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A Generalization of Wiener's Lemma and its Application to Volterra Difference Equations

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1. Introduction

Let X be a Banach space over $\mathbb{C}$ with norm $|\cdot|$. We consider the Volterra difference equation on X

$$x(n+1)=\sum_{j=-\infty}^{n} Q(n-j)x(j), \qquad n\in\mathbb{Z}^+ := \{0,1,2,\dots\}, \tag{1}$$

where $Q(n)$, $n\in\mathbb{Z}^+$, are bounded linear operators on X such that $\sum_{n=0}^{\infty}\|Q(n)\|<\infty$.

In the present paper we will establish a generalization of Wiener's lemma (on absolutely convergent trigonometric series) for operator-valued sequences, and apply the result to get a condition on the characteristic operator associated with Eq. (1) which ensures the summability of the fundamental solution. We also study some stability properties of the zero solution, and moreover show applications of the results to some abstract differential equations with piecewise continuous delays. Our results are generalizations of those in [1, 2] for the case of finite-dimensional X to the case of infinite-dimensional X.

2. A Generalization of Wiener's Lemma

For a Banach space X, we denote by $\mathcal{L}(X)$ the space of all bounded linear operators on X, and define the norm of any T belonging to $\mathcal{L}(X)$ by

$$\|T\|=\sup\{|Tx| : x\in X,\ |x|=1\}.$$

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Let $L^1(\mathbb{Z}^+)$ be the space of all sequences $Q := \{Q(n)\} = (Q(0), Q(1), Q(2), \dots)$ with $Q(j) \in \mathcal{L}(X)$, $j \in \mathbb{Z}^+$, satisfying

$$\sum_{n=0}^{\infty} \|Q(n)\| < \infty.$$

For any Q and W in $L^1(\mathbb{Z}^+)$, we define the product $Q * W$ by

$$(Q * W)(n) = \sum_{k=0}^{n} Q(n-k)W(k), \qquad n \in \mathbb{Z}^+.$$

One can easily see that the space $L^1(\mathbb{Z}^+)$ with the product defined above is a (non-commutative) Banach algebra equipped with norm

$$\|Q\| = \sum_{n=0}^{\infty} \|Q(n)\|.$$

In fact, $L^1(\mathbb{Z}^+)$ possesses the element $e_0 =: e$ defined by

$$e_0(0) = I, \qquad e_0(n) = 0 \quad (n = 1, 2, \dots)$$

as the unit, where I denotes the identity operator on X.

Wiener's lemma [5, p. 226] is generalized as follows.

Theorem 1. *Assume that $Q = \{Q(n)\} \in L^1(\mathbb{Z}^+)$ satisfies the following two conditions:*

(i) $Q(i)Q(j) = Q(j)Q(i)$ for $i, j \in \mathbb{Z}^+$;

(ii) for any $|z| \leq 1$, the operator $\sum_{k=0}^{\infty} Q(k)z^k$ is invertible in $\mathcal{L}(X)$.

Then Q is invertible in $L^1(\mathbb{Z}^+)$; in other words, there exists an $R = \{R(n)\} \in L^1(\mathbb{Z}^+)$ such that

$$Q * R = R * Q = e_0.$$

Outline of proof. Let us consider the subset Ω of $L^1(\mathbb{Z}^+)$ which consists of all the elements of the form $(0, \dots, 0, Q(\cdot), 0, 0, \dots)$, and set $Y = \Gamma(\Gamma(\Omega))$, where $\Gamma(\mathcal{C})$ denotes the centralizer of the set $\mathcal{C}$, that is,

$$\Gamma(\mathcal{C}) = \{W \in L^1(\mathbb{Z}^+) : W * P = P * W \text{ for any } P \in \mathcal{C}\}.$$

Since the set Ω commutes by the condition (i), it follows from [5, p. 280, Theorem 11.22] that $Y = \Gamma(\Gamma(\Omega))$ is a commutative Banach subalgebra containing Ω. Let χ be any character of Y, and set $z_0 = \chi(e_1)$, where $e_1 = (0, I, 0, \dots)$. By virtue of the condition (ii), the element $\sum_{k=0}^{\infty} Q(k)z_0^k$ is invertible in $\mathcal{L}(X)$, and hence $(\sum_{k=0}^{\infty} Q(k)z_0^k, 0, 0, \dots)$ is invertible in $L^1(\mathbb{Z}^+)$ which implies that $(\sum_{k=0}^{\infty} Q(k)z_0^k, 0, 0, \dots)$ is invertible in Y (cf. [5, p.280, Theorem 11.22]). In particular, we get $\chi\big((\sum_{k=0}^{\infty} Q(k)z_0^k, 0, 0, \dots)\big) \neq 0$. Hence

$$\begin{aligned}
& \chi(Q) \\
= \; & \chi((Q(0), Q(1), Q(2), \dots)) \\
= \; & \chi((Q(0), 0, 0, \dots) * e_0 + (Q(1), 0, 0, \dots) * e_1 + (Q(2), 0, 0, \dots) * e_1 * e_1 + \cdots) \\
= \; & \sum_{k=0}^{\infty} \chi((Q(k), 0, 0, \dots))\{\chi(e_1)\}^k \\
= \; & \chi\Big(\Big(\sum_{k=0}^{\infty} Q(k)z_0^k, 0, 0, \dots\Big)\Big) \neq 0,
\end{aligned}$$

which shows that $Q = \{Q(n)\}$ does not belong to any maximal ideal of Y. Then [5, p.265, Theorem 11.5] yields that Q is invertible in Y, and so is it in $L^1(\mathbb{Z}^+)$. □

We now consider the Volterra difference equation (1) on X with $Q(n) \in \mathcal{L}(X)$, $n \in \mathbb{Z}^+$, and let us denote by $\{R(n)\}$ the fundamental solution of Eq. (1). Noticing that $(zI - \tilde{Q}(z))\tilde{R}(z) = zI$ and applying Theorem 1 to $S = \{S(n)\}$ with $S(0) = I$ and $S(n) = -Q(n-1)$, $n = 1, 2, \dots$, we get:

Corollary 1. *Assume that the coefficients $Q = \{Q(n)\} \in L^1(\mathbb{Z}^+)$ in Eq. (1) satisfies the condition (i) in Theorem 1, together with the following condition:*

(ii') for any $|z| \geq 1$, the characteristic operator of Eq. (1) $zI - \sum_{n=0}^{\infty} Q(n)z^{-n}$ is invertible in $\mathcal{L}(X)$.

Then the fundamental solution $\{R(n)\}$ of Eq. (1) is summable, that is, $\{R(n)\} \in L^1(\mathbb{Z}^+)$.

3. Stabilities in Eq. (1)

Let us consider the Banach space $\mathcal{B}$ defined by

$$\mathcal{B} = \{\phi : \mathbb{Z}^- \mapsto X \mid \sup_{\theta \in \mathbb{Z}^-} |\phi(\theta)| < \infty\}$$

equipped with the norm $\|\phi\| = \sup_{\theta \in \mathbb{Z}^-} |\phi(\theta)|$ for $\phi \in \mathcal{B}$. For any $(\tau, \phi) \in \mathbb{Z}^+ \times \mathcal{B}$, Eq. (1) has a unique solution $x(n)$ for $n \geq \tau$ satisfying the initial condition $x(\tau + \theta) \equiv \phi(\theta)$, $\theta \in \mathbb{Z}^-$. We denote this solution by $x(n; \tau, \phi)$. By the variation of constant formula, we get

$$x(n; \tau, \phi) = R(n-\tau)\phi(0) + \sum_{j=\tau}^{n-1} R(n-j-1)\Big(\sum_{s=-\infty}^{-1} Q(j-\tau-s)\phi(s)\Big)$$

for $n \geq \tau$, where we promise that $\sum_{j=\tau}^{\tau-1} = 0$ for $\tau \geq 0$. Using this formula and applying Corollary 1, we obtain:

Theorem 2. *Assume that the coefficients* $Q = \{Q(n)\} \in L^1(\mathbb{Z}^+)$ *in Eq. (1) satisfy the condition (i) in Theorem 1 and that* $Q(n)$, $n \in \mathbb{Z}^+$, *are all compact. Then, for Eq.(1) the following statements are equivalent.*

(i) $(zI - \tilde{Q}(z))^{-1} \in \mathcal{L}(X)$ *for* $|z| \geq 1$.

(ii) $\{R(n)\} \in L^1(\mathbb{Z}^+)$.

(iii) The zero solution of Eq.(1) is uniformly asymptotically stable.

Remark 1. *The implication (i)⇒ (ii) in Theorem 2 holds true under a weker assumption. Indeed, the implication holds true without the compactness condition on* $Q(n)$, $n \in \mathbb{Z}^+$. *Also, the implication (iii)⇒ (i) holds true without the condition (i) in Theorem 1.*

An elememt $\{S(n)\} \in L^1(\mathbb{Z}^+)$ is said to *decay exponentially* if there exist positive constants M and ν with $0 < \nu < 1$ such that $\|S(n)\| \leq M\nu^n$ for $n \in \mathbb{Z}^+$.

Theorem 3. *Let* $Q(n)$, $n \in \mathbb{Z}^+$, *be compact operators, and assume that* $\|R(n)\|$ *tends to zero as* $n \to \infty$. *Then* $R(n)$ *decays exponentially if and only if so does* $Q(n)$.

Outline of "only if" part. Since $(zI - \tilde{Q}(z))\tilde{R}(z) = zI$ for $|z| \geq 1$, by applying the Riesz-Schauder theory to the compact operator $\tilde{Q}(z)$ we can deduce that $\tilde{R}(z)$ is invertible in $\mathcal{L}(X)$ for $|z| \geq 1$. Then there is a positive constant δ such that $\tilde{R}(z)$ is invertible in $\mathcal{L}(X)$ for any z with $|z| \geq 1 - \delta$. Let us consider an analytic function $F(z)$ defined by $F(z) = zI - z\tilde{R}(z)^{-1}$ on the domain $|z| > 1 - \delta$, and denote the Laurent expansion of $F(z)$ by

$$F(z) = \sum_{n \in \mathbb{Z}} b(n) z^n, \qquad |z| > 1 - \delta,$$

where

$$b(n) = \frac{1}{2\pi i} \int_{|z|=L} \frac{F(z)}{z^{n+1}} dz, \qquad L > 1 - \delta.$$

Since $F(z) = \tilde{Q}(z)$ for $|z| \geq 1$ and hence $\sup_{|z|\geq 1} \|F(z)\| = \sup_{|z| \geq 1} \left\| \sum_{n=0}^{\infty} Q(n) z^{-n} \right\| \leq \sum_{n=0}^{\infty} \|Q(n)\| = \|Q\|$, one can derive that $b(n) = 0$ $(n = 1, 2, \dots)$, and hence

$$F(z) = \sum_{n=0}^{\infty} b(-n) z^{-n}, \qquad |z| > 1 - \delta.$$

In particular, the series $\sum_{n=0}^{\infty} b(-n)(1 - \delta/2)^{-n}$ is convergent. Hence we have

$$\|b(-n)\| \leq M_1 \Big(1 - \frac{\delta}{2}\Big)^n, \quad n \in \mathbb{Z}^+$$

for some constant $M_1 > 0$. The uniqueness of the Laurent expansion yields that $Q(n) = b(-n)$, $n \in \mathbb{Z}^+$, and consequently $Q(n)$ must decay exponentially. □

Clearly, the exponential stability implies the uniform asymptotic stability. The converse implication is not always true. Indeed, it follows from Theorem 3 that:

Theorem 4. *Let $Q(n)$, $n \in \mathbb{Z}^+$, be compact operators, and assume that the zero solution of Eq. (1) is uniformly asymptotically stable. Then the zero solution of Eq. (1) is exponentially stable if and only if $Q(n)$ decays exponentially.*

4. Examples and Some Remarks

In what follows, we employ the notation $[\cdot]$ to denote the Gaussian symbol, and consider the differential equation

$$\dot{u}(t) = Au(t) + \sum_{k=0}^{\infty} B(k)u([t-k]), \quad t \geq 0 \tag{2}$$

on a Banach space X, which contains piecewise continuous delays $t-[t-k]$, $k = 0, 1, 2, \dots$. Here and hereafter, we assume that A is the inifinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geq 0$, of bounded linear operators on X, and $B(k)$, $k = 0, 1, 2, \dots$, are bounded linear operators on X such that $\sum_{k=0}^{\infty} \|B(k)\| < \infty$. It is known [3, 6] that Eq. (2) is reduced to the following Volterra difference equation

$$u(n+1) = \sum_{k=0}^{\infty} Q(k)u(n-k), \quad n \in \mathbb{Z}^+, \tag{3}$$

where $Q(k)$, $k \in \mathbb{Z}^+$, are bounded linear operators on X defined by

$$Q(0)x = T(1)x + \int_0^1 T(\tau)B(0)x d\tau, \quad Q(k)x = \int_0^1 T(\tau)B(k)x d\tau, \quad k = 1, 2, \dots \tag{4}$$

for $x \in X$. Sometimes, we call Eq. (3) *the induced Volterra difference equation* of Eq. (2).

It is known ([3, Proposition 1]) that $Q(k)$, $k \in \mathbb{Z}^+$, defined by the relation (4) are compact operators on X whenever $T(t)$ is a compact semigroup on X. In the restricted case where $B(k)$, $k \in \mathbb{Z}^+$ are scalar, that is, $B(k) \equiv b(k)I$, $k \in \mathbb{Z}^+$, for some $b(k) \in \mathbb{C}$, we can determine the spectrum of the characteristic operator $zI - \tilde{Q}(z) := zI - \sum_{k=0}^{\infty} Q(k)z^{-k}$ of Eq. (3).

Proposition 1. *Let $T(t)$ be a compact semigroup on X, and assume that $B(k) \equiv b(k)I$, $k \in \mathbb{Z}^+$, where $b(k)$ is a scalar function satisfying $\sum_{k=0}^{\infty} |b(k)| < \infty$. Then the spectrum of the characteristic operator $zI - \tilde{Q}(z)$ with $|z| \geq 1$ of Eq. (3) is given by*

$$\sigma(zI - \tilde{Q}(z)) = \left(\{z\} \cup \{z - e^{\nu} - \tilde{b}(z) \int_0^1 e^{\nu\tau} d\tau \mid \nu \in \sigma(A)\} \right). \tag{5}$$

Outline of proof. We will give an outline of the proof; see [3, Theorem 3] for the complete proof. By using the continuity of $T(t)$ in $t > 0$ with respect to the operator norm, one can see that

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{k=1}^{n}\{T(1/n)\}^k - \int_0^1 T(\tau)d\tau\right\| = 0. \tag{6}$$

Now, set $S = \{T(t) : \; 0 \le t \le 1\}$. Since S commutes, $\mathcal{A} := \Gamma(\Gamma(S))$ is a commutative Banach algebra containing S, see [5, p. 280, Theorem 11.22]. Here, for any subset Ω of $\mathcal{L}(X)$, $\Gamma(\Omega)$ denotes the centralizer of Ω; that is,

$$\Gamma(\Omega) = \{v \in \mathcal{L}(X) : vw = wv \text{ for every } w \in \Omega\}.$$

Let Δ be the maximal ideal space of $\mathcal{A}$. Let us denote by $\hat{a}$ the Gelfand transform of $a \in \mathcal{A}$. It is known [5, pp. 268–270] that $\hat{a}$ is a function from Δ (which is equipped with the Gelfand topology) into $\mathbf{C}$ with the properties that the range of $\hat{a}$ is the spectrum $\sigma(a)$ of a and that

$$\|\hat{a}\|_\infty \le \|a\|, \quad a \in \mathcal{A},$$

where $\|\hat{a}\|_\infty$ is the maximum of $|\hat{a}(\xi)|$ on Δ. Moreover, the Gelfand transform is a homomorphism mapping $\mathcal{A}$ into a subspace of $C(\Delta; \mathbf{C})$, the space of all the complex valued continuous functions on Δ. Let $|z| \ge 1$, and put

$$a = zI - \tilde{Q}(z) = zI - T(1) - \left(\int_0^1 T(\tau)d\tau\right)\tilde{b}(z)$$

and

$$a_n = zI - W^n - \left(\frac{1}{n}\sum_{k=1}^{n} W^k\right)\tilde{b}(z)$$

for each $n = 1, 2, \ldots$, where $W := T(1/n)$. Then $\{a, a_1, a_2, \ldots\} \subset \mathcal{A}$, and by (6) we get

$$\|\widehat{(a_n)} - \hat{a}\|_\infty \le \|a_n - a\| = \left\|\frac{1}{n}\sum_{k=1}^{n}\{T(1/n)\}^k - \int_0^1 T(\tau)d\tau\right\| |\tilde{b}(z)| \to 0$$

as $n \to \infty$. Thus

$$\lim_{n\to\infty}\widehat{(a_n)}(\xi) = \hat{a}(\xi), \qquad \xi \in \Delta. \tag{7}$$

Observe that $\widehat{(a_n)}(\xi) = z - \left(\hat{W}(\xi)\right)^n - \frac{1}{n}\sum_{k=1}^{n}\left(\hat{W}(\xi)\right)^k \tilde{b}(z)$. Since the operator $T(1/n)$ is compact, the Riesz-Schauder theorem implies that $\sigma(T(1/n)) = P_\sigma(T(1/n)) \cup \{0\}$. Also, it follows from [4, Theorems 2.2.3–2.2.4] that

$$\exp\Big((1/n)\sigma(A)\Big) \subset \sigma(T(1/n)), \qquad P_\sigma(T(1/n)) \cup \{0\} = \exp\Big((1/n)P_\sigma(A)\Big) \cup \{0\}.$$

Therefore we get $\sigma(W) = \sigma(T(1/n)) = \exp\Big((1/n)P_\sigma(A)\Big) \cup \{0\}$. By virtue of these observations, we see that the range of $\widehat{(a_n)}$ is identical with the set

$$\{z\} \cup \left(\{z - e^\nu - \frac{1}{n}\sum_{k=1}^{n} e^{(k/n)\nu}\tilde{b}(z) \mid \nu \in \sigma(A)\}\right).$$

Note that $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} e^{(k/n)\nu} = \int_0^1 e^{\nu\tau}d\tau$. Therefore, combining this fact with (7) we conclude that the set in the right hand side of (5) is identical with the range of $\hat{a}$ which is equal to $\sigma(a) = \sigma(zI - \tilde{Q}(z))$. □

The following corollaries immediately follow from Theorems 2–4 and Proposition 1.

Corollary 2. *Let $T(t)$ be a compact semigroup on X, and assume that $B(k) \equiv b(k)I$, $k \in \mathbb{Z}^+$, where $b(k)$ is a scalar function satisfying $\sum_{k=0}^{\infty}|b(k)| < \infty$. Then the following two statements are equivalent:*

(i) The zero solution of Eq. (3) is uniformly asymptotically stable;

(ii) $z \neq e^\nu + \tilde{b}(z)\int_0^1 e^{\nu\tau}d\tau, \qquad (\forall\ |z| \geq 1,\ \nu \in \sigma(A))$.

Corollary 3. *Let all the conditions in Corollary 2 hold true, and assume that*

$$z \neq e^\nu + \tilde{b}(z)\int_0^1 e^{\nu\tau}d\tau, \qquad (\forall\ |z| \geq 1,\ \nu \in \sigma(A)).$$

Then the zero solution of Eq. (3) is exponentially stable if and only if $b(n)$ decays exponentially.

In the case where the dimension of X is finite or $\{Q(n)\}$ decays exponentially , Theorem 2 and Corollary 1 remain valid without the condition (i) in Theorem 1, that is, the commutative condition on $Q(n)$ (cf. [2, Theorem 2]). In the case where the dimension of X is infinite, it is natural to ask if Theorem 2 and Corollary 1 in this paper remain valid without the commutative condition. Although the authors have not succeeded in answering the question generally, we can partly answer the question. Indeed, let $\mathcal{A}$ be a commutative Banach algebra (containing the identity operator) in $\mathcal{L}(Y)$, where Y is a Banach space, and let us consider all of matrices whose components belong to $\mathcal{A}$. For simplicity, we treat the space $M(\mathcal{A})$ of all 2×2 matrices in the following. Each $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $M(\mathcal{A})$ may be considered as a bounded linear operator on the Banach space $X := Y \oplus Y$. We define the *determinant* $\det T$ of T by

$$\det T = ad - bc.$$

It is easy to see that if $\det T$ is invertible in $\mathcal{L}(Y)$, then T is invertible in $\mathcal{L}(X)$, and the inverse T^{-1} is given by

$$T^{-1} = \begin{pmatrix} d(\det T)^{-1} & -b(\det T)^{-1} \\ -c(\det T)^{-1} & a(\det T)^{-1} \end{pmatrix}.$$

Now, we consider Eq. (1) whose coefficients $Q(n)$ belong to $M(\mathcal{A})$. Notice that the condition (i) in Theorem 1 is not always satisfied. Let $R = \{R(n)\}$ be the fundamental solution of Eq. (1). It is easy to see that $R(n)$ belongs to $M(\mathcal{A})$. Assume that R is summable. Then, for any $|z| \geq 1$ we get $(zI - \tilde{Q}(z))\tilde{R}(z) = zI$, which yields that

$$\det(zI - \tilde{Q}(z)) \cdot \det \tilde{R}(z) = z^2 I.$$

Thus, if R is summable, then the following condition is satisfied;

(ii*) for any $|z| \geq 1$, $\det(zI - \tilde{Q}(z))$ is invertible in $\mathcal{L}(Y)$.

Conversely, assume that the condition (ii*) is satisfied. Define $S = \{S(n)\}$ by the relation

$$S(0) = I, \qquad S(n) = -Q(n-1) \quad n = 1, 2, \dots .$$

By the condition (ii*) and Theorem 1, one can see that there exists an $r \in L^1(\mathbb{Z}^+)$ such that

$$\tilde{r}(z) = [\det \tilde{S}(z)]^{-1}$$

for $|z| \geq 1$. Consequently, each component of $\tilde{R}(z) = (\tilde{S}(z))^{-1}$ is a product of $\tilde{r}(z)$ with the components of $\tilde{S}(z)$; in other words, $R(n)$ is a convolution of $r(n)$ and the component of $S(n)$, and hence $R = \{R(n)\}$ is summable.

Summarizing the above facts, we see that Corollary 1 remains valid without the commutative condition if we replace the condition (ii') by the condition (ii*). Similarly, we can remove the commutative condition in Theorem 2 if the condition (i) in Theorem 2 is replaced by the condition (ii*).

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