数理解析研究所講究録 1308 巻 2003 年 39-52

Monotone Bartlett-type correction for some test statistics under nonnormality

筑波大学・数学系 青嶋 誠 (Makoto Aoshima)
Institute of Mathematics
University of Tsukuba

筑波大学大学院・数理物質科学研究科 榎 広之 (Hiroyuki Enoki)
Graduate School of Pure and Applied Sciences
University of Tsukuba

筑波大学大学院・数理物質科学研究科 伊藤 修 (Osamu Ito)
Graduate School of Pure and Applied Sciences
University of Tsukuba

Suppose that a nonnegative statistic T is asymptotically distributed as a chi-squared distribution with f degrees of freedom, χ_f^2, as a positive number n tends to infinity. We consider monotone transformations to improve chi-squared approximations under nonnormality. The transformations proposed here preserve monotonicity and give transformed statistics whose first three moments are coincident with the ones of χ_f^2 up to $O(n^{-1})$. It may be noted that the proposed transformations can be applied to a wide class of statistics whether an asymptotic expansion of T is available or not. Several examples for applications are presented to demonstrate that the proposed transformations give a significant improvement to the chi-squared approximation when compared to competitors.

Key Words and Phrases: Asymptotic expansion, Bartlett-type correction, chi-squared approximation, monotonicity, nonnormality.

1. INTRODUCTION

Suppose that a nonnegative statistic T is asymptotically distributed as a chi-squared distribution χ_f^2 with f degrees of freedom, as a positive number n tends to infinity. The *Bartlett correction* was originally proposed so that its mean is coincident with the one of χ_f^2 up to the order $O(n^{-1})$. Recently, Fujikoshi (2000) gave different transformations such that the first two moments of transformed statistics are coincident with the ones of χ_f^2 up to $O(n^{-1})$. The latter fact can be stated more concretely as follows: Suppose that the first two moments of T can be expanded as

$$E(T) = f\{1 + n^{-1}c_1 + O(n^{-2})\}, \tag{1.1}$$
$$E(T^2) = f(f+2)\{1 + n^{-1}c_2 + O(n^{-2})\}. \tag{1.2}$$

Then, for the case $\tilde{c}_2 \equiv c_2 - 2c_1 \neq 0$, Fujikoshi (2000) gave the following three transformations:

(i) For $\alpha_0 > 0$ and $n\alpha_0 + \beta_0 > 0$,

$$Y = (n\alpha_0 + \beta_0) \log\left(1 + \frac{1}{n\alpha_0}T\right); \tag{1.3}$$

(ii) For $\alpha_0 < 0$ and $n\alpha_0 + \beta_0 < 0$,

$$Y = T + \frac{1}{n}\left(\frac{\beta_0}{\alpha_0}T - \frac{1}{2\alpha_0}T^2\right); \tag{1.4}$$

(iii) For any α_0, n and β_0,

$$Y = (n\alpha_0 + \beta_0)\left\{1 - \exp\left(-\frac{1}{n\alpha_0}T\right)\right\}; \tag{1.5}$$

with

$$\alpha_0 = 2/\tilde{c}_2, \ \beta_0 = \frac{1}{2}\{(f+2)c_2 - 2(f+4)c_1\}/\tilde{c}_2. \tag{1.6}$$

Then, it holds that Y's are monotone functions of T under each parameter restriction and

$$E(Y) = f + O(n^{-2}), \ E(Y^2) = f(f+2) + O(n^{-2}). \tag{1.7}$$

Further, if T can be expanded as

$$P(T \leq x) = G_f(x) + \frac{1}{n}\sum_{j=0}^{k} a_j G_{f+2j}(x) + O(n^{-2}) \tag{1.8}$$

where k is a positive integer and $G_{f+2j}(\cdot)$ is the distribution function of χ^2_{f+2j}, Y has the asymptotic expansion given by

$$P(Y \leq x) = G_f(x) + O(n^{-2}) \tag{1.9}$$

when $k = 2$. (See also Cordeiro and Ferrari (1998).) However, there exist some test statistics such that the transformations given by (1.3), (1.4) and (1.5) with (1.6) do not work in the sense of (1.9), especially under nonnormality.

It may be noted that Bartlett-type correction, studied by Cordeiro and Ferrari (1991), Kakizawa (1996), Fujikoshi (1997) and Fujisawa (1997), for a statistic with (1.8) depends on the knowledge about k and the coefficients a_j's, and in some cases k is unknown and a_j's must be estimated in a practical use. Further, we often encounter the situations where it is difficult to obtain the coefficients a_j's in (1.8), even though its existence is assured. These situations appear in treating the distributions of multivariate test statistics under nonnormality.

In order to overcome these difficulties, Cordeiro and Ferrari (1998) supposed to obtain the third moment of T as in an expanded form,

$$E(T^3) = f(f+2)(f+4)\{1 + n^{-1}c_3 + O(n^{-2})\} \tag{1.10}$$

adding to (1.1)-(1.2) and they proposed a (2.3)-type transformation beyond the Bartlett correction, depending on the coefficients c_1, c_2 and c_3. So, such a transformation is expected to give an improvement to the chi-squared approximation than do the transformations given by (1.3), (1.4) and (1.5). In general, the problem of deriving (1.1)-(1.2) and (1.10) is more tractable than the one of deriving (1.8). Similarly, the problem of estimating the coefficients c_1, c_2 and c_3 is simpler than the one of estimating the coefficients a_j's. However, unfortunatelly, the transformation proposed by Cordeiro and Ferrari (1998) is not always monotone.

In this paper, we shall consider new transformations given by a different approach from others under the assumptions (1.1)-(1.2) and (1.10). It may be observed that new transformations, proposed in this paper, successfully preserve monotonicity and give a significant improvement to chi-squared approximation as expected. It would lead a broad application with a wide class of statistics, especially under nonnormality, where their asymptotic expansions are quite difficult to access.

This paper is organized as in the following way. In Section 2, we propose monotone transformations beyond the Bartlett correction, which are different from (1.3), (1.4) and (1.5). In Section 3, we give some distributional properties of the proposed transformations when T has an asymptotic expansion (1.8). In Section 4, numerical examples of some test statistics are demonstrated to observe an improvement brought by the proposed transformations beyond the competitors.

2. NEW TRANSFORMATIONS

For a nonnegative statistic T whose asymptotic distribution is χ_f^2, we assume that the first three moments are expanded as in (1.1)-(1.2) and (1.10), respectively. Then, for the transformations Y's given by (1.3), (1.4) and (1.5) with (1.6), it holds that

$$E(Y^3) = f(f+2)(f+4)\{1 + n^{-1}\tilde{c}_3 + O(n^{-2})\}, \tag{2.1}$$

where

$$\tilde{c}_3 = 3(c_1 - c_2) + c_3.$$

Therefore, if $\tilde{c}_2 \neq 0$ and $\tilde{c}_3 = 0$, we have that the differences among the first three moments of Y's and χ_f^2 are $O(n^{-2})$. However, if $\tilde{c}_2 \neq 0$ and $\tilde{c}_3 \neq 0$, in order to keep such an optimum property, we need to consider some other transformations beyond Bartlett correction.

Now we consider the cases $\tilde{c}_2 \neq 0$ and $\tilde{c}_3 \neq 0$. Let us consider the following transformations which were originally given for a statistic with (1.8) when $k = 3$:

(i) For $\alpha > 0$, $n\alpha + \beta > 0$ and $\gamma > 0$,

$$\tilde{T}_1 = (n\alpha + \beta)\log\left\{1 + \frac{1}{n\alpha}\left(T + \frac{\gamma}{n\alpha}T^3\right)\right\} \qquad \text{(Fujikoshi (1997));} \tag{2.2}$$

(ii) For $\alpha < 0$, $n\alpha + \beta < 0$ and $\gamma < 0$,

$$\tilde{T}_2 = T + \frac{1}{n}\left(\frac{\beta}{\alpha}T - \frac{1}{2\alpha}T^2 + \frac{\gamma}{\alpha}T^3\right) \qquad \text{(Cordeiro and Ferrari (1991)).} \tag{2.3}$$

Note that $\tilde{T}_1$ and $\tilde{T}_2$ are monotone increasing functions when the parameters α, β and γ satisfy the parameter restrictions of (i) and (ii), respectively. However, those parameter ristrictions, in which $\alpha\gamma > 0$, are very severe. So, let us propose the following new transformations:

(iii) For any α, β, γ and n,

$$\tilde{T}_3 = (n\alpha + \beta)\left\{1 - \exp\left(-\frac{1}{n\alpha}T - \frac{\gamma}{n^2\alpha^2}T^3 - \frac{9\gamma^2}{20n^3\alpha^3}T^5\right)\right\}; \tag{2.4}$$

(iv) For any α, β, γ and n,

$$\tilde{T}_4 = (n\alpha + \beta)\left(1 - \frac{\beta^2 - \beta + \gamma - 1}{n^2\alpha^2}\right)$$
$$\left\{1 - \exp\left(-\frac{1}{n\alpha}T - \frac{\gamma}{n^2\alpha^2}T^3 - \frac{9\gamma^2}{20n^3\alpha^3}T^5\right)\right\}. \tag{2.5}$$

Note that $\tilde{T}_4 = \{1 - (\beta^2 - \beta + \gamma - 1)/(n^2\alpha^2)\}\tilde{T}_3$, and $\tilde{T}_3$ and $\tilde{T}_4$ preserve monotonicity without parameter restrictions. Further, note that asymptotic expansions of four $\tilde{T}_i$'s described in (i)-(iv) are the same up to $O(n^{-1})$ and they are given by

$$\tilde{T} = T + \frac{1}{n}\left(\frac{\beta}{\alpha}T - \frac{1}{2\alpha}T^2 + \frac{\gamma}{\alpha}T^3\right) + O_p(n^{-2}). \tag{2.6}$$

Originally, $\tilde{T}_3$ is motivated to reduce the amount of the terms of $O_p(n^{-2})$ in (2.6), considering the fact that

$$\exp(-x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \cdots.$$

The expanded $+ - + - \cdots$ terms could be effective to reduce extra terms if $x > 0$. From (2.6), we have

$$E(\tilde{T}) = f\left\{1 + \frac{1}{n}\left(c_1 + \frac{\beta}{\alpha} - \frac{f+2}{2\alpha} + \frac{(f+2)(f+4)\gamma}{\alpha}\right) + O(n^{-2})\right\}, \tag{2.7}$$

$$E(\tilde{T}^2) = f(f+2)\left\{1 + \frac{1}{n}\left(c_2 + \frac{2\beta}{\alpha} - \frac{f+4}{\alpha} + \frac{2(f+4)(f+6)\gamma}{\alpha}\right) + O(n^{-2})\right\} \tag{2.8}$$

and

$$E(\tilde{T}^3) = f(f+2)(f+4)\left\{1 + \frac{1}{n}\left(c_3 + \frac{3\beta}{\alpha} - \frac{3(f+6)}{2\alpha} + \frac{3(f+6)(f+8)\gamma}{\alpha}\right)\right.$$
$$\left. + O(n^{-2})\right\}. \tag{2.9}$$

The coefficient $\{1 - (\beta^2 - \beta + \gamma - 1)/(n^2\alpha^2)\}$ appeared in $\tilde{T}_4$ is motivated to reduce the amount of the terms of $O(n^{-2})$ in (2.7)-(2.9). It might be considered to add some extra term to the inside of $\exp(\cdot)$ so that it works to cancel the terms of $O(n^{-2})$ in (2.7)-(2.9). However, there seems to be difficult to preserve monotonicity. The idea of multiplying the coefficient $\{1-(\beta^2-\beta+\gamma-1)/(n^2\alpha^2)\}$ in $\tilde{T}_4$ aims to reduce the amount of the terms of $O(n^{-2})$ simultaneously. In fact, the effect of coefficient $\{1-(\beta^2-\beta+\gamma-1)/(n^2\alpha^2)\}$ can be seen in Section 4 numerically.

Now, in order to make the terms of order n^{-1} in (2.7)-(2.9) vanish, we need to choose α, β and γ as

$$\begin{aligned} \alpha &= \frac{6}{3\tilde{c}_2-(f+4)\tilde{c}_3}, \\ \beta &= \frac{12(c_2-4c_1)+6f\tilde{c}_2-(f+2)(f+4)\tilde{c}_3}{4\{3\tilde{c}_2-(f+4)\tilde{c}_3\}}, \\ \gamma &= \frac{-\tilde{c}_3}{4\{3\tilde{c}_2-(f+4)\tilde{c}_3\}}, \end{aligned} \tag{2.10}$$

provided that $3\tilde{c}_2-(f+4)\tilde{c}_3 \neq 0$. These results can be summarized as follows:

THEOREM 1. *Suppose that a nonnegative random variate T has an asymptotic chi-squared distribution with f degrees of freedom, and its first three moments are expanded as in (1.1)-(1.2) and (1.10). For the cases that $\tilde{c}_2 \neq 0$, $\tilde{c}_3 \neq 0$ and $3\tilde{c}_2-(f+4)\tilde{c}_3 \neq 0$, let $\tilde{T}$'s be the transformations (2.2)-(2.5) with α, β and γ defined by (2.10). Then, it holds that $\tilde{T}$'s are monotone functions of T and*

$$\begin{aligned} E(\tilde{T}) &= f+O(n^{-2}), \\ E(\tilde{T}^2) &= f(f+2)+O(n^{-2}), \\ E(\tilde{T}^3) &= f(f+2)(f+4)+O(n^{-2}). \end{aligned} \tag{2.11}$$

It is easy to see that the transformation $\tilde{T}_2$ with α, β and γ defined by (2.10) is equivalent to the transformation given by Cordeiro and Ferarri (1998).

Let $t(u)$ be a function of u defined by a relation

$$P(T \le t(u)) = P(\chi_f^2 \le u). \tag{2.12}$$

Note that $P(T \le t(u)) = P(\tilde{T}(T) \le \tilde{T}(t(u)))$ and the distribution of $\tilde{T}(T)$ is close to a chi-squared distribution χ_f^2 in the sense of (2.11). This suggests that an approximation $\tilde{t}(u)$ may be proposed by $\tilde{T}(\tilde{t}(u)) = u$. Since $\tilde{t}(u)$ is an inverse function of $\tilde{T}$, we can express an approximation for (2.2) and (2.3), respectively, as follows:

(i) For $\alpha > 0$, $n\alpha+\beta > 0$ and $\gamma > 0$,

$$\tilde{t}_1(u) = \left(\frac{n^2\alpha^2}{2\gamma^2}\right)^{1/3}\left\{\left(-d_1\gamma+\sqrt{d_1^2\gamma^2+\frac{4\gamma}{27n\alpha}}\right)^{1/3}+\left(-d_1\gamma-\sqrt{d_1^2\gamma^2+\frac{4\gamma}{27n\alpha}}\right)^{1/3}\right\} \tag{2.13}$$

where $d_1 = 1-\exp(\frac{u}{n\alpha+\beta})$;

(ii) For $\alpha < 0$, $n\alpha+\beta < 0$ and $\gamma < 0$,

$$\begin{aligned} \tilde{t}_2(u) = \frac{1}{6\gamma}\Big[1 &+ \left\{1-18(n\alpha+\beta)\gamma+108n\alpha\gamma^2 u+108\gamma\sqrt{d_2}\right\}^{1/3} \\ &+ \left\{1-18(n\alpha+\beta)\gamma+108n\alpha\gamma^2 u-108\gamma\sqrt{d_2}\right\}^{1/3}\Big] \end{aligned} \tag{2.14}$$

where $d_2 = n^2\alpha^2\gamma^2u^2 + \frac{n\alpha\gamma}{3}\left(\frac{1}{18\gamma} - (n\alpha+\beta)\right)u + \frac{(n\alpha+\beta)^2}{108}(16(n\alpha+\beta)\gamma - 1)$. We note that the asymptotic expansions of $\tilde{t}(u)$ given by (2.13)-(2.14) are same up to the order $O(n^{-1})$, and they are given by

$$\tilde{t}(u) = u - \frac{1}{n}\left(\frac{\beta}{\alpha}u - \frac{1}{2\alpha}u^2 + \frac{\gamma}{\alpha}u^3\right) + O(n^{-2}). \tag{2.15}$$

Unfortunately, we cannot describe $\tilde{t}(u)$ explicitly for (2.4) and (2.5). However, those approximate values are available by conducting a numerical computation. It would be enough for a practical use. The accuracy of the approximations to the true percentage point $t(u)$ of T can be evaluated by using

$$P(T \le t(u)) = P(\tilde{T}(T) \le \tilde{T}(t(u))) = P(\tilde{T} \le u). \tag{2.16}$$

3. FURTHER PROPERTIES

In this section, we study some distributional properties of the transformed statistics $\tilde{T} = \tilde{T}(T)$ when a statistic T can be expanded as in (1.8), in addition to the assumptions of Theorem 1. Especially, we examine how much the distributions of $\tilde{T}$ are simplified and close to the distribution of χ^2_f. Before we treat the distributions of $\tilde{T}$, we give the expressions of α, β and γ in (2.10) in terms of the coefficients a_j's. Note that $\sum_{j=0}^k a_j = 0$ to get from (1.8) that

$$\begin{aligned} c_1 &= \frac{2}{f}\sum_{j=1}^k ja_j, \\ c_2 &= \frac{4}{f(f+2)}\sum_{j=1}^k j(j+f+1)a_j, \\ c_3 &= \frac{8}{f(f+2)(f+4)}\sum_{j=1}^k j^2(j+f+1)a_j \\ &\quad + \frac{4}{f(f+2)}\sum_{j=1}^k j(j+f+1)a_j + \frac{2}{f+4}\sum_{j=1}^k ja_j. \end{aligned} \tag{3.1}$$

For the case $k = 2$, we have $\tilde{c}_3 = 0$, and hence the transformations Y's due to Fujikoshi (2000) yield an improvement on approximation of the third moment as well as the first two moments of χ^2_f. Further, we can get (1.9). So, we consider the case $k \ge 3$. First, we note that

$$\begin{aligned} \tilde{c}_2 &= \frac{4}{f(f+2)}\sum_{j=2}^k j(j-1)a_j, \\ \tilde{c}_3 &= \frac{8}{f(f+2)(f+4)}\sum_{j=3}^k j(j-1)(j-2)a_j \end{aligned} \tag{3.2}$$

and hence the expressions of α, β and γ in (2.10) are obtained as

$$\begin{aligned}\alpha &= \frac{-3f(f+2)}{2\sum_{j=2}^{k} j(j-1)(2j-7)a_j}, \\ \beta &= \frac{(f+2)\sum_{j=1}^{k} j(j^2-6j+11)a_j}{2\sum_{j=2}^{k} j(j-1)(2j-7)a_j}, \\ \gamma &= \frac{\sum_{j=3}^{k} j(j-1)(j-2)a_j}{2(f+4)\sum_{j=2}^{k} j(j-1)(2j-7)a_j},\end{aligned} \tag{3.3}$$

provided that $3\tilde{c}_2 - (f+4)\tilde{c}_3 \neq 0$. Especially when $k=3$, (3.3) becomes that

$$\begin{aligned}&\alpha = \frac{1}{4}f(f+2)/(a_2+a_3), \quad \beta = \frac{1}{2}(f+2)a_0/(a_2+a_3), \\ &\gamma = -\frac{1}{2}a_3/\{(f+4)(a_2+a_3)\}.\end{aligned} \tag{3.4}$$

Under the assumption that the distribution of a statistic T can be expanded as in (1.8), Kakizawa (1996) proposed a method for finding a monotone transformation of T. When $k=3$, his method gives the following transformation with (3.4):

$$\begin{aligned}T_K = T &+ \frac{1}{n}\left(\frac{\beta}{\alpha}T - \frac{1}{2\alpha}T^2 + \frac{\gamma}{\alpha}T^3\right) \\ &+ \frac{1}{4n^2}\left\{\frac{\beta^2}{\alpha^2}T - \frac{\beta}{\alpha^2}T^2 + \left(\frac{2\beta\gamma}{\alpha^2} + \frac{1}{3\alpha^2}\right)T^3 - \frac{3\gamma}{2\alpha^2}T^4 + \frac{9\gamma^2}{5\alpha^2}T^5\right\}.\end{aligned} \tag{3.5}$$

Note that the expansion (3.5) is same as in (2.6) up to $O(n^{-1})$.

Now, we consider asymptotic expansions of the distributions of $\tilde{T}$'s with an error term of $O(n^{-2})$. For the purpose, from (2.6) we may deal with

$$\tilde{T} = T + \frac{1}{n}\left(\frac{\beta}{\alpha}T - \frac{1}{2\alpha}T^2 + \frac{\gamma}{\alpha}T^3\right). \tag{3.6}$$

The characteristic function of $\tilde{T}$ can be expanded as

$$\begin{aligned}C(t) &= E(e^{it\tilde{T}}) \\ &= E\left\{e^{itT}\left(1 + \frac{it}{n}\Big(\frac{\beta}{\alpha}T - \frac{1}{2\alpha}T^2 + \frac{\gamma}{\alpha}T^3\Big)\right)\right\} + O(n^{-2}) \\ &= (1-2it)^{-f/2}\left\{1 + \frac{1}{n}\sum_{j=0}^{k} a_j(1-2it)^{-j}\right\} \\ &\qquad + \frac{it}{n}E\left\{e^{itT}\left(\frac{\beta}{\alpha}T - \frac{1}{2\alpha}T^2 + \frac{\gamma}{\alpha}T^3\right)\right\} + O(n^{-2}).\end{aligned}$$

Note that

$$\begin{aligned}&E(Te^{itT}) = f(1-2it)^{-f/2-1} + O(n^{-1}), \\ &E(T^2e^{itT}) = f(f+2)(1-2it)^{-f/2-2} + O(n^{-1}), \\ &E(T^3e^{itT}) = f(f+2)(f+4)(1-2it)^{-f/2-3} + O(n^{-1}).\end{aligned}$$

Using these results, we have

$$C(t) = (1-2it)^{-f/2}\left\{1+\frac{1}{n}\sum_{j=0}^{k}\tilde{a}_j(1-2it)^{-j}+O(n^{-2})\right\}, \tag{3.7}$$

where

$$\begin{aligned}
&\tilde{a}_0 = a_0 - \frac{\beta}{2\alpha}f,\ \ \tilde{a}_1 = a_1 + \frac{1}{4\alpha}(2\beta+f+2)f,\\
&\tilde{a}_2 = a_2 - \frac{1}{4\alpha}(1+2\gamma f+8\gamma)f(f+2),\ \ \tilde{a}_3 = a_3 + \frac{\gamma}{2\alpha}f(f+2)(f+4),\\
&\tilde{a}_j = a_j\ \ (j \geq 4).
\end{aligned} \tag{3.8}$$

Inverting (3.7), we can obtain the following theorem.

THEOREM 2. *Suppose that a nonnegative random variate T has an asymptotic expansion (1.8), and its first three moments can be expanded as in (1.1)-(1.2) and (1.10). Assume that $k \geq 3$ and $\sum_{j=2}^{k} j(j-1)(2j-7)a_j \neq 0$. Then, neglecting the terms of $O(n^{-2})$, $\tilde{T}$'s have the same asymptotic expansion given by*

$$P(\tilde{T} \leq x) = G_f(x) + \frac{1}{n}\sum_{j=0}^{k}\tilde{a}_j G_{f+2j}(x) + O(n^{-2}), \tag{3.9}$$

where the coefficients $\tilde{a}_j$'s are given by (3.8).

Theorem 2 shows that the differences between the asymptotic expansions for T and $\tilde{T}$ appear in only the first four coefficients a_j, $\tilde{a}_j, j = 0,1,2,3$. Further, we can see that the asymptotic expansions for $\tilde{T}$ in the cases $k = 3$ and 4 are considerably simple, and are close to the distribution of χ_f^2. In fact,

(i) The case $k = 3$; $\tilde{a}_j = 0, j = 0,1,2,3$ and

$$P(\tilde{T} \leq x) = G_f(x) + O(n^{-2}). \tag{3.10}$$

(ii) The case $k = 4$; note that

$$\begin{aligned}
&\alpha = \frac{1}{4}f(f+2)/(a_2+a_3-2a_4),\ \ \beta = \frac{1}{2}(f+2)(a_0-a_4)/(a_2+a_3-2a_4),\\
&\gamma = -\frac{1}{2}(a_3+4a_4)/\{(f+4)(a_2+a_3-2a_4)\}.
\end{aligned}$$

Hence, we have that $\tilde{a}_0 = a_4$, $\tilde{a}_1 = -4a_4$, $\tilde{a}_2 = 6a_4$, $\tilde{a}_3 = -4a_4$, $\tilde{a}_4 = a_4$, and

$$\begin{aligned}
P(\tilde{T} \leq x) &= G_f(x) + \frac{a_4}{n}\{G_f(x) - 4G_{f+2}(x) + 6G_{f+4}(x)\\
&\qquad -4G_{f+6}(x) + G_{f+8}\} + O(n^{-2}) \qquad (3.11)\\
&= G_f(x) + \frac{2a_4}{nf}g_f(x)\left\{x - \frac{3}{f+2}x^2 + \frac{3}{(f+2)(f+4)}x^3\right.\\
&\qquad \left. - \frac{1}{(f+2)(f+4)(f+6)}x^4 + O(n^{-2})\right\},
\end{aligned}$$

where $g_f(x)$ is the probability density function of χ^2_f.

It may be noted that the transformations $\widetilde{T}$'s in (2.2)-(2.5) and T_K in (3.5) have removed the terms of $O(n^{-1})$ in the asymptotic expansion (1.8) with $k = 3$. One may refer to Cordeiro and Ferarri (1998) as well. For $k = 4$, we have a simple asymptotic expansion for the distribution of $\widetilde{T}$, which becomes more close to the chi-squared distribution as a_4 becomes close to zero. In many instances, the null distributions of test statistics under nonnormality are expanded in the form (1.8) with $k = 3$.

4. SOME APPLICATIONS

In this section, we shall give the transformations (2.2)-(2.5) for some statistics and examine the accuracy of the approximations to the true percentage point $t(u)$ of T. We conducted simulation experiments as follows: For parameters given in advance, the approximate percentage point was calculated for each monotone transformation. By using these percentage points, we conducted the Monte Carlo simulation with 100,000 ($= R$, say) independent trials for each test statistic. Let t_r ($r = 1, ..., R$) be an observed value of T and $p_r = 1$ (or 0) if t_r is (or is not) larger than the approximate percentage point. On the other hand, let $t_{[1]} \le t_{[2]} \le \cdots \le t_{[R]}$ be the ordered values of t_r and let us define $t_{[0.95R]}$ as an observed value of $t(u)$. We briefly write it $t(u)$. Let $\bar{p} = 100 \sum_{r=1}^{R} p_r/R$ which estimates the test size (5%) with its estimated standard error $s(\bar{p}) = 100\sqrt{(\bar{p}/100)(1 - \bar{p}/100)/R}$. TABLEs I and III give values of the approximate percentage point for each monotone transformation together with the value of $t(u)$. As for the actual test sizes, TABLEs II and IV give values of $\bar{p}$ ($s(\bar{p})$), on the first (second) line in each cell, for each monotone transformation.

EXAMPLE 1. Let $T = (n - q)s_h^2/s_e^2$ be a test statistic for testing the equality of means of q nonnormal populations Π_i ($i = 1, ..., q$) with common variance. Here, s_h^2 and s_e^2 are the sums of squares due to the hypothesis and the error, respectively, based on the sample of the size n_i from Π_i. Let $\rho_i = \sqrt{n_i/n}$, where n is the total sample size. Assume that $\rho_i = O(1)$ as n_j's tend to infinity. Let κ_3 and κ_4 be the third and the fourth cumulants of the standardized variate. Then, under a general condition, an asymptotic expansion for the null distribution of T was given by Fujikoshi, Ohmae and Yanagihara (1999) in the form (1.8) with $k = 3$, $f = q - 1$ and the coefficients given by

$$
\begin{aligned}
a_0 &= \frac{1}{4}(q-1)(q-3) - d_1\kappa_3^2 + d_2\kappa_4, \\
a_1 &= -\frac{1}{2}(q-1)^2 + 3d_1\kappa_3^2 - 2d_2\kappa_4, \\
a_2 &= \frac{1}{4}(q^2-1) - 3d_1\kappa_3^2 + d_2\kappa_4, \\
a_3 &= d_1\kappa_3^2,
\end{aligned}
$$

where

$$
d_1 = \frac{5}{24}\left(\sum_{j=1}^{q}\frac{n}{n_j} - q^2\right) + \frac{1}{12}(q-1)(q-2),
$$

$$d_2 = \frac{1}{8}\left(\sum_{j=1}^{q}\frac{n}{n_j} - q^2\right) - \frac{1}{4}(q-1).$$

We examined performance of our new transfomations under the following three nonnomal models:

(1) χ^2 distribution with 4 degrees of freedom;

(2) Gamma distribution with shape parameter 3 and scale parameter 1/3;

(3) Exponential distribution with scale parameter 1.

TABLE I gives the true percentage point $t(u)$ and the approximate percentage points $t_B(u)$, $t_E(u)$, $t_{1\cdot2}(u)$, $t_3(u)$, $\tilde{t}_{1\cdot2}(u)$, $\tilde{t}_3(u)$, $\tilde{t}_4(u)$ and $t_K(u)$ for the case $q = 3$. Here, u denotes the upper 5% point of χ^2_2, $t_B(u)$ and $t_E(u)$ are computed on the basis of the Bartlett corection and the Cornish-Fisher expansion up to the order $O(n^{-1})$ respectively, and $t_3(u)$, $\tilde{t}_3(u)$, $\tilde{t}_4(u)$ and $t_K(u)$ are computed on the basis of (1.5), (2.4), (2.5) and (3.5) respectively. Note that when $k = 3$, the Cornish-Fisher expansion yields the percentage point $\tilde{t}(u)$ of T in the same form as in (2.15) with (3.4). It means that the transformations T_K and $\tilde{T}$ aim to find an improvement of approximations to $t(u)$ in the terms of $O(n^{-2})$. On the other hand,

$$t_{1\cdot2}(u) = \begin{cases} t_1(u) & \text{if } \alpha_0 > 0 \text{ and } n\alpha_0 + \beta_0 > 0, \\ t_2(u) & \text{if } \alpha_0 < 0 \text{ and } n\alpha_0 + \beta_0 < 0, \end{cases}$$

where $t_1(u)$ and $t_2(u)$ are computed on the basis of (1.3) and (1.4) respectively. Similarly,

$$\tilde{t}_{1\cdot2}(u) = \begin{cases} \tilde{t}_1(u) & \text{if } \alpha > 0,\ n\alpha + \beta > 0 \text{ and } \gamma > 0, \\ \tilde{t}_2(u) & \text{if } \alpha < 0,\ n\alpha + \beta < 0 \text{ and } \gamma < 0, \end{cases}$$

where $\tilde{t}_1(u)$ and $\tilde{t}_2(u)$ are defined by (2.13) and (2.14) respectively.

TABLE I

The percentage points in the case $q = 3$

	Sample sizes			Upper 5% points ($\chi^2_2(0.05) = 5.9915$)								
	n_1	n_2	n_3	$t(u)$	$t_B(u)$	$t_E(u)$	$t_{1\cdot2}(u)$	$t_3(u)$	$\tilde{t}_{1\cdot2}(u)$	$\tilde{t}_3(u)$	$\tilde{t}_4(u)$	$t_K(u)$
	3	3	3	9.803	7.703	7.378	7.958	8.020	-	8.040	8.447	7.943
Model (1)	5	5	5	7.443	6.913	6.823	7.041	7.056	-	7.012	7.122	6.986
$\kappa_3 = \sqrt{2}$	10	10	10	6.503	6.419	6.407	6.476	6.479	-	6.449	6.472	6.443
$\kappa_4 = 3$	3	6	6	7.416	6.913	6.815	7.130	7.179	-	7.116	7.225	7.033
	5	5	10	6.900	6.657	6.609	6.811	6.835	-	6.759	6.812	6.720
	3	3	3	9.937	7.703	7.580	8.176	8.445	-	8.389	8.637	8.719
Model (2)	5	5	5	7.425	6.913	6.945	7.158	7.223	-	7.177	7.241	7.231
$\kappa_3 = 2/\sqrt{3}$	10	10	10	6.399	6.419	6.468	6.530	6.542	-	6.519	6.532	6.528
$\kappa_4 = 2$	3	6	6	7.410	6.913	6.939	7.213	7.317	-	7.228	7.295	7.278
	5	5	10	6.893	6.657	6.702	6.872	6.922	-	6.849	6.883	6.870
	3	3	3	9.262	7.703	6.770	6.966	7.290	-	7.803	8.783	6.689
Model (3)	5	5	5	7.081	6.913	6.458	6.586	6.686	-	6.717	6.949	6.429
$\kappa_3 = 2$	10	10	10	6.305	6.419	6.225	6.291	6.332	-	6.279	6.332	6.218
$\kappa_4 = 6$	3	6	6	7.316	6.913	6.441	6.821	6.850	-	7.112	7.344	6.453
	5	5	10	6.741	6.657	6.329	6.598	6.612	-	6.634	6.735	6.335

100,000 replications

TABLE II
The actual test sizes in the case $q = 3$

	Sample sizes			Nominal 5% test								
	n_1	n_2	n_3	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
	3	3	3	11.343	7.608	8.174	7.182	7.086	-	7.060	6.499	7.205
				0.100	0.084	0.087	0.082	0.081	-	0.081	0.078	0.082
Model (1)	5	5	5	8.199	5.953	6.136	5.719	5.696	-	5.762	5.580	5.821
				0.087	0.075	0.076	0.073	0.073	-	0.074	0.073	0.074
$\kappa_3 = \sqrt{2}$	10	10	10	6.256	5.179	5.200	5.052	5.045	-	5.117	5.059	5.127
				0.077	0.070	0.070	0.069	0.069	-	0.070	0.069	0.070
$\kappa_4 = 3$	3	6	6	7.983	5.890	6.071	5.478	5.412	-	5.502	5.331	5.653
				0.086	0.074	0.076	0.072	0.072	-	0.072	0.071	0.073
	5	5	10	7.095	5.484	5.572	5.171	5.127	-	5.281	5.165	5.348
				0.081	0.072	0.073	0.070	0.070	-	0.071	0.070	0.071
	3	3	3	11.679	7.834	8.050	7.108	6.702	-	6.779	6.437	6.346
				0.102	0.085	0.086	0.081	0.079	-	0.079	0.078	0.077
Model (2)	5	5	5	8.092	5.855	5.788	5.428	5.323	-	5.398	5.294	5.312
				0.086	0.074	0.074	0.072	0.071	-	0.071	0.071	0.071
$\kappa_3 = 2/\sqrt{3}$	10	10	10	5.941	4.945	4.838	4.714	4.686	-	4.739	4.710	4.719
				0.075	0.069	0.069	0.067	0.067	-	0.067	0.067	0.067
$\kappa_4 = 2$	3	6	6	7.961	5.847	5.790	5.313	5.136	-	5.287	5.167	5.195
				0.086	0.074	0.074	0.071	0.070	-	0.071	0.070	0.070
	5	5	10	6.987	5.454	5.365	5.036	4.927	-	5.087	5.014	5.040
				0.081	0.072	0.071	0.069	0.068	-	0.069	0.069	0.069
	3	3	3	10.257	6.809	8.381	8.015	7.432	-	6.673	5.483	8.552
				0.096	0.080	0.088	0.086	0.083	-	0.079	0.072	0.088
Model (3)	5	5	5	7.321	5.294	6.210	5.930	5.719	-	5.665	5.236	6.274
				0.082	0.071	0.076	0.075	0.073	-	0.073	0.070	0.077
$\kappa_3 = 2$	10	10	10	5.830	4.749	5.200	5.029	4.986	-	5.063	4.940	5.217
				0.074	0.067	0.070	0.069	0.069	-	0.069	0.068	0.070
$\kappa_4 = 6$	3	6	6	7.702	5.658	6.553	5.830	5.768	-	5.345	4.969	6.524
				0.084	0.073	0.078	0.074	0.074	-	0.071	0.069	0.078
	5	5	10	6.655	5.138	5.810	5.261	5.238	-	5.191	5.010	5.795
				0.079	0.070	0.074	0.071	0.070	-	0.070	0.069	0.074

100,000 replications

TABLE II gives the actual test sizes denoted by

$$\alpha_1 = P(T > u), \quad \alpha_2 = P(T > t_B(u)), \quad \alpha_3 = P(T > t_E(u)),$$
$$\alpha_4 = P(T > t_{1\cdot 2}(u)), \quad \alpha_5 = P(T > t_3(u)), \quad \alpha_6 = P(T > \tilde{t}_{1\cdot 2}(u)),$$
$$\alpha_7 = P(T > \tilde{t}_3(u)), \quad \alpha_8 = P(T > \tilde{t}_4(u)), \quad \alpha_9 = P(T > t_K(u)),$$

for the case $q = 3$.

In this example, $\tilde{T}_1$ and $\tilde{T}_2$ are not applicable because of $a\gamma < 0$. The reason why there are several α_4 and α_5 values very close to the target (5%) would be caused by the closeness of $\tilde{c}_3$ to 0. In the case when $\tilde{c}_3$ is close to 0, the advantage of using the transformations expanded as (2.6) is seriously influenced by the amount of the terms of $O(n^{-2})$ in (2.11). In fact, we can see from TABLE II that a reduction of the amount of the terms of $O(n^{-2})$ brought by $\tilde{t}_4(u)$ is visible as an improvement of the approximation, especially when n is small.

EXAMPLE 2. We consider chi-squared approximations for the distribution of the score statistic S_R. An asymptotic expansion for the null distribution of S_R was given by Harris (1985) in the form (1.8) with $k = 3$ and the coefficients given by

$$a_0 = \frac{A_2 - A_1 - A_3}{24}, \quad a_1 = \frac{3A_3 - 2A_2 + A_1}{24}, \quad a_2 = \frac{A_2 - 3A_3}{24}, \quad a_3 = \frac{A_3}{24}.$$

The quantities A_1, A_2 and A_3 are usually functions of unknown parameters. Ferrari, Uribe-Opazo and Cordeiro (2002) gave simple formulae of A_1, A_2 and A_3 for two-parameter exponential family models.

Let us consider the gamma distribution with mean $\theta > 0$ and shape parameter $\phi > 0$ $(y > 0)$. In the experiment, our interest is in testing $H_0 : \theta = \theta^{(0)}$ against $H_1 : \theta \neq \theta^{(0)}$, assuming that the shape parameter ϕ is unknown. The Monte Carlo simulation with 100,000 replications was conducted by setting $\theta^{(0)} = 1$, ϕ =0.5, 1.0 and 2.0 and the number of observations was set as n =10, 20, 30 and 40. For each sample, the score statistic were computed as $S_R = n\tilde{\phi}(\bar{y} - 1)^2$, where $\tilde{\phi}$, the MLE of ϕ under H_0, is obtained as a solution to the nonlinear equation

$$\log \tilde{\phi} - \psi(\tilde{\phi}) = \log\left(\frac{\theta^{(0)}}{\bar{y}_g}\right) + \left(\frac{\bar{y} - \theta^{(0)}}{\theta^{(0)}}\right)$$

with $\psi(\phi) = d\log\Gamma(\phi)/d\phi$ the digamma functoin, and $\bar{y}$ and $\bar{y}_g$ are the sample mean and geometric mean of $y_1, \ldots, y_n$. Then, an asymptotic expansion for the null distribution of S_R is the chi-squared distribution with 1 degree of freedom followed by the terms of order n^{-1} with the quantites

$$A_1 = \frac{6\{1 - \phi^2\psi''(\phi) - 2\phi\psi'(\phi)\}}{n\phi\{\phi\psi'(\phi) - 1\}^2}, \quad A_2 = \frac{9\{2\phi\psi'(\phi) - 3\}}{n\phi\{\phi\psi'(\phi) - 1\}} \quad \text{and } A_3 = \frac{20}{n\phi}.$$

See Ferrari, Uribe-Opazo and Cordeiro (2002) for the details.

TABLE III

The percentage points in the case $\theta^{(0)} = 1$

	Sample sizes	Upper 5% points ($\chi_1^2(0.05) = 3.8415$)								
ϕ	n	$t(u)$	$t_B(u)$	$t_E(u)$	$t_{1\cdot 2}(u)$	$t_3(u)$	$\tilde{t}_{1\cdot 2}(u)$	$\tilde{t}_3(u)$	$\tilde{t}_4(u)$	$t_K(u)$
0.5	10	3.613	3.891	3.794	4.044	4.023	-	4.039	3.365	3.391
	20	3.589	3.866	3.823	3.957	4.000	-	3.738	3.620	3.608
	30	3.652	3.858	3.836	3.922	3.941	-	3.743	3.693	3.684
	40	3.722	3.854	3.838	3.903	3.913	-	3.757	3.729	3.723
1.0	10	3.599	3.895	3.813	3.930	3.935	-	3.760	3.587	3.584
	20	3.675	3.868	3.834	3.886	3.887	-	3.752	3.710	3.707
	30	3.730	3.859	3.838	3.871	3.872	-	3.771	3.752	3.751
	40	3.733	3.855	3.840	3.864	3.864	-	3.784	3.774	3.773
2.0	10	3.618	3.884	3.821	3.826	3.831	-	3.740	3.669	3.661
	20	3.737	3.863	3.836	3.833	3.835	-	3.767	3.748	3.746
	30	3.768	3.856	3.839	3.836	3.837	-	3.786	3.778	3.777
	40	3.811	3.852	3.840	3.837	3.838	-	3.798	3.793	3.793

100,000 replications

Similarly to EXAMPLE 1, TABLE III gives the true percentage point and the approximate percentage points for the case $\theta^{(0)} = 1$ and TABLE IV gives the corresponding actual test sizes.

In this example as well, $\tilde{T}_1$ and $\tilde{T}_2$ are not applicable because of $\alpha\gamma < 0$. From these tables, we can see the advantage of using the transformations (2.4)-(2.5) and (3.5). In fact, $\tilde{c}_3$ is not close to 0. Note that $\alpha\gamma < 0$ and $\alpha < 0$. In the case when $\alpha\gamma < 0$ and $\alpha < 0$, the inside of $\exp(\cdot)$ in (2.4) is always positive, so that the using of $\exp(\cdot)$ in (2.4) would cause to increase the amount of the terms of $O_p(n^{-2})$ in (2.6). We can see that the transformation (2.5) with the coefficient $\{1 - (\beta^2 - \beta + \gamma - 1)/(n^2\alpha^2)\}$ produces an improvement successfully in that sense.

TABLE IV

The actual test sizes in the case $\theta^{(0)} = 1$

	Sample sizes	Nominal 5% test								
ϕ	n	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
0.5	10	4.533	4.431	4.616	4.174	3.895	-	4.187	5.612	5.555
		0.066	0.065	0.066	0.063	0.061	-	0.063	0.073	0.072
	20	4.252	4.196	4.278	3.989	3.905	-	4.461	4.757	4.790
		0.064	0.063	0.064	0.062	0.061	-	0.065	0.067	0.068
	30	4.443	4.402	4.455	4.265	4.217	-	4.711	4.877	4.893
		0.065	0.065	0.065	0.064	0.064	-	0.067	0.068	0.068
	40	4.670	4.637	4.679	4.502	4.466	-	4.899	4.985	4.997
		0.067	0.066	0.067	0.066	0.065	-	0.068	0.069	0.069
1.0	10	4.320	4.171	4.394	4.088	4.074	-	4.531	5.033	5.044
		0.064	0.063	0.065	0.063	0.063	-	0.066	0.069	0.069
	20	4.502	4.414	4.516	4.377	4.374	-	4.755	4.896	4.903
		0.066	0.065	0.066	0.065	0.065	-	0.067	0.068	0.068
	30	4.658	4.617	4.668	4.590	4.589	-	4.884	4.934	4.940
		0.067	0.066	0.067	0.066	0.066	-	0.068	0.068	0.069
	40	4.681	4.644	4.688	4.618	4.618	-	4.856	4.877	4.879
		0.067	0.067	0.067	0.066	0.066	-	0.068	0.068	0.068
2.0	10	4.259	4.153	4.337	4.310	4.297	-	4.604	4.835	4.867
		0.064	0.063	0.064	0.064	0.064	-	0.066	0.068	0.068
	20	4.664	4.606	4.685	4.692	4.688	-	4.910	4.957	4.961
		0.067	0.066	0.067	0.067	0.067	-	0.068	0.069	0.069
	30	4.782	4.730	4.789	4.796	4.794	-	4.947	4.978	4.979
		0.067	0.067	0.068	0.068	0.068	-	0.069	0.069	0.069
	40	4.900	4.874	4.904	4.914	4.913	-	5.045	5.061	5.065
		0.068	0.068	0.068	0.068	0.068	-	0.069	0.069	0.069

100,000 replications

Through EXAMPLEs 1 and 2, we have seen how much the distributions of the transformed statistics $\tilde{T}_i$ are close to the one of χ^2_f -variate, or how much the approximate percentage points $\tilde{t}_i(u)$ are close to the true percentage point $t(u)$ of T. It is shown that the proposed transformations of these statistics give a larger improvement to the chi-squared approximation than do the other transformations. Unfortunately, we cannot recommend our transformations in the following cases:

(i) In the case $\alpha > 0$, $\widetilde{T}_3$ has the upper limit $n\alpha + \beta$. Therefore, when u is close to $n\alpha+\beta$, the approximate percentage point $\tilde{t}_3(u)$ cannot hold accuracy seen in EXAMPLEs 1 and 2. Further, when u is over $n\alpha + \beta$, the approximate percentage point $\tilde{t}_3(u)$ cannot be used.

(ii) In the case $\alpha\gamma < 0$, $\widetilde{T}_3$ has an extreme value at $T = \sqrt{-2n\alpha/(3\gamma)}$. Therefore, when u is close to that value, the approximate percentage point $\tilde{t}_3(u)$ cannot hold accuracy seen in EXAMPLEs 1 and 2.

As for (i)-(ii) described above, $\widetilde{T}_4$ has similar natures to $\widetilde{T}_3$. To overcome these difficulties (i)-(ii) simultaneously, the following transformation could be one of the options:

$$\widetilde{T} = \left(n^2 + \frac{\beta}{\alpha}n\right)\log\left(1 + \frac{1}{n^2}T - \frac{1}{2n^3\alpha}T^2 + \frac{\gamma}{n^3\alpha}T^3 + \frac{1}{2n^4}T^2 + \frac{1}{12n^4\alpha^2}T^3 - \frac{3\gamma}{8n^4\alpha^2}T^4 + \frac{9\gamma^2}{20n^4\alpha^2}T^5\right)$$

for any α, β, γ and n. Its asymptotic expansion form is same as in (2.6) up to $O(n^{-1})$. The efficiency of this transformation and its modifications are under investigation.

REFERENCES

[1] G. M. Cordeiro and S. L. de P. Ferrari, A modified score test statistic having chi-squared distribution to order n^{-1}, *Biometrika*, **78** (1991), 573-582.

[2] G. M. Cordeiro and S. L. P. Ferrari, A note on Bartlett-type correction for the first few moments of test statistics, *J. Statist. Plan. Inference*, **71** (1998), 261-269.

[3] S. L. P. Ferrari, M. A. Uribe-Opazo, and G. M. Cordeiro, Bartlett-type corrections for two-parameter exponential family models, *Commun. Statist - Theory Meth.*, **31** (2002), 901-924.

[4] Y. Fujikoshi, A method for improving the large-sample chi-squared approximations to some multivariate test statistics, *Amer. J. Math. Management Sci.*, **17** (1997), 15-29.

[5] Y. Fujikoshi, Transformations with improved chi-squared approximations, *J. Multivariate Analysis*, **72** (2000) 249-263

[6] Y. Fujikoshi, M. Ohmae, and H. Yanagihara, Asymptotic approximations of the null distribution of one-way ANOVA test statistic under nonnormality, *J. Japan Statist.*, **29** (1999), 147-161.

[7] H. Fujisawa, Improvement on chi-squared approximation by monotone transformation, *J. Multivariate Anal.* **60** (1997), 84-89.

[8] P. Harris, An asymptotic expansion for the null distribution of the efficient score statistic, *Biometrika*, **72** (1985), 653-659.

[9] Y. Kakizawa, Higher order monotone Bartlett-type ajustment for some multivariate test statistics, *Biometrika*, **83** (1996), 923-927.