcal{E}(r_k, u, z) = \int_{P_1(z)} |\nabla u_k|^2.$$

By Proposition 2.1 we see that for small r_k

$$\int_{P_1(z)} |\nabla u_k|^2 \leq C(R_0),$$

and from the energy inequality we have

$$\int_{P_{1/2}(z)} |\partial_t u_k|^2 \leq c \int_{P_1(z)} |\nabla u_k|^2 \leq C(R_0).$$

Denote , for fixed constant $\delta > 0$,

$$D_k = \left\{ z \in R^m \times R_- : \begin{array}{l} x_0 + r_k x \in B_\delta(x_0), \\ t_0 + r_k^2 t \in [t_0 - \delta^2, t_0 + \delta^2] \end{array} \right\},$$

then $D_k \to R^m \times R_-$ as $k \to \infty$, since $r_k \to 0$. There is therefore a subsequence (still denoted by r_k) $r_k \to 0$ such that $u_k(x, t) \to v(x, t)$ weakly in $W_{loc}^{1,2}(R^m \times R_-, S^2)$,

$$|\nabla u_k|^2(\cdot, t)dxdt \to |\nabla v|^2(\cdot, t)dxdt + \nu_t dt$$

$$\lvert \partial_t u_k \rvert^2(\cdot, t) dx dt \to \lvert \partial_t v \rvert^2(\cdot, t) dx dt + \mu$$

in the sense of measure as $k \to \infty$, where ν_t is a nonnegative Radon measure in R^m supported in Σ^t, μ is a nonnegative Radon measure in $R^m \times R_-$ supported in Σ, here and in the sequel we use the same notations and results in the sections above.

We have by scaling

$$\int_{t_0-(r_k R_2)^2}^{t_0-(r_k R_1)^2} \int_{R^m} \lambda_1 \Big[\frac{r \partial_r u}{\sqrt{2(t_0-t)}}$$

$$- \sqrt{2(t_0-t)}(\partial_t u - \frac{\lambda_2}{2\lambda_1} u \times \partial_t u)\Big]^2 G_{z_0} dx dt$$

$$\leq \quad \Phi_{z_0}(r_k R_2, u) - \Phi_{z_0}(r_k R_1, u) + \Theta_{z_0}(r_k R_1, r_k R_2, u).$$

Since $z_0 \notin \Sigma_\beta$, we have $\Theta_{z_0}(r_k R_1, r_k R_2, u) \leq C_1(r_k R_2)^\beta$ by Lemma 2.1. Then $\Theta_{z_0}(r_k R_1, r_k R_2, u) \to 0$ as $k \to \infty$. Obviously, for any $R > 0$, $\Phi_{z_0}(r_k R, u) \to \lambda_1^{-m/2}\theta(x_0, t_0)$ as $k \to \infty$. We thus obtain

$$\int_{-R_2^2}^{-R_1^2} \int_{R^m} \lambda_1 \Big[\frac{r \partial_r u_k}{\sqrt{2(-t)}} - \sqrt{2(-t)}(\partial_t u_k$$

$$- \frac{\lambda_2}{2\lambda_1} u_k \times \partial_t u_k)\Big]^2 G_0 dx dt \to 0$$

as $k \to \infty$. That is,

$$\lambda_1 \partial_t u_k - \frac{\lambda_2}{2}(u_k \times \partial_t u_k) - \frac{\lambda_1 r \partial_r u_k}{2(-t)} \to 0 \tag{4.9}$$

strongly in $L^2(R^m \times [-R_2, -R_1], R^3)$ as $k \to \infty$. Furthermore

$$u_k \times \partial_t u_k - u_k \times \frac{x \cdot \nabla u_k}{2(-t)} \to 0 \tag{4.10}$$

strongly in $L^2(R^m \times [-R_2, -R_1], R^3)$. Applying the results in section 4.2 we obtain the blow up formulas

THEOREM 4.6 *If $t_0 \notin \mathcal{T}_\infty$, and $z_0 = (x_0, t_0) \notin \Sigma_\beta$ is a blow up point, then we have two blow up formulas*

$$\int_{R^m}(\lvert \nabla u \rvert^2 div(\zeta) - 2\partial_j v \partial_k v \partial_j \zeta^k) \tag{4.11}$$

$$-2 \int_{R^m}(\lambda_1 \partial_t v - \lambda_2 v \times \partial_t v) \nabla v \zeta$$

$$= \int_{\Sigma^t} \frac{1}{2(-t)}(\lambda_1 x_{\bar{n}} \cdot \zeta_{\bar{n}} - \frac{\lambda_2}{2}\sigma(x)\lvert x_{\bar{n}} \times \zeta_{\bar{n}} \rvert)\nu_t - \int_{\Sigma^t} div_{(\Sigma^t)}(\zeta)\nu_t,$$

$$\int_{t_1}^{t_2} \int_{R^m} 2(\lambda_1 v_t^2 \vartheta + \nabla v \nabla \vartheta v_t) dx dt$$

$$+ \left(\int_{R^m \times t_2} - \int_{R^m \times t_1} \right) |\nabla v|^2 \vartheta dx$$

$$= \int_{\Sigma^t} \vartheta \nu_{t_1} - \int_{\Sigma^t} \vartheta \nu_{t_2} - \frac{\lambda_1^3}{4\lambda_1^2 + \lambda_2^2} \int_{t_1}^{t_2} \int_{\Sigma^t} \frac{\vartheta |x_{\bar{n}}|^2}{t^2} \nu_t$$

$$- \int_{t_1}^{t_2} \int_{\Sigma^t} \frac{1}{2(-t)} (\lambda_1 x_{\bar{n}} \cdot \zeta_{\bar{n}} - \frac{\lambda_2}{2} \sigma(x) |x_{\bar{n}} \times \zeta_{\bar{n}}|) \nu_t, \qquad (4.12)$$

where $\zeta \in C_0^\infty(R^m, R^m)$ and $\vartheta \in C_0^\infty(R^m, R_+)$, $y_{\bar{n}}(x)$ denotes the projective vector of the vector $y(x)$ on the normal plane of Σ^t at x, and $\sigma(x) = \pm 1$ is a sign function determined by some Radon measure.

References

[A]   W. Allard, On the first variation of a varifold, Ann. Math. 95(1972), 417-491

[ASo] L. Ambrosio and H.M. Soner, Level set approach to mean curvature flow in arbitrary codimension, J. Diff. Geom. 43(1996), 693-737

[AS] F.Alouges and A.Soyeur, On global weak solutions for Landau-Lifshitz equations: existence and nonuniqueness. Nonlinear Anal.,TMA, vol 18(11), 1071-1084(1992).

[C]   Y. Chen, Bubbling phenomena and energy identity for Landau-Lifschitz equations, preprint

[CLL] Y.Chen, J.Li, F.H.Lin, Partial regularity for weak heat flows into spheres, Comm. Pure and Applied Math. 4(1995), 429-448

[Ch] X. Cheng, Estimate of the singular set of the evolution problem for harmonic maps, J. D. G., 34(1991), 169-174

[E]   L.C.Evans, Partial regularity for stationary harmonic maps into spheres, Arch. Rational Mech. Anal 116(1991), 101-113

[F]   H. Federer, Geometric measure theory, Springer-Verlag, New York, 1969

[Fe] M.Feldman, Partial regularity for harmonic maps of evolution into sphere, Comm. P. D. E. 19(5&6)(1994),
761-790

[Fo] H.C.Fogedby,Theoretical aspects of mainly low dimensional magnetic systems. Lecture Notes Phys. Vol 131, Berlin, Heidelberg, New York, Springer 1980.

[GH] B.Guo and M.Hong, The Landau-Lifshitz equation of the ferromagnetic spin chain and harmonic maps, Calc. Var. 1, 1993, 311-331.

[H] F.Helein, Regularity des applicaltions faiblement harmoniques entre une surface et une sphere, C. R. Acad.Sci.Pari. 311(1990), 519-524

[I] T.Ilmanen, Elliptic regularization and partial regularity for motion by mean curvature, Memoirs of the Amer. Math. Soc., 520, 994

[LN] M.Lakshmanan and K.Nakamura, Landau-Lifshitz equation of ferromagnetism exact treatment of the Gilbert damping, Phys. Rev. Letter 53(26), 2497-2499(1984).

[LL] L.D.Landau and E.M.Lifshitz, On the theory of the dispersion of magnetic permeability in ferromagnetic bodies. Phys. Z.Sowj. 8, 153 (1935).

[LT] J.Li, G.Tian, The blow up locus of heat flows for harmonic maps, Acta Math. Sinica (English version) 16(2000), 29-62

[LT1] J.Li, G.Tian, A blow up formula for stationary harmonic maps, IMRN 14(1998), 735-755

[Liu] X.Liu, Partial regularity for the Landau-Lifshitz system of the ferromagnetic spin chain, to appear in Comm. PDE

[Liu1] X.Liu, Blow up sets of the Landau-Lifshitz system and quasi-mean curvature flows, preprint

[LW] F.H.Lin and C.Y.Wang, Harmonic and qusi-harmonic spheres, Comm. Anal. Geom., to appear

[P] D.Priess, Geometry of measure in R^n; Distribution, rectifiability, and densities; Ann. Math., 125(1987), 537-643

[Si] L.Simon, Lectures on geometric measure theory, Proc. Center Math. Anal. 3(1983), Australian National Univ. Press

[T] L.A.Takhtalian, Integration of the continuous Heisenberg spin chain through the inverse scattering method, Phys. Letter 15B,3470-3476(1977).

[TW] J.Tjon and J.Wright, Soliton in the continuous Heisenberg chain. Phys. Rev.

[ZGT] Y.Zhou, B.Guo and S.Tan, Existence and uniqueness of smooth solution for system of ferromagnetic chain. Scientia Sinica Ser. A 34, 257-266(1991).