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On positive solutions to some semilinear elliptic equations with nonnegative forcing terms

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§1. Introduction.

In this paper we assume $n \geq 2$ and consider positive solutions to the semilinear elliptic equation involving a forcing term

$$(\mathrm{P})_\kappa \qquad \begin{cases} -\Delta u + u = g(u) + \kappa f_* & \text{in } \mathcal{D}'(\mathbf{R}^n), \\ u \geq 0 \quad \text{a.e. on } \mathbf{R}^n, & u(x) \to 0 \quad \text{as } |x| \to \infty \end{cases}$$

with a positive parameter κ. Here, $\Delta = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^2$ is the Laplacian on $\mathbf{R}^n$, f_* is a given nonnegative forcing term and the nonlinearity function g is given by

$$g(s) = s_+^p \quad \text{for } s \in \mathbf{R} \quad \text{with } p > 1. \tag{1.1}$$

We assume that $f_* \geq 0$ in $\mathcal{D}'(\mathbf{R}^n)$, and hence f_* is a measure on $\mathbf{R}^n$, in general. Though we do not have to take the nonlinearity function exactly in the form (1.1) in our main results, we only treat the case (1.1) in the following, for simplicity.

Then we can observe that, in a suitable situation, problem $(\mathrm{P})_\kappa$ has a solution for small κ, while $(\mathrm{P})_\kappa$ has no solution for large κ. Indeed, the following facts are known. Here, $\underline{p}^* = n/(n-2)$, $p^* = (n+2)/(n-2)$ and

$$\kappa^* = \sup\{\, \kappa > 0 \mid \text{problem } (\mathrm{P})_\kappa \text{ has a solution} \,\}. \tag{1.2}$$

(We agree that $1/0 = \infty$.)

Fact. (I) *(Deng–Li [1], [2]) Let $n \geq 3$ and $f_* \in H^{-1}(\mathbf{R}^n)$ be a non-zero nonnegative function on $\mathbf{R}^n$ satisfying $|x|^{n-2} f_* \in L^\infty(\mathbf{R}^n)$. Then the following properties hold:*
(i) *If $p > 1$, then it holds $0 < \kappa^* < \infty$.*
(ii) *If $1 < p \leq p^*$, then problem $(\mathrm{P})_{\kappa^*}$ has a unique solution.*
(iii) *If $1 < p \leq p^*$ with $3 \leq n \leq 5$ or $1 < p < p^*$ with $n \geq 6$, then problem $(\mathrm{P})_\kappa$ has at least two solutions for any $\kappa \in (0, \kappa^*)$.*
(iv) *If $p = p^*$ with $n \geq 6$, then a solution to $(\mathrm{P})_\kappa$ is unique for small κ, under some symmetry condition on f_*.*
(Here, a solution to $(\mathrm{P})_\kappa$ is in the sense that $u \in H^1(\mathbf{R}^n)$. Also we say that a distribution f on $\mathbf{R}^n$ is non-zero if f is not identically zero on $\mathbf{R}^n$.)

(II) *(Sato [9]) Let $n \geq 2$ and f_* be a non-zero nonnegative finite Radon measure on $\mathbf{R}^n$ with a compact support. If $1 < p < \underline{p}^*$, then the conclusion of* (i)–(iii) *above holds true.*
(We describe the precise definition of solutions later.)

Our main purpose is to discuss the property (ii) above under weaker restriction on p and f_*, including the case where p is *supercritical*, i.e., $p > p^*$. Here, we assume that

f_* has a compact support. In the following, we explain the results containing that of (II).

We denote the norm of $L^q(\mathbf{R}^n)$ by $\|\cdot\|_q$ for $1 \le q \le \infty$, and the norm of $H^1(\mathbf{R}^n)$ by $\|v\|_{1,2} = (\|\nabla v\|_2^2 + \|v\|_2^2)^{1/2}$ for $v \in H^1(\mathbf{R}^n)$. We also denote

$$\begin{cases} L^q_c(\mathbf{R}^n) = \{\, v \in L^q(\mathbf{R}^n) \mid \operatorname{supp} v \text{ is compact } \} \quad (1 \le q \le \infty), \\ C_0(\mathbf{R}^n) = \{\, v \in C(\mathbf{R}^n) \mid v(x) \to 0 \ \text{ as } |x| \to \infty \ \}, \quad BC(\mathbf{R}^n) = (C \cap L^\infty)(\mathbf{R}^n). \end{cases}$$

For a fixed non-zero nonnegative finite Radon measure f_* on $\mathbf{R}^n$, we set

$$\phi_* = E_1 * f_*, \tag{1.3}$$

where E_1 is the (canonical) fundamental solution for $-\Delta + I$ on $\mathbf{R}^n$. Note that $E_1 \in C^\infty(\mathbf{R}^n \setminus \{0\})$ is radial and satisfies

$$\begin{cases} E_1 > 0, \ \dfrac{\partial E_1}{\partial r} < 0 \ \text{ on } \mathbf{R}^n \setminus \{0\}, \quad -\Delta E_1 + E_1 = \delta_0 \ \text{ in } \mathcal{D}'(\mathbf{R}^n). \\ E_1(x) \sim E(x) \ \text{ as } x \to 0, \ \ E_1(x) \sim c_{(n)} \dfrac{e^{-|x|}}{|x|^{(n-1)/2}} \ \text{ as } |x| \to \infty \end{cases} \tag{1.4}$$

(see e.g. [3, Appendix C]). Here, $c_{(n)}$ is a positive constant and E is the fundamental solution for $-\Delta$ on $\mathbf{R}^n$, that is,

$$E(x) = \begin{cases} \dfrac{1}{(n-2)nm(B_1)} \dfrac{1}{|x|^{n-2}} & \text{for } x \in \mathbf{R}^n \setminus \{0\} \quad \text{if } n \ge 3, \\ \dfrac{1}{2\pi} \log \dfrac{1}{|x|} & \text{for } x \in \mathbf{R}^2 \setminus \{0\} \quad \text{if } n = 2. \end{cases} \tag{1.5}$$

(We denote the open ball of radius R centered at the origin in $\mathbf{R}^n$ by B_R, and m is the Lebesgue measure on $\mathbf{R}^n$.) Particularly, $E_1 \in L^q(\mathbf{R}^n)$ for $1 \le q < p^*$, and it holds $\|E_1\|_1 = 1$. Hence, we have that $\phi_* \in L^q(\mathbf{R}^n)$ for $1 \le q < p^*$, in general.

In the following, we assume that

$$\phi_* \in L^{q_0}(\mathbf{R}^n) \setminus \{0\}, \ \ f_* \ge 0 \ \text{ in } \mathcal{D}'(\mathbf{R}^n), \ \ \operatorname{supp} f_* \subset B_{R_*} \tag{A$_*$}$$

for some $q_0 > \max\{p, n(p-1)/2\}$ and $R_* \in (0, \infty)$. Note that, if $u \in L^{q_0}_{\mathrm{loc}}(\mathbf{R}^n)$ satisfies

$$-\Delta u + u = g(u) + \kappa f_* \ \text{ in } \mathcal{D}'(\mathbf{R}^n), \tag{1.6}$$

then we can see that $u \in C^2(\mathbf{R}^n \setminus \operatorname{supp} f_*)$ with the aid of the elliptic regularity argument. So, we define a solution to $(\mathrm{P})_\kappa$ as follows.

Definition 1.1. *Under assumption* (A$_*$), *for* $\kappa \ge 0$, *we call* u *a solution to problem* $(\mathrm{P})_\kappa$ *if*

$$u \in (L^{q_0}_c + C_0)(\mathbf{R}^n), \quad u \ge 0 \ \text{ a.e. on } \mathbf{R}^n \tag{1.7}$$

and u *satisfies* (1.6).

In order to describe our results precisely, we prepare the proposition below.

Proposition 1.1. *Let* u *and* $\bar{u}$ *be non-zero functions on* $\mathbf{R}^n$ *satisfying* (1.7). *Then the following properties hold:*

(i) *There exists a minimizer* $\varphi^1 \in H^1(\mathbf{R}^n) \setminus \{0\}$ *of the minimizing problem*

$$\lambda^1[u] = \inf\left\{ \frac{\|\nabla v\|_2^2 + \|v\|_2^2}{\|g'(u)v^2\|_1} \;\middle|\; v \in H^1(\mathbf{R}^n) \setminus \{0\} \right\}.$$

Particularly, $\lambda^1[u] \in (0,\infty)$.

(ii) *The least eigenvalue of the linearized eigenvalue problem*

$$(\mathrm{L}\,; u)^\lambda \qquad \begin{cases} -\Delta\varphi + \varphi = \lambda g'(u)\varphi & \text{in } \mathcal{D}'(\mathbf{R}^n), \\ \varphi \not\equiv 0 \quad \text{on } \mathbf{R}^n, \quad \varphi(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

is given by $\lambda^1[u]$, *which is a simple eigenvalue. Moreover, the minimizer* φ^1 *is an eigenfunction corresponding to eigenvalue* $\lambda^1[u]$ *satisfying* $\varphi^1 \in C_0(\mathbf{R}^n)$ *and* $\varphi^1 > 0$ *on* $\mathbf{R}^n$ *(or* $\varphi^1 < 0$ *on* $\mathbf{R}^n$*).*

(iii) *If* $(\mathrm{L}\,;u)^\lambda$ *has a positive solution* $\varphi \in H^1(\mathbf{R}^n)\setminus\{0\}$, *then it holds* $\lambda = \lambda^1[u]$.

(iv) *If* $u \le \bar{u}$ *and* $u \not\equiv \bar{u}$ a.e. *on* $\mathbf{R}^n$, *then it holds* $\lambda^1[u] > \lambda^1[\bar{u}]$.

Remark 1.1. (i) *For a solution* u *to problem* $(\mathrm{P})_\kappa$, *the invertibilty of the linearized operator (in a suitable sense) is broken when* $\lambda = 1$ *is an eigenvalue of the linearized eigenvalue problem above. However, if* $\lambda = 1$ *is the least eigenvalue, then the linearized operator is invertible in the 'orthoganal' of* φ^1. *On the other hand, when* $\lambda^1[u] \in (1,\infty)$, *we can see that* $\lambda = 1$ *is not an eigenvalue, and hence the linearized operator is invertible (cf. §4).*

(ii) *The definitin of* $\lambda^1[u]$ *implies the linearization inequality*

$$\lambda^1[u]\|g'(u)v^2\|_1 \le \|\nabla v\|_2^2 + \|v\|_2^2 = \|v\|_{1,2}^2 \quad \text{for } v \in H^1(\mathbf{R}^n). \tag{1.8}$$

Now we set $\bar{p}^* = (n^2 - 8n + 4 + 8(n-1)^{1/2})/((n-2)(n-10)_+)$ and

$$q_*(p) = \begin{cases} p & \text{if } 1 < p < \underline{p}^*, \\ \max\Big\{\dfrac{n}{2}(p-1), \Big(\dfrac{p^*+1}{p-(2-p)/(p')^{1/2}}\Big)'\Big\} & \text{if } \underline{p}^* \le p < \min\{2,\bar{p}^*\}, \\ \max\Big\{\dfrac{n}{2}(p-1), \dfrac{(p^*+1)p}{p^*+1/(p')^{1/2}}\Big\} & \text{if } \max\{\underline{p}^*,2\} \le p < \bar{p}^*, \end{cases} \tag{1.9}$$

where q' is the conjugate exponent of q, i.e. $1/q + 1/q' = 1$ for $q \in [1,\infty]$. Note that $\bar{p}^* > p^*$ if $n \ge 3$, and $q_*(p) \ge \max\{p, n(p-1)/2\}$. Then we can state our results.

Theorem 1.1. *Assume* (A_*) *with* $q_0 \in (q_*(p),\infty)$ *and* $1 < p < \bar{p}^*$. *Then the following properties hold:*

(i) *It holds* $0 < \kappa^* < \infty$.

(ii) *Problem* $(\mathrm{P})_{\kappa^*}$ *has a unique solution* u^*, *and* u^* *satisfies* $\lambda^1[u^*] = 1$.

(iii) *If problem* $(\mathrm{P})_\kappa$ *has a solution* u *satisfying* $\lambda^1[u] = 1$, *then it holds* $\kappa = \kappa^*$.

(iv) *For any* $\kappa \in (0,\kappa^*)$, *problem* $(\mathrm{P})_\kappa$ *has a solution* u_κ *satisfying* $\lambda^1[u_\kappa] \in (1,\infty)$. *Moreover, a solution* u *to* $(\mathrm{P})_\kappa$ *satisfying* $\lambda^1[u] \in (1,\infty)$ *is unique.*

Theorem 1.2. *Assume* (A_*) *with* $q_0 \in (q_*(p),\infty)$ *and* $1 < p < p^*$. *Then, for any* $\kappa \in (0,\kappa^*)$, *problem* $(\mathrm{P})_\kappa$ *has a solution* $\bar{u}_\kappa$ *satisfying* $\bar{u}_\kappa - u_\kappa \in C_0(\mathbf{R}^n)$, $\bar{u}_\kappa - u_\kappa > 0$ *on* $\mathbf{R}^n$ *and* $\lambda^1[\bar{u}_\kappa] \in (0,1)$.

Remark 1.2. (i) *If* $p^* \le p < \bar{p}^*$, *then it holds* $q_*(p) = n(p-1)/2$.

(ii) *If* $1 < p < p^*$, *then it holds* $q_*(p) < p+1 < p^*+1$. *So, our integrability condition is satisfied in the case* $f_* \in H^{-1}(\mathbf{R}^n)$ *with* $n \ge 3$, *because* $\phi_* \in H^1(\mathbf{R}^n) \subset L^{p^*+1}(\mathbf{R}^n)$.

(iii) *The mapping* $p \mapsto q_*(p)$ *is not continuous at* $p = \underline{p}^*$.

§2. Outline of the proof of Theorem 1.1.

In this section we describe the outline of the proof of Theorem 1.1 (i)–(iii). We use the *continuation method* which is essentially due to Keener–Keller [6]. We introduce a new parameter $\tau \in [0,1]$ and consider the problem

$$(\mathrm{P}_\tau)_\kappa \qquad \begin{cases} -\Delta u + u = g(u) - (1-\tau)g(\kappa\phi_*) + \kappa f_* & \text{in } \mathcal{D}'(\mathbf{R}^n), \\ u \geq \kappa\phi_* \quad \text{a.e. on } \mathbf{R}^n, \quad u(x) \to 0 \quad \text{as } |x| \to \infty \end{cases}$$

for $\kappa \geq 0$. Here, the definition of a solution is given in the sense of Definition 1.1. When u is a solution to $(\mathrm{P}_\tau)_\kappa$, we call u a *strictly minimal solution*, a *turning solution* or a *nonminimal solution* to $(\mathrm{P}_\tau)_\kappa$ if $\lambda^1[u] > 1$, $\lambda^1[u] = 1$ or $\lambda^1[u] < 1$, respectively. (Formally, we define $\lambda^1[0] = \infty$ and call $u \equiv 0$ also a strictly minimal solution to $(\mathrm{P}_\tau)_0$.)

Remark 2.1. (i) *Problems* $(\mathrm{P})_\kappa$ *and* $(\mathrm{P}_1)_\kappa$ *are equivalent for* $\kappa \geq 0$.
(ii) *For* $\tau \in [0,1]$, $u \equiv 0$ *is a solution to* $(\mathrm{P}_\tau)_0$.
(iii) *For* $\kappa \geq 0$, $u = \kappa\phi_*$ *is a solution to* $(\mathrm{P}_0)_\kappa$.

For the proof of Theorem 1.1 it is significant to find a turning solution to $(\mathrm{P})_\kappa$ for some κ, which is equivalent to find a solution to $(\mathrm{Q}_1)^*$ in the sense below.

Definition 2.1. *For* $\tau \in [0,1]$, *we call* $(u, \varphi\,; \kappa)$ *a solution to* $(\mathrm{Q}_\tau)^*$ *if* u *is a solution to* $(\mathrm{P}_\tau)_\kappa$ *and* φ *is a positive solution to* $(\mathrm{L}\,; u)^1$. *Then we set*

$$T^* = \{\, \tau \in [0,1] \mid \textit{problem } (\mathrm{Q}_\tau)^* \textit{ has a solution} \,\}. \tag{2.1}$$

Remark 2.2. *If there exists* κ *such that problem* $(\mathrm{P}_\tau)_\kappa$ *has a solution* u *satisfying* $\lambda^1[u] = 1$, *then it holds* $\tau \in T^*$. *Indeed, we can easily see that* $(u, \varphi^1; \kappa)$ *is a solution to* $(\mathrm{Q}_\tau)^*$, *where* φ^1 *is a positive solution to* $(\mathrm{L}\,; u)^{\lambda^1[u]} = (\mathrm{L}\,; u)^1$ *obtained by* Proposition 1.1.

Theorem 1.1 (i)–(iii) is obtained by two propositions below.

Proposition 2.1. *Under assumption* (A_*), *if problem* $(\mathrm{Q}_\tau)^*$ *has a solution* $(u, \varphi\,; \kappa)$, *then the following properties hold:*
(i) *A solution to problem* $(\mathrm{P}_\tau)_\kappa$ *is unique.*
(ii) *Problem* $(\mathrm{P}_\tau)_{\bar\kappa}$ *has no solution for* $\bar\kappa > \kappa$, *provided that* $\tau \in (0,1]$. *Particularly,* $\kappa = \sup\{\, \bar\kappa > 0 \mid \textit{problem } (\mathrm{P}_\tau)_{\bar\kappa} \textit{ has a solution} \,\}$.

Proposition 2.2. *Under assumption* (A_*) *with* $q_0 \in (q_*(p), \infty)$, *it holds* $T^* = [0,1]$.

The proof of Proposition 2.2 consists of three steps below:

Step 1. T^* *is non-empty.*
Step 2. T^* *is open in* $[0,1]$.
Step 3. T^* *is closed in* $[0,1]$.

Now we give the proof of Step 1.

Lemma 2.1. *Under assumption* (A_*), *it holds* $0 \in T^*$, *and hence* T^* *is non-empty.*

Proof. Note that $\kappa\phi_*$ is a solution to $(\mathrm{P}_0)_\kappa$ for any $\kappa > 0$. Then we see from Proposition 1.1 that

$$0<\lambda^1[\kappa\phi_*]=\inf\Big\{\frac{\|\nabla v\|_2^2+\|v\|_2^2}{\|g'(\kappa\phi_*)v^2\|_1}\ \Big|\ v\in H^1(\mathbf{R}^n)\backslash\{0\}\Big\}$$
$$=\frac{1}{\kappa^{p-1}}\inf\Big\{\frac{\|\nabla v\|_2^2+\|v\|_2^2}{\|p\phi_*^{p-1}v^2\|_1}\ \Big|\ v\in H^1(\mathbf{R}^n)\backslash\{0\}\Big\}=\frac{1}{\kappa^{p-1}}\lambda^1[\phi_*]\quad\text{for }\kappa>0.$$

By choosing $\kappa_0^*=\lambda^1[\phi_*]^{1/(p-1)}$ we have that $\lambda^1[\kappa_0^*\phi_*]=1$, and the assertion follows from Remark 2.2. q.e.d.

Other steps will be proved in the following sections.

§3. Minimal solutions.

In this section we explain the construction of a solution to problem $(\mathrm{P}_\tau)_\kappa$ by using the *supersolution–subsolution method*. We introduce the notation below:

$$(3.1)\qquad \begin{cases} u_{\tau,\kappa}^k=\sum_{j=0}^{k}\phi_{\tau,\kappa}^j\ (k\geq 0),\ \ \phi_{\tau,\kappa}^0=\kappa\phi_*,\ \ \phi_{\tau,\kappa}^k=E_1*g_{\tau,\kappa}^{k-1}\ (k\geq 1),\\ g_{\tau,\kappa}^0=\tau g(\kappa\phi_*),\ \ g_{\tau,\kappa}^k=g(u_{\tau,\kappa}^k)-g(u_{\tau,\kappa}^{k-1})\ (k\geq 1).\end{cases}$$

Roughly speaking, if the sequence $\{u_{\tau,\kappa}^k\}_{k=0}^\infty$ converges to a function u in a suitable sense, then u is a solution to $(\mathrm{P}_\tau)_\kappa$.

Remark 3.1. (i) *It holds* $\phi_{\tau,0}^k=0$ $(k\geq 0)$, *which corresponds to that* $u\equiv 0$ *is a solution to* $(\mathrm{P}_\tau)_0$.
(ii) *It holds* $\phi_{0,\kappa}^k=0$ $(k\geq 1)$, *which corresponds to that* $u=\kappa\phi_*$ *is a solution to* $(\mathrm{P}_0)_\kappa$.

By choosing $q_0>q_*(p)$ small if neccesary, we may assume that $1/q_{k_*-1}>0>1/q_{k_*}$ for some $k_*\in\mathbf{N}$, where

$$\frac{1}{q_k}=\frac{1}{q_0}-\alpha_*k\ \ (k\geq 0)\quad\text{and}\quad \alpha_*=\frac{2}{n}-\frac{p-1}{q_0}\,(\in(0,1)).$$

Then the boot-strap argument works, and we can show that $g'(u_{\tau,\kappa}^k)\in L^{q_0/(p-1)}(\mathbf{R}^n)$ $(k\geq 0)$ and the following properties inductively, because $0\leq g_{\tau,\kappa}^k\leq g'(u_{\tau,\kappa}^k)\phi_{\tau,\kappa}^k$ a.e. on $\mathbf{R}^n$ $(k\geq 0)$.

Lemma 3.1. *Under assumption* (A_*), *the following properties hold:*
(i) $\phi_{\tau,\kappa}^k\in(L^1\cap L^{q_k})(\mathbf{R}^n)$ $(0\leq k\leq k_*-1)$.
(ii) $\phi_{\tau,\kappa}^k\in(L^1\cap C_0)(\mathbf{R}^n)$ $(k\geq k_*)$.
(iii) $0\leq\phi_{\tau,\kappa}^k\leq\phi_{\bar\tau,\bar\kappa}^k$ *a.e. on* $\mathbf{R}^n$ *for* $\tau\leq\bar\tau$, $\kappa\leq\bar\kappa$ $(k\geq 0)$.

Now we put $u=u_{\tau,\kappa}^{k_*}+w$. Then we can show the following lemma.

Lemma 3.2. *Under assumption* (A_*), $u=u_{\tau,\kappa}^{k_*}+w$ *is a solution to* $(\mathrm{P}_\tau)_\kappa$ *if and only if* $w\in C_0(\mathbf{R}^n)$ *and*

$$(3.2)\qquad w=E_1*[g(u_{\tau,\kappa}^{k_*}+w)-g(u_{\tau,\kappa}^{k_*-1})]\geq 0\quad\text{on }\mathbf{R}^n.$$

So, we define a supersolution to $(\mathrm{P}_\tau)_\kappa$ as follows. Note that a solution to $(\mathrm{P}_\tau)_\kappa$ is also a supersolution to $(\mathrm{P}_\tau)_\kappa$.

Definition 3.1. *We call* $\widetilde{u} = u^{k_*}_{\tau,\kappa} + \widetilde{w}$ *a supersolution to problem* $(\mathrm{P}_\tau)_\kappa$ *if* $\widetilde{w} \in C_0(\mathbf{R}^n)$ *and*

$$\widetilde{w} \geq E_1 * [g(u^{k_*}_{\tau,\kappa} + \widetilde{w}) - g(u^{k_*-1}_{\tau,\kappa})] \geq 0 \quad \text{on } \mathbf{R}^n. \tag{3.3}$$

If problem $(\mathrm{P}_\tau)_\kappa$ has a supersolution $\widetilde{u} = u^{k_*}_{\tau,\kappa} + \widetilde{w}$, then we have that $0 \leq \sum_{j=k_*+1}^{k} \phi^j_{\tau,\kappa} \leq \widetilde{w}$ on $\mathbf{R}^n$ $(k \geq k_* + 1)$, inductively. Moreover, we have the proposition below.

Proposition 3.1. *Under assumption* (A_*), *suppose that problem* $(\mathrm{P}_\tau)_\kappa$ *has a supersolution* $\widetilde{u} = u^{k_*}_{\tau,\kappa} + \widetilde{w}$. *Then* $w = \sum_{j=k_*+1}^{\infty} \phi^j_{\tau,\kappa}$ *converges uniformly on* $\mathbf{R}^n$. *Moreover,* $u = u^{k_*}_{\tau,\kappa} + w$ *is a solution to* $(\mathrm{P}_\tau)_\kappa$ *satisfying*

$$0 \leq w \leq \widetilde{w}, \ (\kappa\phi_* \leq)\, u^{k_*}_{\tau,\kappa} \leq u \leq \widetilde{u} \quad \text{a.e. on } \mathbf{R}^n. \tag{3.4}$$

We call u, obtained by the proposition above, a *minimal solution* to $(\mathrm{P}_\tau)_\kappa$.

Remark 3.2. (i) *In the proposition above,* $u^{k_*}_{\tau,\kappa}$ *is a subsolution to* $(\mathrm{P}_\tau)_\kappa$.
(ii) *A strictly minimal solution to* $(\mathrm{P}_\tau)_\kappa$ *is a minimal solution to* $(\mathrm{P}_\tau)_\kappa$. *(We can prove this fact by using* Proposition 4.1.*)*
(iii) *If* $\overline{u} = u^{k_*}_{\tau,\overline{\kappa}} + \overline{w}$ *is a solution to* $(\mathrm{P}_\tau)_{\overline{\kappa}}$, *then* $\widetilde{u} = u^{k_*}_{\tau,\kappa} + \overline{w}$ *is a supersolution to* $(\mathrm{P}_\tau)_\kappa$ *for any* $\kappa < \overline{\kappa}$. *Particularly, if* $(\mathrm{P}_\tau)_\kappa$ *has a solution, then* $(\mathrm{P}_\tau)_\kappa$ *also has a minimal solution. Moreover,* Theorem 1.1 (ii) *implies* Theorem 1.1 (iv) *by virtue of* Proposition 1.1 (iv).

§4. Invertibility of linearized operators.

In this section, by using the compactness of supp f_*, we describe the invertibility of the linearized operators of a given solutioin to problem $(\mathrm{P}_\tau)_\kappa$ in a precise sense. This property is useful for the proof of Step 2 and related properties (cf. §5). Now we introduce a radial function $e_1 \in C^\infty(\mathbf{R}^n)$ satisfying

$$e_1(x) = \begin{cases} 1 & \text{for } 0 \leq |x| \ll 1, \\ E_1(x) & \text{for } |x| \gg 1, \end{cases} \quad \frac{\partial e_1}{\partial r} \leq 0 \quad \text{on } \mathbf{R}^n \backslash \{0\}. \tag{4.1}$$

Since supp $f_* \subset B_{R_*}$, we can show the decay properties of solutions below.

Lemma 4.1. *Under assumption* (A_*), *the following properties hold:*
(i) *It holds* $\phi^k_{\tau,\kappa}/e_1 \in L^\infty(\mathbf{R}^n \backslash \overline{B_{R_*}})$ $(k \geq 0)$. *Particularly,* $\phi^k_{\tau,\kappa}/e_1 \in BC(\mathbf{R}^n)$ $(k \geq k_*)$.
(ii) *If* $u = u^{k_*}_{\tau,\kappa} + w$ *is a non-zero solution to* $(\mathrm{P}_\tau)_\kappa$ *and* φ *is a solution to* $(\mathrm{L}\,; u)^\lambda$ *with some* $\lambda \in (0, \infty)$, *then* $w/e_1, \varphi/e_1 \in BC(\mathbf{R}^n)$.

For a non-zero solution u to $(\mathrm{P}_\tau)_\kappa$ we define

$$\Phi[u]\xi = \frac{1}{e_1} E_1 * [g'(u)\xi e_1] \quad \text{for } \xi \in BC(\mathbf{R}^n), \tag{4.2}$$

and consider the invertibility of the operator $I - \lambda\Phi[u]$ in $BC(\mathbf{R}^n)$ or its closed subspace. The following lemma is the key point of the argument in this section, which can be proved by the similar way to [8, Proposition 4.1].

Lemma 4.2. *Assume* (A_*), $\nu \in (0,1)$ *and* $\bar{q} \in ((q_0/(p-1))', \infty)$. *If u is a solution to* $(\mathrm{P}_\tau)_\kappa$, *then the operator* $\Psi_\nu[u] : L^{\bar{q}}(\mathbf{R}^n) \to L^{\bar{q}}(\mathbf{R}^n)$ *is compact, where*

$$\Psi_\nu[u]\psi = \frac{1}{e_1^{1-\nu}} E_1 * [g'(u)e_1^{1-\nu}\psi] \quad \text{for } \psi \in L^{\bar{q}}(\mathbf{R}^n). \tag{4.3}$$

Now we denote

$$[\phi] = \{\, a\phi \mid a \in \mathbf{R} \,\} \quad \text{and} \quad [\phi]_q^{\perp} = \left\{ \psi \in L^{q'}(\mathbf{R}^n) \,\middle|\, \int_{\mathbf{R}^n} \psi\phi\, dm = 0 \right\} \quad \text{for } \phi \in L^q(\mathbf{R}^n).$$

With the aid of Fredholm's alternative, we can see the lemma below.

Lemma 4.3. *Assume* (A_*), $\nu \in (0,1)$ *and* $\bar{q} \in ((q_0/(p-1))', \infty)$. *Let u be a non-zero solution to* $(\mathrm{P}_\tau)_\kappa$ *and* φ^1 *be a positive solution to* $(\mathrm{L}\,; u)^{\lambda^1[u]}$. *Then the following properties hold:*

(i) *There hold*

$$\mathrm{Ker}\,(I - \lambda^1[u]\Psi_\nu[u]) = [\overline{\psi}_\nu] \quad \text{and} \quad (I - \lambda^1[u]\Psi_\nu[u])(L^{\bar{q}}(\mathbf{R}^n)) = [\overline{\psi}_\nu^*]_{\bar{q}'}^{\perp},$$

where $\overline{\psi}_\nu = \varphi^1/e_1^{1-\nu} \in L^{\bar{q}}(\mathbf{R}^n)$ *and* $\overline{\psi}_\nu^* = g'(u)\varphi^1 e_1^{1-\nu} \in L^{\bar{q}'}(\mathbf{R}^n)$. *Particularly, operator* $\Phi_\nu^1[u] = (I - \lambda^1[u]\Psi_\nu[u])|_{[\overline{\psi}_\nu^*]_{\bar{q}'}^{\perp}} : [\overline{\psi}_\nu^*]_{\bar{q}'}^{\perp} \to [\overline{\psi}_\nu^*]_{\bar{q}'}^{\perp}$ *is invertible.*

(ii) *If* $\lambda^1[u] \in (1,\infty)$, *then operator* $I - \Psi_\nu[u] : L^{\bar{q}}(\mathbf{R}^n) \to L^{\bar{q}}(\mathbf{R}^n)$ *is also invertible.*

Note that $1/\alpha_* > (q_0/(p-1))'$. Now we assume $\nu \in (0, \min\{1, p-1\})$, $\bar{q} \in (1/\alpha_*, \infty)$ and define

$$J^1[u]\eta = \frac{1}{e_1^\nu} \Phi_\nu^1[u]^{-1}(e_1^\nu \eta) \quad \text{for } \eta \in \Lambda^1[u], \tag{4.4}$$

where

$$\Lambda^1[u] = \left\{ \eta \in BC(\mathbf{R}^n) \,\middle|\, \int_{\mathbf{R}^n} g'(u)\varphi^1 \eta e_1\, dm = 0 \right\}. \tag{4.5}$$

(Since $e_1^\nu \eta \in [\overline{\psi}_\nu^*]_{\bar{q}'}^{\perp}$ for $\eta \in \Lambda^1[u]$, operator $J^1[u]$ is well-defined.) We also define

$$J[u]\xi = \frac{1}{e_1^\nu}[I - \Psi_\nu[u]]^{-1}(e_1^\nu \xi) \quad \text{for } \xi \in BC(\mathbf{R}^n), \tag{4.6}$$

provided that $\lambda^1[u] \in (1,\infty)$. With the aid of Lemma 4.3 and the estimate

$$\left\| \frac{1}{e_1} E_1 * [g'(u)e_1^{1-\nu}\psi] \right\|_\infty \le c_{\nu,\bar{q}}[u] \|\psi\|_{\bar{q}} \quad \text{for } \psi \in L^{\bar{q}}(\mathbf{R}^n), \tag{4.7}$$

we can show the following proposition.

Proposition 4.1. *Assume* (A_*), $\nu \in (0, \min\{1, p-1\})$ *and* $\bar{q} \in (1/\alpha_*, \infty)$. *Let u be a non-zero solution to* $(\mathrm{P}_\tau)_\kappa$ *and* φ^1 *be a positive solution to* $(\mathrm{L}\,; u)^{\lambda^1[u]}$. *Then the following properties hold:*

(i) *Operator* $\Phi[u] : BC(\mathbf{R}^n) \to BC(\mathbf{R}^n)$ *is bounded.*

(ii) *There hold*

$$\mathrm{Ker}\,(I - \lambda^1[u]\Phi[u]) = \left[\frac{\varphi^1}{e_1}\right] \quad \text{and} \quad (I - \lambda^1[u]\Phi[u])(BC(\mathbf{R}^n)) \subset \Lambda^1[u].$$

Moreover, $J^1[u]$ *is a bounded right inverse operator of* $(I - \lambda^1[u]\Phi[u])|_{\Lambda^1[u]} : \Lambda^1[u] \to$

(iii) *If* $\lambda^1[u] > 1$, *then* $J[u]$ *is a bounded right inverse operator of* $I - \Phi[u] : BC(\mathbf{R}^n) \to BC(\mathbf{R}^n)$.

For the proof of (ii), we set

$$\widetilde{\Phi}^1[u]\eta = \frac{1}{e_1} E_1 * [g'(u)(J^1[u]\eta)e_1] \quad \text{for } \eta \in \Lambda^1[u]. \tag{4.8}$$

By virtue of (4.7) and the boundedness of $\Phi^1_\nu[u]^{-1}$ on $[\overline{\psi}^*_\nu]^\perp_{\overline{q}'}$, we can see that $\widetilde{\Phi}^1[u] : \Lambda^1[u] \to BC(\mathbf{R}^n)$ is bounded. Then we have that $J^1[u] = I|_{\Lambda^1[u]} + \lambda^1[u]\widetilde{\Phi}^1[u]$ and $J^1[u]$ is also bounded in $\Lambda^1[u]$. So, there hold $\widetilde{\Phi}^1[u] = \Phi[u]J^1[u]$ and $(I - \lambda^1[u]\Phi[u])J^1[u] = I|_{\Lambda^1[u]}$. Similarly, we can prove assertion (iii).

We can prove Step 2 by using the proposition above, which will be shown in the next section. Now we give the proof of Proposition 2.1.

Proof of Proposition 2.1. (i) Suppose that $\overline{u}$ is another solution to $(\mathrm{P}_\tau)_\kappa$ and put $\xi = (\overline{u} - u)/e_1$. Then it holds $\xi \in BC(\mathbf{R}^n)$ and, from the convexity of g, we have that

$$(I - \Phi[u])\xi = \frac{1}{e_1} E_1 * [g(\overline{u}) - g(u) - g'(u)(\overline{u} - u)] \geq 0 \quad \text{on } \mathbf{R}^n.$$

On the other hand, Proposition 4.1 (ii) implies that $(I - \Phi[u])\xi \in \Lambda^1[u]$, and it follows $\overline{u} \equiv u$ on $\mathbf{R}^n$.

(ii) Suppose that $\tau \in (0,1]$ and problem $(\mathrm{P}_\tau)_{\overline{\kappa}}$ has a solution $\overline{u} = u^{k_*}_{\tau,\overline{\kappa}} + \overline{w}$ for some $\overline{\kappa} > \kappa$. Then $\widetilde{u} = u^{k_*}_{\tau,\kappa} + \overline{w}$ is a supersolution to $(\mathrm{P}_\tau)_\kappa$ by virtue of Remark 3.2 (iii), and we have that $u \leq \widetilde{u}$ a.e. on $\mathbf{R}^n$ with the aid of (i) and Proposition 3.1. By putting $\xi = (\widetilde{u} - u)/e_1$, it holds $\xi \in BC(\mathbf{R}^n)$ and, by virtue of (3.2) and the convexity of g, we have that

$$(I - \Phi[u])\xi \geq \frac{1}{e_1} E_1 * [g'(u)(\phi^{k_*}_{\tau,\kappa} - \phi^{k_*}_{\tau,\overline{\kappa}})] > 0 \quad \text{on } \mathbf{R}^n.$$

This contradicts that $(I - \Phi[u])\xi \in \Lambda^1[u]$. q.e.d.

§5. Openness of T^*.

In this section we give the sketch of the proof of Step 2 and construct strictly minimal solutions near a given one along the parameters κ and τ. Note that, if problem $(\mathrm{Q}_\tau)^*$ has a solution, then it is unique up to constant multiplication of φ^1. So we denote a solution to $(\mathrm{Q}_\tau)^*$ by $(u^*_\tau, \varphi^*_\tau; \kappa^*_\tau)$ for $\tau \in T^*$, and set $w^*_\tau = u^*_\tau - u^{k_*}_{\tau,\kappa^*_\tau}$. Then it holds from Proposition 2.1 that

$$\kappa^*_\tau = \sup\{\, \kappa > 0 \mid \text{problem } (\mathrm{P}_\tau)_\kappa \text{ has a solution} \,\} \quad \text{for } \tau \in T^* \setminus \{0\}. \tag{5.1}$$

When $(\mathrm{P}_\tau)_\kappa$ has a solution, we denote the minimal solution to $(\mathrm{P}_\tau)_\kappa$ by $u_{\tau,\kappa}$, and set $w_{\tau,\kappa} = u_{\tau,\kappa} - u^{k_*}_{\tau,\kappa}$. Moreover, we denote $\lambda_{\tau,\kappa} = \lambda^1[u_{\tau,\kappa}]$ and a positive solution to $(\mathrm{L}\,; u_{\tau,\kappa})^{\lambda_{\tau,\kappa}}$ by $\varphi_{\tau,\kappa}$, provided that $\kappa > 0$.

First we show the openness of T^* in $[0,1]$. That is, for any given $\tau \in T^*$, we construct a solution to $(\mathrm{Q}_{\tau+\varepsilon})^*$ for $|\varepsilon| \ll 1$. By using Proposition 4.1 (ii), we can show the proposition below, which implies Step 2. Here, we omit the precise proof. Note that $u^*_0 = \kappa^*_0\phi_*$ and $w^*_0 = 0$ for the case $\tau = 0$.

Proposition 5.1. *Under assumption* (A_*), *the following properties hold:*
(i) *There exists a positive constant* ε_0 *such that* $\varepsilon^2 \in T^*$ *for* $\varepsilon \in [0, \varepsilon_0]$. *Moreover, a solution to* $(\mathrm{Q}_{\varepsilon^2})^*$ *is expressed by*

$$(5.2)\qquad \begin{cases} (w^*_{\varepsilon^2}, \varphi^*_{\varepsilon^2}\,; \kappa^*_{\varepsilon^2}) = (\varepsilon(\sigma_0^\varepsilon)^{1/2}\varphi_0^* + \varepsilon^2\xi_0^\varepsilon e_1, \varphi_0^* + \varepsilon\eta_0^\varepsilon e_1\,; \kappa_0^* - \varepsilon\rho_0^\varepsilon), \\ (\xi_0^\varepsilon, \eta_0^\varepsilon\,; \sigma_0^\varepsilon, \rho_0^\varepsilon) \in \Lambda^1[u_0^*]^2 \times (0,\infty)^2. \end{cases}$$

(ii) *If* $\tau \in T^*\backslash\{0,1\}$, *then there exists a positive constant* ε_τ *such that* $\tau + \varepsilon \in T^*$ *for* $\varepsilon \in [-\varepsilon_\tau, \varepsilon_\tau]$. *Moreover, a solution to* $(\mathrm{Q}_{\tau+\varepsilon})^*$ *is expressed by*

$$(5.3)\qquad \begin{cases} (w^*_{\tau+\varepsilon}, \varphi^*_{\tau+\varepsilon}\,; \kappa^*_{\tau+\varepsilon}) = (w_\tau^* + \varepsilon(\sigma_\tau^\varepsilon\varphi_\tau^* + \xi_\tau^\varepsilon e_1), \varphi_\tau^* + \varepsilon\eta_\tau^\varepsilon e_1\,; \kappa_\tau^* - \varepsilon\rho_\tau^\varepsilon), \\ (\xi_\tau^\varepsilon, \eta_\tau^\varepsilon\,; \sigma_\tau^\varepsilon, \rho_\tau^\varepsilon) \in \Lambda^1[u_\tau^*]^2 \times (\mathbf{R}\times(0,\infty)). \end{cases}$$

If $1 \in T^*$, *then the same statement holds with* $\tau = 1$ *and* $[-\varepsilon_1, 0]$ *instead of* $[-\varepsilon_\tau, \varepsilon_\tau]$.

Moreover, we can show the lemma below, by using Proposition 4.1 (ii)–(iii).

Lemma 5.1. *Assume* (A_*) *and that* $(\mathrm{P}_\tau)_\kappa$ *has a strictly minimal solution* $u_{\tau,\kappa}$. *Then the following properties hold:*
(i) *If* $\kappa > 0$, *then there exists a positive constant* $\widetilde{\varepsilon}_{\tau,\kappa}$ *such that problem* $(\mathrm{P}_\tau)_{\kappa+\varepsilon}$ *has a strictly minimal solution* $u_{\tau,\kappa+\varepsilon}$ *for* $\varepsilon \in [-\widetilde{\varepsilon}_{\tau,\kappa}, \widetilde{\varepsilon}_{\tau,\kappa}]$.
(ii) *If* $\tau \in (0,1)$, *then there exists a positive constant* $\bar{\varepsilon}_{\tau,\kappa}$ *such that problem* $(\mathrm{P}_{\tau+\varepsilon})_\kappa$ *has a strictly minimal solution* $u_{\tau+\varepsilon,\kappa}$ *for* $\varepsilon \in [-\bar{\varepsilon}_{\tau,\kappa}, \bar{\varepsilon}_{\tau,\kappa}]$. *If* $\tau = 0$ *or* $\tau = 1$, *then the same statement holds with* $[0, \bar{\varepsilon}_{0,\kappa}]$ *or* $[-\bar{\varepsilon}_{1,\kappa}, 0]$ *instead of* $[-\widetilde{\varepsilon}_{\tau,\kappa}, \widetilde{\varepsilon}_{\tau,\kappa}]$, *respectively.*

Remark 5.1. *Also in the case* $\kappa = 0$, *we can construct a strictly minimal solution* $u_{\tau,\varepsilon}$ *to* $(\mathrm{P}_\tau)_\varepsilon$ *for* $0 < \varepsilon \ll 1$ *near the solution* $u_{\tau,0} \equiv 0$ *to* $(\mathrm{P}_\tau)_0$.

§6. A priori estimate.

Next we show the closedness of T^*. In this section we give the a priori estimate for solutions to $\{(\mathrm{Q}_\tau)^*\}_{\tau\in T^*}$ under assumption (A_*) with $q_0 > q_*(p)$. Since $1 < p < \bar{p}^*$, we have that $q_*(p) < \bar{q}_*(p)$ and we may assume that $q_0 \in (q_*(p), \bar{q}_*(p))$, where

$$(6.1)\qquad \bar{q}_*(p) = \begin{cases} p^* & \text{if } 1 < p < \underline{p}^*, \\ \dfrac{p^*+1}{1 - 1/(p')^{1/2}} & \text{if } \underline{p}^* \le p < \bar{p}^*. \end{cases}$$

Our purpose in this section is the following.

Proposition 6.1. *Assume* (A_*) *with* $q_0 \in (q_*(p), \bar{q}_*(p))$. *Then* $\{w_\tau^*\}_{\tau\in T^*}$ *is uniformly bounded and equi-continuous on* $\mathbf{R}^n$.

We denote the *translation operator* by τ_z for $z \in \mathbf{R}^n$, that is,

$$\tau_z v(x) = v(x-z) \quad \text{for } x \in \mathbf{R}^n.$$

By using the elliptic regularity argument similarly in Lemma 3.1, we can show the lemma below.

Lemma 6.1. *Under assumption* (A_*), *if* $\{w_\tau^*\tau_z[e_1^\nu]\}_{\tau\in T^*, z\in\mathbf{R}^n}$ *is bounded in* $L^q(\mathbf{R}^n)$ *for some* $q \in [q_0, \infty)$ *and sufficiently small* $\nu \in (0,1)$, *then* $\{w_\tau^*\}_{\tau\in T^*}$ *is uniformly bounded and equi-continuous on* $\mathbf{R}^n$.

By using Proposition 4.1 (ii), we can obtain the a priori estimate for $\{\kappa_\tau^*\}_{\tau\in T^*}$.

Lemma 6.2. *Under assumption* (A_*), *if* $\tau, \overline{\tau} \in T^*$ *and* $\tau < \overline{\tau}$, *then it holds* $\kappa_\tau^* > \kappa_{\overline{\tau}}^*$. *Particularly,*

$$0 < \kappa_\tau^* \le \kappa_0^* \quad \text{for } \tau \in T^*. \tag{6.2}$$

Combining with Lemma 3.1 (iii) we have that $\{u_{\tau,\kappa_\tau^*}^{k_*-1}\}_{\tau\in T^*}$ and $\{\phi_{\tau,\kappa_\tau^*}^{k_*}\}_{\tau\in T^*}$ are bounded in $L^{q_0}(\mathbf{R}^n)$ and $L^\infty(\mathbf{R}^n)$, respectively. Note that, for $\tau \in T^*$, there holds

$$\begin{cases} -\Delta w_\tau^* + w_\tau^* = g(u_\tau^*) - g(u_{\tau,\kappa_\tau^*}^{k_*-1}) & \text{in } \mathcal{D}'(\mathbf{R}^n), \\ w_\tau^* = E_1 * [g(u_\tau^*) - g(u_{\tau,\kappa_\tau^*}^{k_*-1})] > 0 & \text{on } \mathbf{R}^n \end{cases} \tag{6.3}$$

and (1.8) implies that

$$\|g'(u_\tau^*)v^2\|_1 \le \|\nabla v\|_2^2 + \|v\|_2^2 = \|v\|_{1,2}^2 \quad \text{for } v \in H^1(\mathbf{R}^n), \tag{6.4}$$

because $\lambda^1[u_\tau^*] = 1$. Now we show the assumption of Lemma 6.1. We devide into two cases.

Case 1. $1 < p < \underline{p}^*$.

Since $q_0 < \underline{p}^*$, there exists a positive constant $\bar{c}_\nu$ such that

$$\|(E_1 * v)\tau_z[e_1^\nu]\|_{q_0} \le \bar{c}_\nu \|v\tau_z[e_1^\nu]\|_1 \quad \text{for } v \in L^1(\mathbf{R}^n). \tag{6.5}$$

Proof of Proposition 6.1 (i). The case $1 < p < \underline{p}^*$.

We multiply the first expression of (6.3) by $\tau_z[e_1^\nu]$ and integrate over $\mathbf{R}^n$. By using integration by part and Young's inequality, we see that

$$\begin{aligned} \|(g(u_\tau^*) - g(u_{\tau,\kappa_\tau^*}^{k_*-1}))\tau_z[e_1^\nu]\|_1 &= \int_{\mathbf{R}^n} w_\tau^* \tau_z[-\Delta[e_1^\nu] + e_1^\nu]dm \le c_\nu \|w_\tau^*\tau_z[e_1^\nu]\|_1 \\ &\le c_\nu \Big\| \Big(\frac{1}{p}(\varepsilon w_\tau^*)^p + \frac{1}{p'}\Big(\frac{1}{\varepsilon}\Big)^{p'}\Big)\tau_z[e_1^\nu]\Big\|_1 \le \frac{c_\nu}{p}\varepsilon^p \|g(u_\tau^*)\tau_z[e_1^\nu]\|_1 + \frac{c_\nu}{p'\varepsilon^{p'}}\|e_1^\nu\|_1 \end{aligned}$$

for any $\varepsilon > 0$. Combining with (6.2) and Lemma 3.1 (iii) we have that

$$\Big(1 - \frac{c_\nu}{p}\varepsilon^p\Big)\|g(u_\tau^*)\tau_z[e_1^\nu]\|_1 \le \|g(u_{1,\kappa_0^*}^{k_*-1})\|_1 + \frac{c_\nu}{p'\varepsilon^{p'}}\|e_1^\nu\|_1.$$

On the other hand, we see from (6.3) and (6.5) that

$$\|w_\tau^*\tau_z[e_1^\nu]\|_{q_0} \le \|E_1 * [g(u_\tau^*)]\tau_z[e_1^\nu]\|_{q_0} \le \bar{c}_\nu \|g(u_\tau^*)\tau_z[e_1^\nu]\|_1 \quad \text{for all } \tau \in T^*,\ z \in \mathbf{R}^n,$$

and hence $\{w_\tau^*\tau_z[e_1^\nu]\}_{\tau\in T^*, z\in\mathbf{R}^n}$ is bounded in $L^{q_0}(\mathbf{R}^n)$ by choosing $\varepsilon > 0$ small. Then the assertion follows from Lemma 6.1. q.e.d.

Case 2. $\underline{p}^* \le p < \overline{p}^*$.

We are going to obtain the boundedness of $\{(w_\tau^*\tau_z[e_1^\nu])^r\}_{\tau\in T^*, z\in\mathbf{R}^n}$ in $H^1(\mathbf{R}^n)$ for some $r \in [r_0, \infty)$ by multiplying the first expression of (6.3) by $p(w_\tau^*)^{2r-1}\tau_z[e_1^{2r\nu}]$ and integrating over $\mathbf{R}^n$, where $r_0 = q_0/(p^* + 1)$. Once we obtain such estimate, Sobolev's inequality implies that $\{w_\tau^*\tau_z[e_1^\nu]\}_{\tau\in T^*, z\in\mathbf{R}^n}$ is bounded in $L^{(p^*+1)r}(\mathbf{R}^n)$ with $(p^*+1)r \in [q_0, \infty)$, and the assertion follows from Lemma 6.1.

First we observe the inequalities below, which are concerned with the nonlinearity function g. We use these inequalities with $s = u_{\tau,\kappa_\tau^*}^{k_*-1}(x)$ and $t = [w_\tau^* + \phi_{\tau,\kappa_\tau^*}^{k_*}](x)$ for

Lemma 6.3. (i) *It holds*

$$p(g(s+t)-g(s)) \le g'(s+t)t + (g'(s+t)-g'(s))s \quad \textit{for } s,t \ge 0.$$

(ii) *When* $p<2$, *it holds*

$$(g'(s+t)-g'(s))s \le pst^{p-1} \quad \textit{for } s,t \ge 0.$$

(iii) *When* $p \ge 2$, *for any* $\varepsilon>0$, *there exists a positive constant* $C(\varepsilon)$ *such that*

$$(g'(s+t)-g'(s))s \le \varepsilon g'(s+t)t + C(\varepsilon)s^p \quad \textit{for } s,t \ge 0.$$

By using (ii)–(iii) of the lemma above, (6.4), Sobolev's inequality. Hölder's inequality and Young's inequality, we can show the estimate below.

Lemma 6.4. *Assume* (A_*), $p \ge \underline{p}^*$ *and that* $r \in [r_0,\infty)$ *satisfies*

$$\frac{1}{r} \ge \frac{1}{2-p}\Big(\frac{1}{r_0}-(p^*-1)\Big) \ \textit{if } p<2, \quad \frac{1}{r} \ge \frac{p}{r_0}-(p^*-1) \ \textit{if } p \ge 2. \tag{6.6}$$

Then, for any $\varepsilon>0$, *there exists a positive constant* $C^*_{\nu,r}(\varepsilon)$ *such that*

$$\|g'(u^*_\tau)\phi^{k_*}_{\tau,\kappa^*_\tau}(w^*_\tau)^{2r-1}\tau_z[e_1^{2r\nu}]\|_1 \le \varepsilon\|(w^*_\tau\tau_z[e_1^\nu])^r\|^2_{1,2} + C^*_{\nu,r}(\varepsilon), \tag{6.7}$$

$$\|(g(u^*_\tau)-g(u^{k_*-1}_{\tau,\kappa^*_\tau}))u^{k_*-1}_{\tau,\kappa^*_\tau}(w^*_\tau)^{2r-1}\tau_z[e_1^{2r\nu}]\|_1 \le \varepsilon\|(w^*_\tau\tau_z[e_1^\nu])^r\|^2_{1,2} + C^*_{\nu,r}(\varepsilon) \tag{6.8}$$

$$\textit{for all } \tau \in T^*,\ z \in \mathbf{R}^n.$$

By making use of the estimate above, we can show the following lemma.

Lemma 6.5. *Assume* (A_*), $p \ge \underline{p}^*$ *and that* $r \in [r_0,\infty)$ *satisfies* (6.6) *and*

$$p\Big(\frac{2}{r}-\frac{1}{r^2}\Big) > 1. \tag{6.9}$$

Then $\{(w^*_\tau\tau_z[e_1^\nu])^r\}_{\tau\in T^*, z\in\mathbf{R}^n}$ *is bounded in* $H^1(\mathbf{R}^n)$, *provided that* $\nu \in (0,1)$ *is sufficiently small.*

Proof. We multiply the first expression of (6.3) by $p(w^*_\tau)^{2r-1}\tau_z[e_1^{2r\nu}]$ and integrate over $\mathbf{R}^n$. Then we can see from (6.4), Lemmas 6.3 and Lemma 6.4 that

$$\begin{aligned}
&p\Big(\frac{2}{r}-\frac{1}{r^2}-c\nu^2\Big)\|(w^*_\tau\tau_z[e_1^\nu])^r\|^2_{1,2} \le p\int_{\mathbf{R}^n}(-\Delta w^*_\tau + w^*_\tau)(w^*_\tau)^{2r-1}\tau_z[e_1^{2r\nu}]dm\\
&= p\|(g(u^*_\tau)-g(u^{k_*-1}_{\tau,\kappa^*_\tau}))(w^*_\tau)^{2r-1}\tau_z[e_1^{2r\nu}]\|_1\\
&\le \|g'(u^*_\tau)(w^*_\tau\tau_z[e_1^\nu])^{2r}\|_1 + \|g'(u^*_\tau)\phi^{k_*}_{\tau,\kappa^*_\tau}(w^*_\tau)^{2r-1}\tau_z[e_1^{2r\nu}]\|_1\\
&\quad + \|(g(u^*_\tau)-g(u^{k_*-1}_{\tau,\kappa^*_\tau}))u^{k_*-1}_{\tau,\kappa^*_\tau}(w^*_\tau)^{2r-1}\tau_z[e_1^{2r\nu}]\|_1\\
&\le (1+2\varepsilon)\|(w^*_\tau\tau_z[e_1^\nu])^r\|^2_{1,2} + 2C^*_{\nu,r}(\varepsilon) \quad \text{for all } \tau\in T^*,\ z\in\mathbf{R}^n.
\end{aligned}$$

Because of (6.9), $\{(w^*_\tau\tau_z[e_1^\nu])^r\}_{\tau\in T^*, z\in\mathbf{R}^n}$ is bounded in $H^1(\mathbf{R}^n)$ by choosing ν and ε small. q.e.d.

Note that (6.9) is equivalent to that $(p^*+1)r < \bar{q}_*(p)$. Moreover, if $\underline{p}^* \le p < \overline{p}^*$ and $q_0 \in (q_*(p),\bar{q}_*(p))$, then there exists $r \in [r_0,\infty)$ satisfying (6.6) and (6.9), and the assumption of Lemma 6.1 holds. Therefore, Proposition 6.1 has proved.

§7. Plane reflection method.

In this section we explain the *plane reflection method*, which is useful for the control of the behavior of solutions at infinity. Here, we use the compactness of $\operatorname{supp} f_*$ essentially. Step 3 can be proved by using the argument below, together with the a priori estimate obtained in the previous section.

Definition 7.1. *Let $\omega \in S^{n-1}$ and $a \in (0,\infty)$.*
(i) *We set*

$$H^{\omega,a} = \{\, x \in \mathbf{R}^n \mid x{\cdot}\omega < a \,\} \quad \text{and} \quad x^{\omega,a} = x + 2(a - x{\cdot}\omega)\omega \quad \text{for } x \in \mathbf{R}^n.$$

(ii) *We say that a function v on $\mathbf{R}^n$ satisfies condition* $(\mathrm{H})^{\omega,a}$ *if*

$$v(x) \geq v(x^{\omega,a}) \quad \text{for a.e. } x \in H^{\omega,a}.$$

Remark 7.1. (i) *Note that $x^{\omega,a}$ is the reflection point of x about the hyperplane $\partial H^{\omega,a}$, and hence $(x^{\omega,a})^{\omega,a} = x$.*
(ii) *If $\operatorname{supp} f_* \subset B_{R_*}$, then ϕ_* satisfies condition $(\mathrm{H})^{\omega,a}$ for any $\omega \in S^{n-1}$ and $a \geq R_*$. (This fact can be proved by the similar way to* Lemma 7.1 (i).*)*

The next lemma is the key point of the argument in this section.

Lemma 7.1. *Assume* (A_*) *and that ϕ_* satisfies condition $(\mathrm{H})^{\omega,a}$ for fixed $\omega \in S^{n-1}$ and $a > 0$. Then the following properties hold:*
(i) *For any $\tau \in [0,1]$ and $\kappa \geq 0$, $\phi^k_{\tau,\kappa}$ satisfies condition $(\mathrm{H})^{\omega,a}$ $(k \geq 0)$.*
(ii) *If $(\mathrm{P}_\tau)_\kappa$ has a solution, then $u_{\tau,\kappa}$ and $w_{\tau,\kappa}$ also satisfy condition $(\mathrm{H})^{\omega,a}$.*

Proof. (i) Let $k \geq 0$ and suppose that $\phi^0_{\tau,\kappa}, \phi^1_{\tau,\kappa}, \ldots, \phi^k_{\tau,\kappa}$ satisfy condition $(\mathrm{H})^{\omega,a}$. Then we can see that $g^k_{\tau,\kappa} = g(u^k_{\tau,\kappa}) - g(u^{k-1}_{\tau,\kappa})$ also satisfies condition $(\mathrm{H})^{\omega,a}$ by virtue of the convexity of g. Since

$$\text{(7.1)} \qquad \begin{cases} |x - y^{\omega,a}| = |x^{\omega,a} - y|, \ |x - y| = |x^{\omega,a} - y^{\omega,a}| & \text{for } x, y \in \mathbf{R}^n, \\ |x - y| \leq |x^{\omega,a} - y| \quad \text{for } x, y \in H^{\omega,a}, \end{cases}$$

we can obtain

$$\begin{aligned}
&\phi^{k+1}_{\tau,\kappa}(x) - \phi^{k+1}_{\tau,\kappa}(x^{\omega,a}) = E_1 * g^k_{\tau,\kappa}(x) - E_1 * g^k_{\tau,\kappa}(x^{\omega,a}) \\
&= \int_{H^{\omega,a}} (E_1(x-y) - E_1(x^{\omega,a} - y)) g^k_{\tau,\kappa}(y)\,dm(y) \\
&\quad - \int_{\mathbf{R}^n \setminus \overline{H^{\omega,a}}} (E_1(x - y^{\omega,a}) - E_1(x^{\omega,a} - y^{\omega,a})) g^k_{\tau,\kappa}(y)\,dm(y) \\
&= \int_{H^{\omega,a}} (E_1(x-\eta) - E_1(x^{\omega,a} - \eta))(g^k_{\tau,\kappa}(\eta) - g^k_{\tau,\kappa}(\eta^{\omega,a}))\,dm(\eta) \\
&\geq 0 \quad \text{for a.e. } x \in H^{\omega,a},
\end{aligned}$$

by using (3.1) and the change of variables $\eta = y^{\omega,a}$ for $y \in \mathbf{R}^n \setminus H^{\omega,a}$. Therefore, $\phi^{k+1}_{\tau,\kappa}$ also satisfies condition $(\mathrm{H})^{\omega,a}$.
(ii) It is trivial from Proposition 3.1. q.e.d.

moreover, we can show the lemma below.

Lemma 7.2. *Assume* (A_*) *and that a function v on $\mathbf{R}^n$ satisfies condition $(\mathrm{H})^{\omega,a}$ for any $\omega \in S^{n-1}$ and $a \in [R_*,\infty)$. Then the following properties hold:*

(i) $v(x) \geq v(x+t\omega)$ *if* $x \in \mathbf{R}^n \setminus H^{\omega, R_*}$, $\omega \in S^{n-1}$, $t > 0$.
(ii) $v(x) \geq v(y)$ *if* $|y| \geq 4|x| + 3(2+\sqrt{2})R_*$, $|x| \geq \sqrt{2}R_*$.
(iii) *There exists a limit* $v_\infty = \lim_{r\to\infty} S[v](r) = \lim_{|x|\to\infty} v(x) \in [-\infty, \infty)$ *and there holds* $v \geq v_\infty$ *on* $\mathbf{R}^n$, *where*

$$S[v](r) = \frac{1}{nm(B_1)} \int_{S^{n-1}} v(r\omega)d\sigma(\omega) \quad \text{for } r > 0 \tag{7.2}$$

and $d\sigma$ *is the surface element of* S^{n-1}.

Proof. (i) It is trivial from the definition.

(ii) We fix $\omega \in S^{n-1}$ and $r \geq \sqrt{2}R_*$ arbitrarily. For any $\varpi \in S^{n-1}$ satisfying $\omega \cdot \varpi = 0$ we can show that

$$v(r\omega) \geq v(\alpha\omega + \beta\varpi) \quad \text{for } (\alpha, \beta) \in K(r\,; R_*) \tag{7.3}$$

by using (i), where

$$\begin{cases} K(r\,;R) = \bigcup_{j=1}^{5} \Big\{ (\alpha,\beta) \in \mathbf{R}\times[0,\infty) \;\Big|\; \Big(\sin\frac{j\pi}{4}\Big)\alpha - \Big(\cos\frac{j\pi}{4}\Big)\beta \geq l_j(r\,;R) \Big\}, \\ l_1(r\,;R) = \frac{1}{\sqrt{2}}r, \ \ l_2(r\,;R) = r + R, \ \ l_3(r\,;R) = \sqrt{2}\,r + (1+\sqrt{2})R, \\ l_4(r\,;R) = 2r + (3+\sqrt{2})R, \ \ l_5(r\,;R) = 2\sqrt{2}\,r + 3(1+\sqrt{2})R. \end{cases}$$

Since

$$\mathbf{R}^n \setminus B_{4r+3(2+\sqrt{2})R_*} \subset \bigcup_{\substack{\varpi\in S^{n-1} \\ \omega\cdot\varpi=0}} \{ \alpha\omega + \beta\varpi \in \mathbf{R}^n \mid (\alpha,\beta) \in K(r\,;R_*) \}, \tag{7.4}$$

the assertion follows.

(iii) We see from (i) that v is nonincreasing in the radial direction in $\mathbf{R}^n \setminus B_{R_*}$, and hence $S[v]$ is also nonincreasing in $[R_*, \infty)$. So, there exists a limit $v_\infty = \lim_{r\to\infty} S[v](r)$ and it holds $S[v] \geq v_\infty$ on $[R_*, \infty)$. We can also see from (ii) that $v \geq v_\infty$ on $\mathbf{R}^n$, and it follows $\lim_{|x|\to\infty} v(x) = v_\infty$. q.e.d.

§8. Closedness of T^*.

In this this section we prove the closedness of T^* by using the argument in §6 and §7.

Proposition 8.1. *Assume* (A$_*$) *with* $q_*(p) < q_0 < \bar{q}_*(p)$ *and* $1 < p < \bar{p}^*$. *Then* T^* *is closed in* $[0, 1]$.

Proof. Suppose that $\{\tau_i\}_{i=1}^\infty \subset T^*$ and $\tau_i \to \tau$ as $i \to \infty$. We have to show that $\tau \in T^*$, and we may assume that $0 \notin \{\tau_i\}_{i=1}^\infty \cup \{\tau\}$ by virtue of Step 1. Then, for $i \in \mathbf{N}$, $u^*_{\tau_i}$ is a minimal solution to $(\mathrm{P}_{\tau_i})_{\kappa^*_{\tau_i}}$ satisfying $\lambda^1[u^*_{\tau_i}] = 1$. So the following properties hold from (6.4) and Lemma 7.1 (ii):

$$w^*_{\tau_i} = E_1 * [g(u^*_{\tau_i}) - g(u^{k_*-1}_{\tau_i,\kappa^*_{\tau_i}})] > 0 \quad \text{on } \mathbf{R}^n, \tag{a$_i$}$$

$$\|g'(u^*_{\tau_i})v^2\|_1 \leq \|\nabla v\|_2^2 + \|v\|_2^2 \quad \text{for all } v \in H^1(\mathbf{R}^n), \tag{b$_i$}$$

(c)$_i$ $\qquad w^*_{\tau_i}$ satisfies condition (H)$^{\omega,a}$ for any $\omega \in S^{n-1}$ and $a \in [R_*, \infty)$.

Because of Lemma 6.2 and Proposition 6.1 we can apply the Ascoli–Arzelà theorem to $\{w^*_{\tau_i}\}_{i=1}^{\infty}$ on any compact subset of $\mathbf{R}^n$. So, by choosing a subsequence, we may assume that

$$\kappa^*_{\tau_i} \to \kappa \quad \text{and} \quad w^*_{\tau_i} \to w \quad \text{locally uniformly on } \mathbf{R}^n \quad \text{as } i \to \infty$$

for some $\kappa \geq 0$ and $w \in BC(\mathbf{R}^n)$. With the aid of the dominated convergence theorem we can show the properties below by letting $i \to \infty$ in (a)$_i$, (b)$_i$, (c)$_i$:

(a) $\qquad w = E_1 * [g(u) - g(u^{k_*-1}_{\tau,\kappa})] \geq 0$ on $\mathbf{R}^n$,

(b) $\qquad \|g'(u)v^2\|_1 \leq \|\nabla v\|_2^2 + \|v\|_2^2$ for all $v \in H^1(\mathbf{R}^n)$,

(c) $\qquad w$ satisfies condition (H)$^{\omega,a}$ for any $\omega \in S^{n-1}$ and $a \in [R_*, \infty)$,

where $u = u^{k_*}_{\tau,\kappa} + w$.

(i) We see from (c) and Lemma 7.2 that there exists a limit $w_\infty = \lim_{r\to\infty} S[w](r) = \lim_{|x|\to\infty} w(x)$ and it holds $w \geq w_\infty$ on $\mathbf{R}^n$. From (a) we have that

$$S[w](r) = \frac{1}{nm(B_1)} \int_{S^{n-1}} E_1 * [g(u) - g(u^{k_*-1}_{\tau,\kappa})](r\omega) d\sigma(\omega) \to g(w_\infty) = w_\infty^p \quad \text{as } r \to \infty,$$

and hence it follows either $w_\infty = 0$ or $w_\infty = 1$. If $w_\infty = 1$, then $g'(u) \geq g'(w) \geq g'(w_\infty) = p$ on $\mathbf{R}^n$ and (b) implies that

$$(p-1)\|v\|_2^2 \leq \|g'(u)v^2\|_1 - \|v\|_2^2 \leq \|\nabla v\|_2^2 \quad \text{for all } v \in H^1(\mathbf{R}^n).$$

This means that Poincaré's inequality on $\mathbf{R}^n$ holds, which is a contradiction. So, we have that $w_\infty = 0$ and u is a solution to $(\mathrm{P}_\tau)_\kappa$.

(ii) We have from (b) that $\lambda^1[u] \in [1, \infty)$. Now we suppose that $\lambda^1[u] \in (1, \infty]$. Then u is a strictly minimal solution to $(\mathrm{P}_\tau)_\kappa$ and there exists $\bar{\kappa} > \kappa$ such that $(\mathrm{P}_\tau)_{\bar{\kappa}}$ also has a strictly minimal solution by virtue of Lemma 5.1 (i) and Remark 5.1. By using Lemma 5.1 (ii) there exists $\bar{\varepsilon} > 0$ such that $(\mathrm{P}_{\tau+\varepsilon})_{\bar{\kappa}}$ has a strictly minimal solution for $|\varepsilon| \leq \bar{\varepsilon}$. So, for sufficiently large i, we have that $\kappa^*_{\tau_i} < \bar{\kappa}$ and $|\tau_i - \tau| \leq \bar{\varepsilon}$, so that $(\mathrm{P}_{\tau_i})_{\bar{\kappa}}$ has a solution, which contradicts (5.1). Therefore, we obtain $\lambda^1[u] = 1$ and $\tau \in T^*$.

q.e.d.

Thus we have proved Step 3 and Theorem 1.1 holds true.

§9. Existence of nonminimal solutions.

In the final section we assume $1 < p < p^*$ and find a nonminimal solution $\bar{u}_{\tau,\kappa}$ to $(\mathrm{P}_\tau)_\kappa$ when a strictly minimal solution $u_{\tau,\kappa}$ exists. We are going to find a solution $\bar{u}$ in the form $\bar{u} = u + v$ with $v > 0$ on $\mathbf{R}^n$, when u is a strictly minimal solution. So we have to find a positive solution v to

$$-\Delta v + v = g(u+v) - g(u) \quad \text{in } \mathcal{D}'(\mathbf{R}^n). \tag{9.1}$$

This problem is equivalent to find a nontrivial critical point of the functional

$$I[u](v) = \frac{1}{2}(\|\nabla v\|_2^2 + \|v\|_2^2) - \|\Gamma(u,v)\|_1 \quad \text{for } v \in H^1(\mathbf{R}^n), \tag{9.2}$$

(9.3) $\Gamma(s,t) = G(s+t_+) - G(s) - g(s)t_+,\ \gamma(s,t) = g(s+t_+) - g(s)$ for $s \geq 0$, $t \in \mathbf{R}$

and

(9.4) $$G(s) = \int_0^s g(t)dt = \frac{1}{p+1}s_+^{p+1} \quad \text{for } s \in \mathbf{R}.$$

Here, we call v a *critical point* of $I[u]$ if $I[u]'(v) = 0$, where $I[u]'$ is the Fréchet derivative of $I[u]$.

Proposition 9.1. *Assume $1 < p < p^*$ and that u satisfies* (1.7) *and* $\lambda^1[u] \in (1,\infty]$. *Then functional $I[u] : H^1(\mathbf{R}^n) \to \mathbf{R}$ has a (nontrivial) critical point $v \in H^1(\mathbf{R}^n)\setminus\{0\}$.*

This proposition is proved by using the mountain pass theorem with the aid of the concentration compactness argument. Here, we only describe the key point of the proof. Note that $I[0]$ is the functional corresponding to the *problem at infinity.* Since $1 < p < p^*$, $\lambda^1[u] \in (1,\infty]$ and

(9.5) $$G(t) < \Gamma(s,t) \quad \text{for } s > 0,\ t \in \mathbf{R},$$

we can show the lemma below.

Lemma 9.1. *Assume $1 < p < p^*$ and that u satisfies* (1.7) *and* $\lambda^1[u] \in (1,\infty]$. *Then the following properties hold:*

(i) *Functional $I[u] : H^1(\mathbf{R}^n) \to \mathbf{R}$ is of class C^1 and its derivative is given by*

(9.6) $$<I[u]'(v), \phi> = \int_{\mathbf{R}^n} (\nabla v \cdot \nabla \phi + v\phi - \gamma(u,v)\phi)dm \quad \text{for } v, \phi \in H^1(\mathbf{R}^n).$$

(ii) *The origin (in $H^1(\mathbf{R}^n)$) is a local minimum of $I[u]$ and satisfies $I[u](0) = 0$.*

(iii) *There exists $\tilde{v} \in H^1(\mathbf{R}^n)\setminus\{0\}$ such that $I[u](\tilde{v}) \leq I[0](\tilde{v}) < 0$.*

Now we denote $\mathcal{P} = \{\, P \in C([0,1]; H^1(\mathbf{R}^n)) \mid P(0) = 0,\ P(1) = \tilde{v} \,\}$ and set

(9.7) $$c[u] = \inf_{P\in\mathcal{P}} \max_{t\in[0,1]} I[u](P(t)).$$

Note that $c[u] > 0$ under the assumption of Lemma 9.1.

Definition 9.1. *Let $c \in \mathbf{R}$ and u satisfies* (1.7). *We call $\{v_j\}_{j=1}^\infty \subset H^1(\mathbf{R}^n)$ a Palais–Smale sequence for $I[u]$ at level c if*

$$I[u](v_j) \to c \quad \text{and} \quad I[u]'(v_j) \to 0 \quad \text{as } j \to \infty.$$

Then we say that $I[u]$ satisfies condition $(\mathrm{PS})_c$, *which is called Palais–Smale condition at level c, if any Palais–Smale sequence for $I[u]$ at level c contains a convergent subsequence in $H^1(\mathbf{R}^n)$.*

It is well-known that there exists a critcal point $\bar{u}_0$ of $I[0]$ satisfying $I[0](\bar{u}_0) = c[0]$. By using this fact and the concentration compactness argument as in [14, Chapter 8], we can show the following lemma.

Lemma 9.2. *Assume $1 < p < p^*$ and that u is non-zero and satisfies* (1.7) *and* $\lambda^1[u] \in (1,\infty)$. *Then the following properties hold:*

(i) *For any $c > 0$, any Palais–Smale sequence for $I[u]$ at level c is bounded in $H^1(\mathbf{R}^n)$.*

(ii) $0 < c[u] < c[0]$.

(iii) *Functional* $I[u]$ *satisfies condition* $(\mathrm{PS})_{c[u]}$.

From two lemmas above we can apply the mountain pass theorem to $I[u]$ and prove Proposition 9.1. Moreover, we can obtain a nonminimal solution $\bar{u}_{\tau,\kappa}$ to $(\mathrm{P}_\tau)_\kappa$ by putting $u = u_{\tau,\kappa}$ provided that $\lambda^1[u_{\tau,\kappa}] \in (1, \infty)$. Particularly, $\bar{u}_{1,\kappa} = \bar{u}_\kappa$ is a solution required in Theorem 1.2.

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