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Sequential Point Estimation of the Powers of an Exponential Scale Parameter

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1. Introduction

Let $X_1, X_2, X_3, \ldots$ be independent and identically distributed (i.i.d.) random variables with the probability density function (p.d.f.)

$$f_{\mu,\sigma}(x) = \frac{1}{\sigma} \exp\left(-\frac{x-\mu}{\sigma}\right) I_{(x \geq \mu)} ,$$

where both $\mu \in (-\infty, \infty)$ and $\sigma \in (0, \infty)$ are unknown and $I(A)$ denotes the indicator function of the set A. For any given $r \neq 0$ we want to estimate the power of the scale parameter σ^r. Let $\delta_n = \delta_n(X_1, \ldots, X_n)$ be an estimator of σ^r based on a random sample $X_1, \ldots, X_n$ of size n. Then as a loss function we use the squared error loss defined by $L_n = (\delta_n - \sigma^r)^2$. The risk associated with the estimator δ_n is given by $R_n = R_n(\delta_n) = E(L_n)$. Let $w > 0$ be a preassigned error bound for the risk. We want to find the smallest sample size $n = n_0$ which satisfies that $R_n \leq w$. In bounded risk point estimation problems the error bound w is assumed to be sufficiently small, so that we suppose that the sample size n is sufficiently large. For $n \geq 2$ set

$$T_n = \min\{X_1, \ldots, X_n\} \quad \text{and} \quad \sigma_n = (n-1)^{-1}\sum_{i=1}^{n}(X_i - T_n).$$

We can show that $c_n\sigma_n^r$ with $c_n = (n-1)^r\Gamma(n-1)/\Gamma(r+n-1)$ is the uniformly minimum variance unbiased estimator of σ^r provided $n > \max\{1, 1-r\}$. We use, however, σ_n^r as an estimator of σ^r, because σ_n^r is approximately equal to $c_n\sigma_n^r$ for large n and the calculation of σ_n^r is easier than that of $c_n\sigma_n^r$. Our goal is to find an asymptotically smallest sample size n_0 satisfying that $R_n = E(\sigma_n^r - \sigma^r)^2 \leq w$.

Estimation of σ and σ^2 are of great importance. For $r = 1$, namely, for the estimation of the standard deviation by σ_n, Isogai, Saito and Uno (1999) dealt with this bounded risk point estimation problem. Minimum risk point estimation problems for $r = 1$ were considered by Mukhopadhyay and Ekwo (1987), Ghosh and Mukhopadhyay (1989) and Isogai and Uno (1994). Starr and Woodroofe (1972) treated the same problem for $r = 2$. Uno and Isogai (2002) proposed a fully sequential procedure for the estimation of σ^r with normal scale parameter σ. For a review one may refer to Mukhopadhyay (1988), Ghosh and Sen (1991) and Ghosh, Mukhopadhyay and Sen (1997). Sometimes, it is of interest to measure mean $\lambda = \mu + \sigma$ in $\sigma-$units and hence to estimate $\mu\sigma^{-1}$. For a normal distribution with mean μ and variance σ^2 both unknown, Sriram (1990) considered the sequential point estimation problem for $\mu\sigma^{-1}$ by using an estimator of σ^r with $r = -1$.

We shall now compute the risk $R_n = E(\sigma_n^r - \sigma^r)^2$ to find n_0. We can show that for $n > \max\{2, 1-2r\}$.

$$R_n < \infty \quad \text{and} \quad R_n = r^2\sigma^{2r}n^{-1} + O(n^{-2}) \quad \text{as} \quad n \to \infty.$$

Ignoring the order term above, we can find the asymptotically smallest sample size n_0 satisfying that $R_n \leq w$. Suppose

$$r^2\sigma^{2r}n^{-1} \leq w, \quad \text{or equivalently,} \quad n \geq \frac{r^2\sigma^{2r}}{w} = n^* \quad \text{(say)}. \tag{1.1}$$

For simplicity n^* is assumed to be an integer. Then $n_0 = n^*$ is the asymptotically best fixed sample size if σ is known. Unfortunately, the asymptotically best fixed sample size procedure n_0 cannot be used since σ is unknown. Further there is no fixed sample size procedure satisfying our condition. Thus we need to find a sequential sampling rule.

In Section 2 we shall propose a fully sequential procedure for this estimation problem and give two theorems concerning the second order approximation to its average sample size and risk associated with our procedure. We shall also consider a class of sequential estimators derived on the basis of the idea of bias-correction and compare them from the point of view of risk. In Section 3 we shall provide brief simulation results.

2. Results

In this section we shall propose a fully sequential procedure N motivated by the form of n^* in (1.1) and give two theorems concerning the second order approximation to its average sample size $E(N)$ and risk $R_N = E(\sigma_N^r - \sigma^r)^2$ as $w \to 0$. We shall also consider a class of sequential estimators, including the ordinary estimator σ_N^r, based on the idea of bias-correction. The comparison will be made from the point of view of risk. It will turn out that we can find an appropriate sequential estimator to reduce the risk associated with the ordinary one.

In this paper we propose the stopping rule defined by

$$N = N_w(r) = \inf\left\{ n \geq m : n \geq \frac{r^2 \sigma_n^{2r}}{w} l_n \right\}, \tag{2.1}$$

where m is a starting sample size satisfying that $m > \max\{2,\, 1-2r\}$ and l_x is a given positive function of x on $(0,\infty)$ such that

$$l_x = 1 + \frac{l_0}{x} + o\left(\frac{1}{x}\right) \quad \text{as } x \to \infty \quad \text{with a constant } l_0.$$

We can show that $P(N_w(r) < \infty) = 1$ for all $w > 0$ and $r \neq 0$. Once the sampling stops at the Nth stage, we estimate σ^r by σ_N^r. Then the risk associated with σ_N^r is given by $R_N = E(\sigma_N^r - \sigma^r)^2$. The following two theorems are concerned with the second order approximation to the average sample size and risk.

Theorem 2.1. If $m > m_1(r)$, then as $w \to 0$

$$E(N) = n^* + \rho + l_0 - r(2r+1) + o(1),$$

where

$$m_1(r) = \begin{cases} 1 + 6r & \text{if} \quad r > 0 \\ 1 - 2r & \text{if} \quad r < 0 \end{cases}$$

and ρ is a certain constant with $0 \leq \rho \leq \frac{1}{2} + 2r^2$.

Theorem 2.2. If $m > m_2(r)$, then as $w \to 0$

$$n^* \left(\frac{R_N}{w} - 1 \right) = 11r^2 + 8r + 1 + \frac{3}{4}(r-1)^2 - \rho - l_0 + o(1),$$

where

$$m_2(r) = \begin{cases} \max\{1 + 10r, 7 + 8r\} & \text{if} \quad r > 0 \\ 7 - 14r & \text{if} \quad r < 0. \end{cases}$$

Remark 2.1. (i) If we take an arbitrary constant l_0 such that

$$l_0 > 11r^2 + 8r + 1 + \frac{3}{4}(r-1)^2 - \rho, \tag{2.2}$$

then from Theorem 2.2 we have that $R_N < w$ for sufficiently small $w > 0$. Thus our condition on the risk is satisfied for sufficiently small w.
(ii) Theorem 2.1 of Isogai, Saito and Uno (1999) with $a = 0$ and $b = 1$ is the same as Theorems 2.1 and 2.2 with $r = 1$ except for the condition on the starting sample size. The difference of this condition is caused by the fact that this paper deals with all powers. Further the methods of the proofs are different.

We shall here evaluate the bias of σ_N^r.

Proposition 2.1. If $m > m_3(r)$, then as $w \to 0$

$$E(\sigma_N^r) - \sigma^r = -\frac{1}{2}\text{sign}(r)(3r+1)(n^*)^{-1/2}w^{1/2} + o(w),$$

where

$$\text{sign}(r) = \begin{cases} 1 & \text{if} \quad r > 0 \\ -1 & \text{if} \quad r < 0 \end{cases} \quad \text{and} \quad m_3(r) = \begin{cases} \max\{1 + 6r, 3 + 3r\} & \text{if} \quad r > 0 \\ 3 - 5r & \text{if} \quad r < 0. \end{cases}$$

Taking Proposition 2.1 into account, we consider a class of sequential estimators $\{\sigma_N^r(k), k \in (-\infty, \infty)\}$ for σ^r defined by

$$\sigma_N^r(k) = \sigma_N^r + k N^{-1/2} w^{1/2}.$$

Then we get the following proposition concerning the bias of $\sigma_N^r(k)$.

Proposition 2.2. If $m > m_3(r)$, then as $w \to 0$

$$E(\sigma_N^r(k)) = \sigma^r + \{k - \tfrac{1}{2}\text{sign}(r)(3r+1)\}(n^*)^{-1/2}w^{1/2} + o(w).$$

For $k = \frac{1}{2}\text{sign}(r)(3r+1)$, $\sigma_N^r(k)$ is a second-order asymptotically unbiased estimator.

We shall now compare the risk of $\sigma_N^r(k)$ with that of σ_N^r. Let $R_N(k) = E(\sigma_N^r(k) - \sigma^r)^2$.

Theorem 2.3. If $m > m_2(r)$, then as $w \to 0$

$$\frac{n^*}{w}(R_N(k) - R_N) = k^2 - \text{sign}(r)(5r+1)k + o(1).$$

Remark 2.2 Let $k = \frac{1}{2}\text{sign}(r)(5r+1)$ for $r \neq 0$. Then we have

$$R_N(k) < R_N \quad \text{for sufficiently small } w > 0 \quad \text{if } r \neq -\tfrac{1}{5}.$$

Thus bias-correction is asymptotically effective to reduce the risk for all $r \neq 0$ with using $\sigma_N^r + \frac{1}{2}\text{sign}(r)(5r+1)N^{-1/2}w^{1/2}$ which is not a second-order asymptotically unbiased estimator.

3. Simulation Results

We shall give brief simulation results which are based on 100,000 repetitions. We choose the constant l_0 satisfying the inequality in (2.2) in Tables 1–3. In Tables 4 and 5 we choose l_0 such that the average sample size $E(N)$ approximately equals the optimal one n^*. Since we do not know any approximate value of ρ between 0 and $\frac{1}{2} + 2r^2$, we use here $\rho = 0$ or $\rho = \frac{1}{2} + 2r^2$ as ρ. From these simulation results we might need to improve the stopping rule N in (2.1).

Table 1. $\rho = 0, \quad l_0 > 11r^2 + 8r + 1 + \frac{3}{4}(r-1)^2 - \rho$

$n^* = 100$	$r = -1$	$r = 1$	$r = 2$
$\mu = 0,\ \sigma = 1$	$w = 0.01$	$w = 0.01$	$w = 0.04$
	$m = 22$	$m = 16$	$m = 24$
$l_n = 1 + l_0/n$	$l_0 = 8$	$l_0 = 21$	$l_0 = 62$
	$k = 2$	$k = 3$	$k = 5.5$
$E(N)$	108.630260	116.346200	137.330710
$E(\sigma_N^r)$	0.991965	0.983436	0.951511
$E(\sigma_N^r(k))$	1.011400	1.011567	1.048849
R_N/w	0.963153	0.984520	0.949165
$R_N(k)/w$	0.934459	0.921439	0.800041
$n^*(R_N(k) - R_N)/w$	-2.869400	-6.308015	-14.912360

Table 2. $r = -1, \quad \rho = \frac{1}{2} + 2r^2 = 2.5, \quad l_0 > 11r^2 + 8r + 1 + \frac{3}{4}(r-1)^2 - \rho = 4.5$

$n^* = 40$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 2$
$\mu = 0, \quad l_0 = 5$	$w = 0.1$	$w = 0.025$	$w = 0.00625$
$m = 22, \quad k = 2$	$\sigma^r = 2$	$\sigma^r = 1$	$\sigma^r = 0.5$
$E(N)$	45.752540	45.700810	45.738710
$E(\sigma_N^r)$	1.965109	0.982012	0.491141
$E(\sigma_N^r(k))$	2.061243	1.030103	0.515186
R_N/w	0.918271	0.914489	0.925572
$R_N(k)/w$	0.865381	0.859792	0.870948
$n^*(R_N(k) - R_N)/w$	-2.115595	-2.187878	-2.184982

Table 3. $r = 1, \quad \rho = \frac{1}{2} + 2r^2 = 2.5, \quad l_0 > 11r^2 + 8r + 1 + \frac{3}{4}(r-1)^2 - \rho = 17.5$

$n^* = 40$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 2$
$\mu = 0, \quad l_0 = 18$	$w = 0.00625$	$w = 0.025$	$w = 0.1$
$m = 16, \quad k = 3$	$\sigma^r = 0.5$	$\sigma^r = 1$	$\sigma^r = 2$
$E(N)$	52.049260	52.088870	52.200480
$E(\sigma_N^r)$	0.481075	0.962627	1.928253
$E(\sigma_N^r(k))$	0.514786	1.030025	2.062843
R_N/w	1.004848	1.002715	0.984277
$R_N(k)/w$	0.870552	0.871007	0.862504
$n^*(R_N(k) - R_N)/w$	-5.371849	-5.268332	-4.870901

Table 4. $r = -1, \quad \rho = \frac{1}{2} + 2r^2 = 2.5, \quad l_0 \leq r(2r+1) - \rho = -1.5$

$n^* = 40$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 2$
$\mu = 0, \quad l_0 = -2$	$w = 0.1$	$w = 0.025$	$w = 0.00625$
$m = 22, \quad k = 2$	$\sigma^r = 2$	$\sigma^r = 1$	$\sigma^r = 0.5$
$E(N)$	39.236610	39.303210	39.232770
$E(\sigma_N^r)$	1.951794	0.976822	0.487953
$E(\sigma_N^r(k))$	2.056594	1.029155	0.514149
R_N/w	1.034014	1.018080	1.026389
$R_N(k)/w$	0.942045	0.931146	0.935053
$n^*(R_N(k) - R_N)/w$	-3.678757	-3.477360	-3.653437

Table 5. $r = 1, \quad \rho = \frac{1}{2} + 2r^2 = 2.5, \quad l_0 \leq r(2r+1) - \rho = 0.5$

$n^* = 40$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 2$
$\mu = 0, \quad l_0 = 0$	$w = 0.00625$	$w = 0.025$	$w = 0.1$
$m = 16, \quad k = 3$	$\sigma^r = 0.5$	$\sigma^r = 1$	$\sigma^r = 2$
$E(N)$	37.647620	37.596930	37.616820
$E(\sigma_N^r)$	0.468557	0.936478	1.873344
$E(\sigma_N^r(k))$	0.509244	1.017909	2.036156
R_N/w	1.470935	1.470930	1.473031
$R_N(k)/w$	1.107491	1.103551	1.106660
$n^*(R_N(k) - R_N)/w$	-14.537765	-14.695145	-14.654831

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