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CELLULAR BASES OF THE TWO-PARAMETER VERSION OF THE CENTRALISER ALGEBRA FOR THE MIXED TENSOR REPRESENTATIONS OF THE QUANTUM GENERAL LINEAR GROUP

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ABSTRACT. An explicit combinatorial construction is given for the cellular bases, in the sense of Graham and Lehrer, for the centraliser algebra for the mixed tensor representations of the quantum general linear group.

1. INTRODUCTION

Let $\mathfrak{g} = gl(k, \mathbb{C})$ and V denote the natural representation of $U_{\hat{q}}(\mathfrak{g})$. If V^* is the dual space of V considered as a $U_{\hat{q}}(\mathfrak{g})$-module then the mixed tensor representation of $U_{\hat{q}}(\mathfrak{g})$ is defined to be the rational representation $T^{m,n} = V^{\otimes m} \otimes (V^*)^{\otimes n}$. Schur-Weyl duality in this context has been considered by Kosuda and Murakami in [7] where they constructed a generalised Hecke algebra $H^k_{m,n}(\hat{q})$ such that the action of $H^k_{m,n}(\hat{q})$ on $T^{m,n}$ generates $\mathrm{End}_{U_{\hat{q}}(\mathfrak{g})}(T^{m,n})$. Subsequently, Leduc [8] has defined a two parameter version $A_{m,n}(\hat{r}, \hat{q})$ of the generalised Hecke algebra of Kosuda and Murakami, from which $H^k_{m,n}(q)$ is recovered by making the specialisation $\hat{r} = \hat{q}^k$.

The main purpose of this paper is to give an explicit combinatorial construction of the representations of $A_{m,n}(\hat{r}, \hat{q})$; it is shown that each cellular basis, in terms of Graham and Lehrer, for the tensor product of (classical) Iwahori-Hecke algebras $H_m(\hat{q}) \otimes H_n(\hat{q})$ will give rise to a cellular structure on $A_{m,n}(\hat{r}, \hat{q})$. By this means, we produce for instance, an analogue of the Murphy basis [11] for the algbebra $A_{m,n}(\hat{r}, \hat{q})$, along with naturally defined cell modules, the basis of which will be indexed by certain multi-tableau. By Graham and Lehrer, these cell modules (which generalise the Specht modules from the classical theory of the representations of the symmetric group), will be absolutely irreducible for generic parameters $\hat{r}$ and $\hat{q}$ and, in the non-generic setting will have a radical defined in terms of a certain associative, symmetric bilinear form.

The irreducible representations of $\mathcal{A}_{m,n}(\hat{r}, \hat{q})$ have also been constructed by Kosuda [5] by means of an analogue of the Kazhdan-Lusztig basis of the Iwahori-Hecke algebra of type A, though without reference to Graham and Lehrer's machinery of cellular bases. It being that the Kazhdan-Lusztig basis for the Iwahori-Hecke algebra of type A is cellular, the procedures given below, which explicitly relate cellular structures on $\mathcal{A}_{m,n}(\hat{r}, \hat{q})$ to cellular structures on the Iwahori-Hecke algebras, allow us to again recover the results of [5].

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2. PRELIMINARIES

In this section we establish the basic notation and state some known results which will be used subsequently. A reference for the material presented in this section is [9].

2.1. **The Symmetric Group.** Let $\mathfrak{S}_n$ denote the symmetric group acting on the integers $\{1,2,\dots,n\}$ on the right. The elementary transpositions in $\mathfrak{S}_n$ are the elements

$$S = \{s_i = (i,i+1) \mid 1 \le i < n\}.$$

The elementary transpositions, together with the relations

$$\begin{aligned} &s_i^2 = 1 && \text{for } 1 \le i < n \\ &s_is_j = s_js_i && \text{for } 2 \le |i-j| \text{ and } 1 \le i,j < n \\ &s_is_{i+1}s_i = s_{i+1}s_is_{i+1} && \text{for } 1 \le i < n-1 \end{aligned}$$

give a presentation for $\mathfrak{S}_n$ as a Coxeter group. Let w be a permutation in $\mathfrak{S}_n$. An expression $w = s_{i_1}s_{i_2}\dots s_{i_k}$ for w in terms of elementary transpositions is said to be *reduced* if w cannot be written as a proper sub-expression of $s_{i_1}s_{i_2}\dots s_{i_k}$. In this case we say w is a permutation with *length* k and write $l(w) = k$. Note that while there are usually several reduced expressions for w, the length of w will not depend on this choice. The length function on $\mathfrak{S}_n$ is determined by the properties

$$l(s_iw) = \begin{cases} l(w)+1 & \text{if } (i)w < (i+1)w, \\ l(w)-1 & \text{otherwise;} \end{cases} \tag{1}$$

and,

$$l(ws_i) = \begin{cases} l(w)+1 & \text{if } (i)w^{-1} < (i+1)w^{-1}, \\ l(w)-1 & \text{otherwise,} \end{cases} \tag{2}$$

together with the normalizing condition $l(1_{\mathfrak{S}_n}) = 0$.

2.2. **Compositions and Tableaux.** Let $k \ge 0$ be an integer. A partition of k is a non-increasing sequence $\nu = (\nu_1,\nu_2,\dots)$ of integers such satisfying $\sum_{i\ge1}\nu_i = k$. We will write $\nu \vdash k$ to denote the fact that ν is a partition of k. If ν is a partition it will also be convenient to write $|\nu| = k$ whenever $\sum_{i\ge1}\nu_i = k$. If μ,ν are partitions of k, then write $\mu \trianglerighteq \nu$ and say μ *dominates* ν, if $\sum_{i=1}^{j}\mu_k \ge \sum_{i=1}^{j}\nu_k$ for all $j \ge 0$. The fact that $\mu \trianglerighteq \nu$ and $\mu \ne \nu$ will be denoted by $\mu \triangleright \nu$.

The diagram of a partition $\nu \vdash k$ is the set of nodes

$$[\nu] = \{\, (i,j) \mid 1 \le j \le \nu_i \text{ and } i \ge 1 \,\} \subset \mathbb{N}\times\mathbb{N}$$

Let $\nu \vdash k$. A ν-tableau is a bijection $\mathfrak{t} : [\nu] \to \{1,2,\dots,k\}$; equivalently a ν-tableau $\mathfrak{t}$ may be regarded as a labeling of the nodes of $[\nu]$ by the integers $1,2,\dots,k$. For example, if $k = 7$ and $\nu = (4,2,1)$, then

$$\mathfrak{t} = \begin{array}{|c|c|c|c|} \hline 2 & 4 & 6 & 7 \\ \hline 1 & 3 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array} \tag{3}$$

is a ν-tableau. The super-standard tableau $\mathfrak{t}^\nu$ is the unique ν-tableau in which has as its entries the integers $1,2,\dots,k$ appearing in increasing sequence from left to right and top to bottom. In case $k = 7$ and $\nu = (4,2,1)$ we have

$$\mathfrak{t}^\nu = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \tag{4}$$

A ν-tableau $\mathfrak{t}$ is said to be *row standard* if the entries of each row of $\mathfrak{t}$ increase when read from left to right and a row standard ν-tableau $\mathfrak{t}$ is said to be *standard* if the entries of each column of $\mathfrak{t}$ increase when read from top to bottom. The tableau of (3) is row standard but not standard. We will denote by $\operatorname{Std}(\nu)$ the collection of standard ν-tableaux.

Let $\nu \vdash k$ be a partition. The symmetric group $\mathfrak{S}_k$ acts from the right on the set of ν-tableaux by permuting entries. Let, for example, $n = 5$ and $\nu = (3,2)$; if $\mathfrak{t} = \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$, then $\mathfrak{t}(1,2)(4,5) = \begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1&5\\\cline{1-2}\end{array}$. If $\mathfrak{t}$ is a ν-tableau, then $d(\mathfrak{t}) \in \mathfrak{S}_k$ is the permutation defined by the equation $\mathfrak{t}^\nu d(\mathfrak{t}) = \mathfrak{t}$. The Young subgroup of $\mathfrak{S}_\nu \cong \mathfrak{S}_{\nu_1} \times \mathfrak{S}_{\nu_2} \times \ldots \mathfrak{S}_{\nu_k}$ will be the row stabiliser of $\mathfrak{t}^\nu$ in $\mathfrak{S}_k$; that is

$$\mathfrak{S}_\nu = \langle\, s_i \mid i, i+1 \text{ are in the same row of } \mathfrak{t}^\nu \,\rangle.$$

For example, when $\nu = (4,2,1)$ and $\mathfrak{t}^\nu$ is given by (4), then $\mathfrak{S}_\nu = \langle s_1, s_2, s_3\rangle \times \langle s_5\rangle$.

A multi-partition of k is a tuple of partitions $\nu = (\nu^{(1)}, \nu^{(2)}, \ldots)$ satisfying the condition that $\sum_{i\geq 0} |\nu^{(i)}| = k$. The diagram of the multi-partition $\nu = (\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(k)})$ is defined simply to be the corresponding tuple of diagrams $[\nu] = ([\nu^{(1)}], [\nu^{(2)}], \ldots, [\nu^{(k)}])$. Since our definitions of multi-tableaux will vary according to context, we will confine ourselves here to the definition of diagram of a multi-partition and postpone the introduction of multi-tableaux.

2.3. **The Iwahori-Hecke Algebra of the Symmetric Group.** Let R be a domain and q^2 be an invertible element in R. The Iwahori-Hecke algebra $\mathcal{H}_{R,n}(q^2)$ associated with $\mathfrak{S}_n$ is the unital associative R-algebra generated by the elements $\{X_i \mid 1 \leq i < n\}$ subject to the relations

$$\begin{aligned}
&(X_i - q^2)(X_i + 1) = 0 && \text{for } 1 \leq i < n,\\
&X_iX_{i+1}X_i = X_{i+1}X_iX_{i+1} && \text{for } 1 \leq i \leq n-2, \text{ and,}\\
&X_iX_j = X_jX_i && \text{for } 2 \leq |i-j| \text{ and } 1 \leq i,j < n.
\end{aligned}$$

If w is a permutation in $\mathfrak{S}_n$ with reduced expression $w = s_{i_1} \ldots s_{i_k}$, the element X_w of $\mathcal{H}_{R,n}(q^2)$ is defined by

$$X_w = X_{i_1} \ldots X_{i_k}.$$

By Matsumoto's Theorem (Theorem 1.8 of [9]), X_w is a well defined element of $\mathcal{H}_{R,n}(q^2)$. The next statement follows from (1) and (2) together with the defining relations for $\mathcal{H}_{R,n}(q^2)$.

Lemma 2.1. *If $w \in \mathfrak{S}_n$ and s is an elementary transposition, then*

$$X_wX_s = \begin{cases} X_{ws} & \text{if } l(ws) > l(w),\\ q^2X_{ws} + (q^2-1)X_w & \text{if } l(ws) < l(w);\end{cases}$$

and,

$$X_sX_w = \begin{cases} X_{sw} & \text{if } l(sw) > l(w),\\ q^2X_{sw} + (q^2-1)X_w & \text{if } l(sw) < l(w).\end{cases}$$

The next statement is Lemma 2.3 of [11].

Lemma 2.2. *Let $*, \dagger, \#$ be the maps defined by:*

$$\begin{aligned}
* &: X_w \mapsto X_{w^{-1}}\\
\dagger &: X_w \mapsto (-q^2)^{l(w)}X_w^{-1}\\
\sharp &: X_w \mapsto (-q^2)^{l(w)}X_{w^{-1}}^{-1},
\end{aligned}$$

for each $w \in \mathfrak{S}_n$, extended to $\mathcal{H}_{R,n}(q^2)$ by linearity. Then $*$ and $\dagger$ are R-algebra anti-involutions of $\mathcal{H}_{R,n}(q^2)$ and $\sharp$ is an R-algebra automorphism of $\mathcal{H}_{R,n}(q^2)$.

2.4. **The Murphy Basis for the Iwahori-Hecke Algebra.** In [11] Murphy gives a nice basis for $\mathcal{H}_{R,n}(q^2)$ indexed by pairs of standard tableaux, a basis which allows him to define a filtration on $\mathcal{H}_{R,n}(q^2)$ by two-sided ideals and to describe the representations of $\mathcal{H}_{R,n}(q^2)$. In this section we recall Murphy's construction and refer the reader to [11] or [9] for the details.

For a partition $\lambda \vdash n$, Murphy defines the element $m_\lambda \in \mathcal{H}_{R,n}(q^2)$ by

$$m_\lambda = \sum_{w \in \mathfrak{S}_\lambda} X_w,$$

and associates to each pair $\mathfrak{s}, \mathfrak{t}$ of standard λ-tableaux the element

$$m_{\mathfrak{s}\mathfrak{t}} = X^*_{d(\mathfrak{s})} m_\lambda X_{d(\mathfrak{t})}.$$

Let N^λ denote the R-submodule of $\mathcal{H}_{R,n}(q^2)$ generated by the elements

$$\{m_{\mathfrak{s}\mathfrak{t}} = X^*_{d(\mathfrak{s})} m_\mu X_{d(\mathfrak{t})} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\mu) \text{ and } \mu \trianglerighteq \lambda\}$$

and $\check{N}^\lambda$ be the R-submodule of N^λ generated by

$$\{m_{\mathfrak{s}\mathfrak{t}} = X^*_{d(\mathfrak{s})} m_\mu X_{d(\mathfrak{t})} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\mu) \text{ and } \mu \triangleright \lambda\}.$$

The following result is due to Murphy (Theorem 4.17 and Theorem 4.18 of [11] or Theorem 3.2 of [9]).

Theorem 2.3. *The Iwahori-Hecke algebra $\mathcal{H}_{R,n}(q^2)$ has a free R-basis*

$$\mathscr{M} = \{\, m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\lambda) \text{ and } \lambda \vdash n\}.$$

Moreover, the following hold:

(1) *The R-linear map determined by $m_{\mathfrak{s}\mathfrak{t}} \mapsto m_{\mathfrak{t}\mathfrak{s}}$, for all $m_{\mathfrak{s}\mathfrak{t}} \in \mathscr{M}$, is an algebra anti-involution of $\mathcal{H}_{R,n}(q^2)$.*

(2) *Suppose that $h \in \mathcal{H}_{R,n}(q^2)$ and that $\mathfrak{t} \in \operatorname{Std}(\lambda)$. Then there exist $a_\mathfrak{v} \in R$, for $\mathfrak{v} \in \operatorname{Std}(\lambda)$, such that*

$$m_{\mathfrak{s}\mathfrak{t}} h \equiv \sum_{\mathfrak{v} \in \operatorname{Std}(\lambda)} a_\mathfrak{v} m_{\mathfrak{s}\mathfrak{v}} \mod \check{N}^\lambda \tag{5}$$

for all $\mathfrak{s} \in \operatorname{Std}(\lambda)$.

The crucial point about (5) is that the elements $\mathfrak{v}$ and $a_\mathfrak{v}$ depend on $\mathfrak{t}$ and h but not on $\mathfrak{s}$. Also, as a consequence of Theorem 2.3, both N^λ and $\check{N}^\lambda$ are two sided ideals of $\mathcal{H}_{R,n}(q^2)$ and the dominance order on partitions gives rise to a filtration of $\mathcal{H}_{R,n}(q^2)$ by two-sided ideals.

The right *Specht module* S^λ is defined to be the $\mathcal{H}_{R,n}(q^2)$-submodule of $N^\lambda/\check{N}^\lambda$ generated by the elements

$$\{\, \check{N}^\lambda + m_{\mathfrak{t}^\lambda\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\lambda)\,\}. \tag{6}$$

By the last item of Theorem 2.3, the set (6) is a free R-basis for S^λ. For $\mathfrak{s} \in \operatorname{Std}(\lambda)$, let $m_\mathfrak{s}$ denote the element $\check{N}^\lambda + m_{\mathfrak{t}^\lambda\mathfrak{s}} \in S^\lambda$. Murphy defines a symmetric bilinear form $\langle\ ,\ \rangle : S^\lambda \times S^\lambda \to R$ by setting

$$\langle m_\mathfrak{s}, m_\mathfrak{t}\rangle m_\lambda \equiv m_{\mathfrak{t}^\lambda\mathfrak{s}} m^*_{\mathfrak{t}^\lambda\mathfrak{t}} \mod \check{N}^\lambda.$$

Since $\langle\ ,\ \rangle$ satisfies the condition $\langle m_\mathfrak{s}, m_\mathfrak{t} h\rangle = \langle m_\mathfrak{s} h^*, m_\mathfrak{t}\rangle$ for all $h \in \mathcal{H}_{R,n}(q^2)$, it follows that the set $\operatorname{rad}(S^\lambda) = \{\, a \in S^\lambda \mid \langle a, b\rangle = 0 \text{ for all } b \in S^\lambda\,\}$ will be a right $\mathcal{H}_{R,n}(q^2)$-module. Consequently Murphy defines $D^\lambda = S^\lambda/\operatorname{rad}(S^\lambda)$. The first item below is Theorem 6.2 of [11] while the second item is Theorem 6.3 of [11].

Theorem 2.4. *Let R be a field. Then*

(1) *Then either $D^\lambda = 0$ or D^λ is an absolutely irreducible $\mathcal{H}_{R,n}(q^2)$-module.*

(2) *The collection* $\{ D^\lambda \mid \lambda \vdash n \text{ and } D^\lambda \neq 0 \}$ *is a complete set of pairwise non-isomorphic absolutely irreducible* $\mathcal{H}_{R,n}(q^2)$*-modules.*

2.5. **Cellular Algebras.** The definition of a cellular algebra, due to Graham and Lehrer in [2] was motivated by Kazhdan-Lusztig theory. In this section we state the main results of Graham and Lehrer and refer the reader to the exposition in [9] for a more thorough treatment. For an equivalent but basis free approach to the subject, the reader is referred to a work of König and Xi [3].

Definition 2.1. Let R be a domain and A a unital associative R algebra with a free R basis. Let Λ be a finite set with partial order $\leq$ and suppose that for each $\lambda \in \Lambda$ there is a finite index set $\mathcal{I}(\lambda)$ such that there exists a set

$$\mathscr{C} = \{ c^\lambda_{\mathfrak{v}\mathfrak{u}} \in A \mid \mathfrak{v}, \mathfrak{u} \in \mathcal{I}(\lambda) \text{ and } \lambda \in \Lambda \}$$

which is an R-basis for A. For $\lambda \in \Lambda$, let $\breve{A}^\lambda$ denote the R-submodule of A generated by the elements

$$\{ c^\mu_{\mathfrak{v}\mathfrak{u}} \mid \mathfrak{v}, \mathfrak{u} \in \mathcal{I}(\mu) \text{ where } \mu \in \Lambda \text{ and } \lambda < \mu \}.$$

Then $(\Lambda, \mathscr{C})$ is a *cellular basis* and A a *cellular algebra* if

(1) the R-linear map $* : A \to A$ determined by $* : c^\lambda_{\mathfrak{v}\mathfrak{u}} \mapsto c^\lambda_{\mathfrak{u}\mathfrak{v}}$ for all $\lambda \in \Lambda$ and $\mathfrak{u}, \mathfrak{v} \in \mathcal{I}(\lambda)$ is an algebra anti-automorphism of A; and,

(2) if $\lambda \in \Lambda, \mathfrak{v} \in \mathcal{I}(\lambda)$ and $a \in A$, then there exist $\alpha_\mathfrak{t} \in R$, for $\mathfrak{t} \in \mathcal{I}(\lambda)$, such that

$$c^\lambda_{\mathfrak{u}\mathfrak{v}} a \equiv \sum_{\mathfrak{t} \in \mathcal{I}(\lambda)} \alpha_\mathfrak{t} c^\lambda_{\mathfrak{u}\mathfrak{t}} \mod \breve{A}^\lambda \tag{7}$$

for all $\mathfrak{u} \in \mathcal{I}(\lambda)$.

The essential feature of the expression (7) is that the elements $\mathfrak{t} \in \mathcal{I}(\lambda)$ and the constants $\alpha_\mathfrak{t}$ are determined entirely by a and $\mathfrak{v}$ and are independent of $\mathfrak{u}$.

Examples of cellular algebras include Ariki-Koike algebras (including the Iwahori-Hecke algebras), the Brauer and Temperly-Lieb algebras (Theorem 4.10 and Theorem 6.7 of [2]) and the Birman-Murakami-Wenzl algebras (Theorem 3.11 of [12]). Note that a cellular algebra may have more than one cellular basis; the Murphy basis, for instance, makes the Iwahori-Hecke algebra into a cellular algebra, as does the Kazhdan-Lusztig basis for the Iwahori-Hecke algebra (see, for example, Theorem 5.5 of [2]).

For $\lambda \in \Lambda$, denote by A^λ the R-submodule of A generated by the elements $c^\mu_{\mathfrak{v}\mathfrak{u}}$ where $\mathfrak{v}, \mathfrak{u} \in \mathcal{I}(\mu)$ and $\mu \geq \lambda$. Observe that $\breve{A}^\lambda \subseteq A^\lambda$ and that $A^\lambda/\breve{A}^\lambda$ has an R-basis given by $\breve{A}^\lambda + c^\lambda_{\mathfrak{v}\mathfrak{u}}$ where $\mathfrak{v}, \mathfrak{u} \in \mathcal{I}(\lambda)$. The next statement is now a straightforward consequence of the definitions (Lemma 2.3 of [9]).

Lemma 2.5. *Let* $(\mathscr{C}, \Lambda)$ *be a cellular basis for* A *and* λ *be an element of* Λ.

(1) *Suppose that* $\mathfrak{u} \in \mathcal{I}(\lambda)$ *and that* $a \in A$. *Then for all* $\mathfrak{v} \in \mathcal{I}(\lambda)$,

$$a^* c^\lambda_{\mathfrak{u}\mathfrak{v}} \equiv \sum_{\mathfrak{t} \in \mathcal{I}(\lambda)} \alpha_\mathfrak{t} c^\lambda_{\mathfrak{t}\mathfrak{v}} \mod \breve{A}^\lambda$$

where, for each $\mathfrak{t}$, $\alpha_\mathfrak{t}$ *is the element of* R *determined by* (7).

(2) *The* R*-modules* A^λ *and* $\breve{A}^\lambda$ *are two-sided ideals of* A.

(3) *If* $\mathfrak{s}, \mathfrak{t} \in \mathcal{I}(\lambda)$, *then there are* $\alpha_{\mathfrak{s}\mathfrak{t}} \in R$ *such that for any* $\mathfrak{u}, \mathfrak{v} \in \mathcal{I}(\lambda)$,

$$c^\lambda_{\mathfrak{v}\mathfrak{s}} c^\lambda_{\mathfrak{t}\mathfrak{u}} \equiv \alpha_{\mathfrak{s}\mathfrak{t}} c^\lambda_{\mathfrak{v}\mathfrak{u}} \mod \breve{A}^\lambda. \tag{8}$$

The second item of Lemma 2.5 shows that there is a filtration of A by the ideals A^λ; indeed, the posets of ideals A^λ ordered by containment is isomorphic to the poset $(\Lambda, \leq)$. The third item shows that each of the quotients $A^\lambda/\check{A}^\lambda$ is equipped with a bilinear form; this bilinear form will be defined below.

Let $\lambda \in \Lambda$ be fixed. For $\mathfrak{v} \in \mathcal{I}(\lambda)$, define $C^\lambda_\mathfrak{v}$ to be the R-submodule of $A/\check{A}^\lambda$ generated by the elements $\{\check{A}^\lambda + c^\lambda_{\mathfrak{v}\mathfrak{u}} \mid \mathfrak{u} \in \mathcal{I}(\lambda)\}$. By (7), the algebra A has a well-defined action on $C^\lambda_\mathfrak{v}$ by right multiplication. Moreover, under this action $C^\lambda_\mathfrak{v} \cong C^\lambda_\mathfrak{u}$ whenever $\mathfrak{v}, \mathfrak{u} \in \mathcal{I}(\lambda)$. Given the latter observation, the right *cell module* C^λ is defined to be the right A-module which is free as an R-module with basis $\{c^\lambda_\mathfrak{v} \mid \mathfrak{v} \in \mathcal{I}(\lambda)\}$ and right A-action given by

$$c^\lambda_\mathfrak{v} a = \sum_\mathfrak{t} \alpha_\mathfrak{t} c^\lambda_\mathfrak{t}$$

where the $\alpha_\mathfrak{t}$ are given by (7). Then the map $C^\lambda_\mathfrak{u} \to C^\lambda$ defined by $c^\lambda_{\mathfrak{u}\mathfrak{v}} + \check{A}^\lambda \mapsto c^\lambda_\mathfrak{v}$ is an isomorphism of right A-modules. The left cell module $C^{*\lambda}$ is defined to be the left A-module which is free as an R-module with basis $\{c^\lambda_\mathfrak{v} \mid \mathfrak{v} \in \mathcal{I}(\lambda)\}$ and left A-action given by

$$a^* c^\lambda_\mathfrak{v} = \sum_\mathfrak{t} \alpha_\mathfrak{t} c^\lambda_\mathfrak{t}$$

where $\alpha_\mathfrak{t}$ are once more determined by (7). With this definition, it is easy to see that $C^{*\lambda} \cong \operatorname{Hom}_R(C^\lambda, R)$ as left A-modules. As a right A-module we have the decompositon

$$A/\check{A} \cong C^{*\lambda} \otimes_R C^\lambda \cong \bigoplus_{\mathfrak{v} \in \mathcal{I}(\lambda)} C^\lambda_\mathfrak{v}.$$

By Lemma 2.5 there is a bilinear form $\langle\ ,\ \rangle : C^\lambda \times C^\lambda \to R$

$$\langle c^\lambda_\mathfrak{s}, c^\lambda_\mathfrak{t} \rangle = \alpha_{\mathfrak{s}\mathfrak{t}} \text{ for all } \mathfrak{s}, \mathfrak{t} \in \mathcal{I}(\lambda),$$

where $\alpha_{\mathfrak{s}\mathfrak{t}}$ are determined by (8). The following statements follow readily from the definitions (Proposition 2.9 of [9]).

Proposition 2.6. *Let $\lambda \in \Lambda$ and $a \in A$. Then*

(1) $\langle c^\lambda_\mathfrak{u}, c^\lambda_\mathfrak{v} \rangle = \langle c^\lambda_\mathfrak{v}, c^\lambda_\mathfrak{u} \rangle$ *for all* $\mathfrak{u}, \mathfrak{v} \in \mathcal{I}(\lambda)$.
(2) $\langle c^\lambda_\mathfrak{u} a, c^\lambda_\mathfrak{v} \rangle = \langle c^\lambda_\mathfrak{v}, c^\lambda_\mathfrak{u} a^* \rangle$ *for all* $\mathfrak{u}, \mathfrak{v} \in \mathcal{I}(\lambda)$.
(3) $b\, c^\lambda_{\mathfrak{u}\mathfrak{v}} = \langle b, c^\lambda_\mathfrak{u} \rangle c^\lambda_\mathfrak{v}$ *for all* $\mathfrak{u}, \mathfrak{v} \in \mathcal{I}(\lambda)$ *and* $b \in C^\lambda$.

The *radical* of the module C^λ is defined to be

(9) $$\operatorname{rad}(C^\lambda) = \{ a \in C^\lambda \mid \langle a, b \rangle = 0 \text{ for all } b \in C^\lambda \}.$$

By the second item of Proposition 2.6, $\operatorname{rad}(C^\lambda)$ is an A-submodule of C^λ, motivating the definition $D^\lambda = C^\lambda / \operatorname{rad}(C^\lambda)$.

Proposition 2.7. *Let R be a field and let $\lambda \in \Lambda$.*

(1) *If $D^\lambda \neq 0$, then $D^\lambda = 0$ or D^λ is absolutely irreducible.*
(2) *The intersection of the maximal submodules of C^λ is equal to* $\operatorname{rad}(C^\lambda)$.

In principle at least, the following Theorem of Graham and Lehrer (Theorem 2.19 of [9]) allows us to classify the simple A-modules.

Theorem 2.8. *Suppose that R is a field. Then*

$$\{ D^\lambda \mid \lambda \in \Lambda \text{ and } D^\lambda \neq 0 \}$$

is a complete set of pairwise non-isomorphic irreducible A-modules.

Graham and Lehrer also give the following equivalences (Corollary 2.21 of [9]).

Theorem 2.9. *Suppose that R is a field. Then the following are equivalent.*

(1) *A is (split) semisimple.*
(2) $C^\lambda = D^\lambda$ *for all* $\lambda \in \Lambda$.
(3) $\operatorname{rad}(C^\lambda) = 0$ *for all* $\lambda \in \Lambda$.

Proposition 2.10. *Let A and B be R-algebras which are cellular with respect to the bases $(\mathscr{C}^{(1)}, \Lambda^{(1)})$ and $(\mathscr{C}^{(2)}, \Lambda^{(2)})$ respectively. Then $C = A \otimes B$ is cellular with the basis $(\mathscr{C}, \Lambda)$, where*

$$\mathscr{C} = \{a^{\lambda^{(1)}}_{\mathfrak{vu}} \otimes b^{\lambda^{(2)}}_{\mathfrak{ts}} \mid a^{\lambda^{(1)}}_{\mathfrak{vu}} \in \mathscr{C}^{(1)} \text{ and } b^{\lambda^{(2)}}_{\mathfrak{ts}} \in \mathscr{C}^{(2)}\},$$

and $\Lambda = \{(\lambda^{(1)}, \lambda^{(2)}) \mid \lambda^{(1)} \in \Lambda^{(1)}$ and $\lambda^{(2)} \in \Lambda^{(2)}\}$ is ordered by $(\lambda^{(1)}, \lambda^{(2)}) \le (\mu^{(1)}, \mu^{(2)})$ if $\lambda^{(1)} \le \mu^{(1)}$ and $\lambda^{(2)} \le \mu^{(2)}$.

Proof. Since $\mathscr{C}$ is a basis for C over R and the map $* : a^{\lambda^{(1)}}_{\mathfrak{vu}} \otimes b^{\lambda^{(2)}}_{\mathfrak{ts}} \mapsto a^{\lambda^{(1)}}_{\mathfrak{uv}} \otimes b^{\lambda^{(2)}}_{\mathfrak{st}}$ defines an algebra anti-involution of C, we must now verify that $(\mathscr{C}, \Lambda)$ satisfies the condition (2) of Definition 2.1. Let $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda$. If $a \in A$ and $b \in B$, then there exist $\alpha_{\mathfrak{h}}, \beta_{\mathfrak{g}} \in R$, for $\mathfrak{h} \in \mathcal{I}(\lambda^{(1)})$ and $\mathfrak{g} \in \mathcal{I}(\lambda^{(2)})$, together with $\check{a} \in \check{A}^{\lambda^{(1)}}$ and $\check{b} \in \check{B}^{\lambda^{(2)}}$ such that

$$\begin{aligned}
(a^{\lambda^{(1)}}_{\mathfrak{vu}} \otimes b^{\lambda^{(2)}}_{\mathfrak{ts}})(a \otimes b) &= a^{\lambda^{(1)}}_{\mathfrak{vu}} a \otimes b^{\lambda^{(2)}}_{\mathfrak{ts}} b = \Big(\sum_{\mathfrak{h}} \alpha_{\mathfrak{h}} a^{\lambda^{(1)}}_{\mathfrak{vh}} + \check{a}\Big) \otimes \Big(\sum_{\mathfrak{g}} \beta_{\mathfrak{g}} b^{\lambda^{(2)}}_{\mathfrak{th}} + \check{b}\Big) \\
&= \sum_{\mathfrak{h}} \alpha_{\mathfrak{h}} a^{\lambda^{(1)}}_{\mathfrak{vh}} \otimes \sum_{\mathfrak{g}} \beta_{\mathfrak{g}} b^{\lambda^{(2)}}_{\mathfrak{th}} + \sum_{\mathfrak{h}} \alpha_{\mathfrak{h}} a^{\lambda^{(2)}}_{\mathfrak{vh}} \otimes \check{b} + \sum_{\mathfrak{g}} \beta_{\mathfrak{g}} \check{a} \otimes b^{\lambda^{(2)}}_{\mathfrak{th}} \\
&\equiv \sum_{\mathfrak{h}} \alpha_{\mathfrak{h}} a^{\lambda^{(1)}}_{\mathfrak{vh}} \otimes \sum_{\mathfrak{g}} \beta_{\mathfrak{g}} b^{\lambda^{(2)}}_{\mathfrak{th}} \quad \operatorname{mod}\ (A^{\lambda^{(1)}} \otimes \check{B}^{\lambda^{(2)}} + \check{A}^{\lambda^{(1)}} \otimes B^{\lambda^{(2)}}).
\end{aligned}$$

Now observe that $(\lambda^{(1)}, \lambda^{(2)}) < (\mu^{(1)}, \mu^{(2)})$ if $\lambda^{(1)} < \mu^{(1)}$ and $\lambda^{(2)} \le \mu^{(2)}$ or $\lambda^{(1)} \le \mu^{(1)}$ and $\lambda^{(2)} < \mu^{(2)}$; thus if $\lambda = (\lambda^{(1)}, \lambda^{(2)})$, then $\check{C}^{\lambda}$ is generated as an R-module by

$$\{a^{\mu^{(1)}}_{\mathfrak{vu}} \otimes b^{\mu^{(2)}}_{\mathfrak{ts}} \mid \lambda^{(1)} < \mu^{(1)} \text{ and } \lambda^{(2)} \le \mu^{(2)} \text{ or } \lambda^{(1)} \le \mu^{(1)} \text{ and } \lambda^{(2)} < \mu^{(2)}\}$$

and we have shown that

$$(a^{\lambda^{(1)}}_{\mathfrak{vu}} \otimes b^{\lambda^{(2)}}_{\mathfrak{ts}})(a \otimes b) \equiv \sum_{\mathfrak{h},\mathfrak{g}} \alpha_{\mathfrak{h}} \beta_{\mathfrak{g}}\, a^{\lambda^{(1)}}_{\mathfrak{vh}} \otimes b^{\lambda^{(2)}}_{\mathfrak{th}} \quad \operatorname{mod}\ \check{C}^{\lambda}.$$

Since C is generated as an R algebra by $a \otimes b$, for $a \in A$ and $b \in B$, this completes the proof of the Proposition. □

3. The Algebra $\mathcal{A}_{m,n}(r, q)$

While the algebra $A_{m,n}(\hat{r}, \hat{q})$ is an associative algebra over a field $\kappa = \mathbb{C}(\hat{r}, \hat{q})$, rather than working over the rational function field κ, we produce cellular bases $\{ b_i \}$ for a generic algebra over an appropriate localization R of a polynomial ring over $\mathbb{Z}$ and then obtain bases for $A_{m,n}(\hat{r}, \hat{q})$ by specializations to κ.

Let r, q be indeterminates over $\mathbb{Z}$ and R be the localization of $\mathbb{Z}[r^{\pm 1}, q^{\pm 1}]$ at $(q^2 - 1)$. Define the element z in R as

$$z = \frac{r - r^{-1}}{q - q^{-1}}$$

and let m, n be non-negative integers. The generic algebra $\mathcal{A}_{m,n}(r, q)$ is the unital associative algebra with generators

$$\{ T_i, \hat{T}_j, E \mid 1 \le i < m, 1 \le j < n \}$$

subject to the the following relations:

$$\begin{array}{ll}
(T_i - q^2)(T_i + 1) = 0 & \text{for } 1 \le i < m \\
(\hat{T}_i - q^2)(\hat{T}_i + 1) = 0 & \text{for } 1 \le i < n \\
T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} & \text{for } 1 \le i < m \\
\hat{T}_i\hat{T}_{i+1}\hat{T}_i = \hat{T}_{i+1}\hat{T}_i\hat{T}_{i+1} & \text{for } 1 \le i < n \\
T_iT_j = T_jT_i & \text{for } |i-j| \ge 2 \text{ and } 1 \le i < m \\
\hat{T}_i\hat{T}_j = \hat{T}_j\hat{T}_i & \text{for } |i-j| \ge 2 \text{ and } 1 \le i < n \\
T_i\hat{T}_j = \hat{T}_jT_i & \text{for } 1 \le i < m \text{ and } 1 \le j < n \\
ET_i = T_iE & \text{for } 1 < i < m \\
E\hat{T}_i = \hat{T}_iE & \text{for } 1 < i < n \\
ET_1^{\pm 1}E = (qr)^{\pm 1}E & \\
E\hat{T}_1^{\pm 1}E = (qr)^{\pm 1}E & \\
ET_1^{-1}\hat{T}_1ET_1 = ET_1^{-1}\hat{T}_1E\hat{T}_1 & \\
T_1ET_1^{-1}\hat{T}_1ET_1 = \hat{T}_1ET_1^{-1}\hat{T}_1E. &
\end{array}$$

From the fact that $T_1 - q^2T_1^{-1} = (q^2-1)$ we have

$$\begin{aligned}
ET_1E - q^2ET_1^{-1}E &= (q^2-1)E^2 \\
qrE - (qr)^{-1}E &= (q^2-1)E^2
\end{aligned}$$

which yields $E^2 = zE$. For $2 \le i \le m$ and $2 \le j \le n$ we define the elements $E_{i,j}$ recursively by $E_{1,1} = E$ and

$$E_{i,k} = T_{i-1}E_{i-1,k}T_{i-1} \qquad \text{for } 1 \le k \le n$$

and

$$E_{k,j} = \hat{T}_{j-1}^{-1}E_{k,j-1}\hat{T}_{j-1}^{-1} \qquad \text{for } 1 \le k \le m.$$

The following additional relations can be deduced from the defining relations:

$$\begin{aligned}
E_{i,j}T_i^{\pm 1}E_{i,j} &= (qr)^{\pm 1}E_{i,j}, \\
E_{i,j}T_{i-1}^{\pm 1}E_{i,j} &= (qr)^{\pm 1}E_{i,j}, \\
E_{i,j}\hat{T}_j^{\pm 1}E_{i,j} &= (qr)^{\pm 1}E_{i,j}, \\
E_{i,j}\hat{T}_{j-1}^{\pm 1}E_{i,j} &= (qr)^{\pm 1}E_{i,j}, \\
E_{i,j}E_{k,l} &= E_{k,l}E_{i,j} \quad \text{if } i \ne k \text{ and } j \ne l, \\
E_{i,j}E_{i+1,j+1}T_i &= E_{i,j}E_{i+1,j+1}\hat{T}_j.
\end{aligned}$$

In each case above the indices are chosen from all values of $1 \le i < m$ and $1 \le j < n$ for which the given expression makes sense.

Observe that there is an algebra anti-involution of $\mathcal{A}_{m,n}(r,q)$, defined on generators by $* : T_w \mapsto T_{w^{-1}}$, $* : \hat{T}_v \mapsto \hat{T}_{v^{-1}}$ and $* : E \mapsto E$ and that the map $*$ fixes $E_{i,j}$ for $1 \le i \le m$ and $1 \le j \le n$.

4. Specializations of $\mathcal{A}_{m,n}(r,q)$

Later we will use the specializations of $\mathcal{A}_{m,n}(r,q)$ to a field $\kappa = \mathbb{C}(\hat{r},\hat{q})$.

Definition 4.1. Let $\phi : R \to \mathbb{C}(\hat{r},\hat{q})$ be the ring homomorphism given by $\phi : r \mapsto \hat{r}$ and $\phi : q \mapsto \hat{q}$. Then $A_{m,n}(\hat{r},\hat{q})$ is the κ-algebra $\mathcal{A}_{m,n}(r,q) \otimes_R \kappa$.

5. The Walled Brauer Algebras

An (m, n)-*Brauer diagram* is a graph consisting of two horizontal rows of $m + n$ vertices, together with "wall" between the m-th pair and the $m+1$-st pair of vertices, such that (i) each vertex is incident to exactly one edge; (ii) every edge connecting vertices in the same row must cross the wall, and; (iii) no edge connecting vertices in different rows crosses the wall. Figure 1 is example of a $(5, 3)$-Brauer diagram.

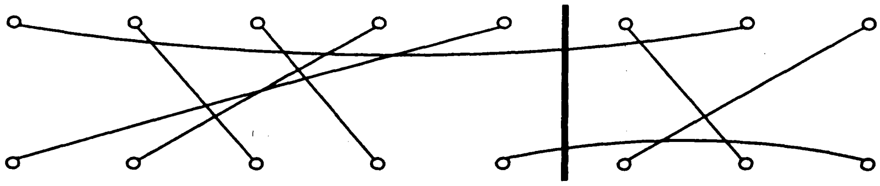

Figure 1. A $(5, 3)$-Brauer diagram.

Let y be an indeterminate over $\mathbb{Z}$. The generic *walled Brauer algebra* $\mathcal{B}_{m,n}(y)$ is the $\mathbb{Z}[y]$-span of the (m, n)-diagrams equipped with the usual product for multiplying Brauer-diagrams (for example see [1]). Using Schur-Weyl duality, Benkart et al, have given a description the representations of the walled Brauer algebras over a field of characteristic zero in [1].

The algebra $\mathcal{A}_{m,n}(r, q)$ may be regarded as a two parameter deformations of the algebra $\mathcal{B}_{m,n}(y)$. In particular, Leduc has shown, in Corollary 2.15 of [8], the following.

Theorem 5.1. *Suppose $\hat{q}$ is not a root of unity and that $\hat{r} \neq \hat{q}^k$, for $k < m + n$. If $\hat{y}$ is either an indeterminate over $\mathbb{C}$ or an integer, $m + n \leq \hat{y}$, then the algebras $A_{m,n}(\hat{r}, \hat{q}) = \mathcal{A}_{m,n}(r, q) \otimes_R \kappa$ and $B_{m,n}(\hat{y}) = \mathcal{B}_{m,n}(y) \otimes_{\mathbb{Z}[y]} \mathbb{C}(\hat{y})$ are semi-simple and have the same numerical invariants. Moreover,*

$$A_{m,n}(\hat{r}, \hat{q}) \equiv \bigoplus_{f=0}^{\min\{m,n\}} \bigoplus_{\lambda \in \Gamma_f} C_{f,\lambda}$$

where $C_{f,\lambda}$ is a full matrix ring and Γ_f is the set of bi-partitions

$$\Gamma_f = \left\{ (\lambda^{(1)}, \lambda^{(2)}) \,\middle|\, \lambda^{(1)} \vdash m - f \text{ and } \lambda^{(2)} \vdash n - f \right\} \tag{10}$$

for each integer $0 \leq f \leq \min\{m, n\}$.

6. Cellular Bases for $\mathcal{A}_{m,n}(r, q)$

To construct cellular bases for the algebra $\mathcal{A}_{m,n}(r, q)$, let, as in Theorem 5.1, f denote an integer $0 \leq f \leq \min\{m, n\}$ and let Γ_f be the set of bi-partitions given by (10). For the purposes of this chapter a multi-partition ν of $m + n$ will be an ordered tuple of partions $(\nu^{(1)}, \dots, \nu^{(4)})$ where $\nu^{(1)} = \nu^{(3)} = (1^f)$ and $(\nu^{(2)}, \nu^{(4)}) \in \Gamma_f$ for an integer $0 \leq f \leq \min\{m, n\}$. The diagram $[\nu]$ is the ordered tuple of diagrams $([\nu^{(1)}], \dots, [\nu^{(4)}])$ and a ν-multi-tableau $\mathfrak{t}$ is pair of bijections $[\nu^{(1)}] \cup [\nu^{(2)}] \to \{1, \dots, m\}$ and $[\nu^{(3)}] \cup [\nu^{(4)}] \to \{1, \dots, n\}$ such that the nodes $[\nu^{(1)}] \cup [\nu^{(2)}]$ are labelled by the integers $\{1, \dots, m\}$ and the nodes $[\nu^{(3)}] \cup [\nu^{(4)}]$ are labelled by the integers $\{1, \dots, n\}$. For example, if $m = 7$, $n = 6$ and $\nu = ((1^2), (3, 2), (1^2), (2, 1^2))$ then

$$\left(\ytableaushort{1,2}\,, \ytableaushort{345,67}\,, \ytableaushort{1,2}\,, \ytableaushort{34,5,6} \right) \quad \text{and} \quad \left(\ytableaushort{3,6}\,, \ytableaushort{147,25}\,, \ytableaushort{6,4}\,, \ytableaushort{15,2,3} \right)$$

are both ν-multi-tableaux. If $\mathfrak{t}$ is a multi-tableau we will write $\mathfrak{t}^{(i)}$ for the labelled diagram $\mathfrak{t}[\nu^{(i)}]$ for $i = 1, \dots, 4$. If $\nu = (\nu^{(1)}, \dots, \nu^{(4)})$ is a multi-partition, the multi-tableau $\mathfrak{t}^\nu$ will have the integers $1, 2 \dots, f$ appear sequentially from top to bottom in $[\nu^{(1)}]$ and $[\nu^{(3)}]$ while the integers $f+1, \dots, m$ appear from left to right and top to bottom in $[\nu^{(2)}]$ and the integers $f+1, \dots, n$ appear from left to right and top to bottom in $[\nu^{(4)}]$. In the example where $\nu = ((1^2), (3,2), (1^2), (2,1^2))$, then $\mathfrak{t}^\nu$ is the multi-tableaux

$$\mathfrak{t}^\nu = \left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 6 & 7 \\ \cline{1-2} \end{array}, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} \right) \tag{11}$$

A ν-multi-tableau $\mathfrak{t}$ is *row standard* if the entries in each row of $\mathfrak{t}^{(i)}$ increase from left to right for $i = 2, 4$. A row standard ν-multi-tableau is *standard* if $\mathfrak{t}^{(1)} = \mathfrak{t}^{(3)} = \mathfrak{t}^{\nu^{(1)}}$ and the entries in each column of $\mathfrak{t}^{(2)}$ and $\mathfrak{t}^{(4)}$ increase read from top to bottom. Denote by $\operatorname{Std}(\nu)$ the collection of standard ν-multi-tableaux.

Given permutations $w \in \mathfrak{S}_m$ and $v \in \mathfrak{S}_n$, we let $\mathfrak{t}^\nu(w,v)$ denote the multi-tableaux obtained by allowing w to permute the entries of $\mathfrak{t}^{\nu^{(1)}}$ and $\mathfrak{t}^{\nu^{(2)}}$ and v to permute the entries of $\mathfrak{t}^{\nu^{(3)}}$ and $\mathfrak{t}^{\nu^{(4)}}$. For example, if $w = (1,7)(2,5,3,6)$ and $v = (1,2)(4,6,5)$ then, referring to the multi-tableaux (11), we have

$$\mathfrak{t}^\nu(w,v) = \left(\begin{array}{|c|}\hline 7 \\ \hline 5 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 6 & 4 & 3 \\ \hline 2 & 1 \\ \cline{1-2} \end{array}, \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 3 & 6 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \right).$$

Given a multi-partition ν, write $\mathfrak{S}_\nu$ for the direct product $\mathfrak{S}_{\nu^{(2)}} \times \mathfrak{S}_{\nu^{(4)}}$ where $\mathfrak{S}_{\nu^{(2)}}$ is the row stabiliser in $\mathfrak{S}_m$ of $\mathfrak{t}^{\nu^{(2)}}$ and $\mathfrak{S}_{\nu^{(4)}}$ is the row stabiliser of $\mathfrak{t}^{\nu^{(4)}}$ in $\mathfrak{S}_n$. Referring to the above example where $\mathfrak{t}^\nu$ is the multi-tableaux (11),

$$\mathfrak{S}_\nu = \langle s_3, s_4, s_6 \rangle \times \langle s_3 \rangle.$$

For $0 < f \le \min\{m,n\}$, let $\mathfrak{D}_f$ denote the diagonal subgroup of $\mathfrak{S}_f \times \mathfrak{S}_f$ in $\mathfrak{S}_m \times \mathfrak{S}_n$ given by

$$\mathfrak{D}_f = \langle (s_i, s_i) \mid 1 \le i < f \rangle$$

and set $\mathfrak{D}_f = \langle 1 \rangle$ when $f = 0$.

The next statement generalises the well known result giving a set of distinguished coset representatives for a parabolic subgroup of the symmetric group (Proposition 3.3 of [9]).

Proposition 6.1. *Let $0 \le f \le \min\{m,n\}$ and $\nu = (\nu^{(1)}, \dots, \nu^{(4)})$ be a multi-partition of $m+n$ with $\nu^{(i)} = (1^f)$, for $i = 1, 3$. If*

$$\mathscr{D}_\nu = \left\{ (w,v) \in \mathfrak{S}_m \times \mathfrak{S}_n \;\middle|\; \begin{array}{c} (\mathfrak{t}_1, \dots, \mathfrak{t}_4) = \mathfrak{t}^\nu(w,v) \textit{ is row standard and } \mathfrak{t}_1 \\ \textit{is increasing read from top to bottom} \end{array} \right\},$$

then $\mathscr{D}_\nu$ is a complete set of right coset representatives for $\mathfrak{D}_f\mathfrak{S}_\nu$ in $\mathfrak{S}_m \times \mathfrak{S}_n$. Moreover, if $(w,v) \in \mathscr{D}_\nu$, then $l(uw) = l(u) + l(w)$ and $l(tv) = l(t) + l(v)$ for all $(u,t) \in \mathfrak{S}_\nu$.

Proof. Suppose that $\mathfrak{D}_f\mathfrak{S}_\nu(v,u) = \mathfrak{D}_f\mathfrak{S}_\nu(w,t)$ and let $\mathfrak{s} = \mathfrak{t}^\nu(v,u)$ and $\mathfrak{u} = \mathfrak{t}^\nu(w,t)$. Then the permutation of rows which takes $\mathfrak{s}^{(1)}$ to $\mathfrak{u}^{(1)}$ also takes $\mathfrak{s}^{(3)}$ to $\mathfrak{u}^{(3)}$, while $\mathfrak{s}^{(2)}$ and $\mathfrak{u}^{(2)}$ (resp. $\mathfrak{s}^{(4)}$ and $\mathfrak{u}^{(4)}$) differ by a reordering of the entries of each row. Therefore $\mathscr{D}_\nu$ is a compete set of coset representatives for $\mathfrak{D}_f\mathfrak{S}_\nu$ in $\mathfrak{S}_m \times \mathfrak{S}_n$. Now fix $(w,v) \in \mathscr{D}_\nu$; then $\mathfrak{t}^\nu(w,v)$ is row standard so $(j)w < (j+1)w$ (resp. $(k)v < (k+1)v$) whenever j and $j+1$ are in the same row of $\mathfrak{t}^{\nu^{(2)}}$ (resp. k and $k+1$ are in the same row of $\mathfrak{t}^{\nu^{(4)}}$). Thus $l(s_j w) = l(w) + 1$ (resp. $l(s_k v) = l(v) + 1$) whenever $(s_j, 1) \in \mathfrak{S}_\nu$ (resp. $(1, s_k) \in \mathfrak{S}_\nu$).

Now suppose that $(u,t) \in \mathfrak{S}_\nu$ and that $1 < l(t)$. Then $t = s_k t'$ and $l(t) = l(t') + 1$ for some s_k with $(1, s_k) \in \mathfrak{S}_\nu$; therefore $(k)t' < (k+1)t'$. Now $(k)t'$ and $(k+1)t'$

belong to the same row of $\mathfrak{t}^{\nu^{(4)}}$, so $(k)t'v < (k+1)t'v$ and hence $l(s_k t'v) = l(t'v)+1$. By induction therefore $l(tv) = l(s_k t'v) = l(t'v) + 1 = l(t') + l(v) + 1 = l(t) + l(v)$. If $1 < l(u)$, then an identical argument shows that $l(uw) = l(u) + l(w)$. □

For $0 \le f \le \min\{m,n\}$, let B^f denote the ideal in $\mathcal{A}_{m,n}(r,q)$ generated by the element

$$\prod_{i=1}^{f} E_{i,i}$$

and set $B^f = \{0\}$ if $f > m$ or $f > n$. We thus obtain a filtration of $\mathcal{A}_{m,n}(r,q)$

$$\mathcal{A}_{n,n}(r,q) = B^0 \supseteq B^1 \supseteq \cdots \supseteq \{0\} \tag{12}$$

by by two sided ideals.

By Theorem 2.3 and Proposition 2.10, the algebra $\mathcal{H}_{R,m-f}(q^2) \otimes \mathcal{H}_{R,n-f}(q^2)$, for $0 \le f \le \min\{m,n\}$, is cellular. By identifying $\mathcal{H}_{R,m-f}(q^2)$ with the subalgebra of $\mathcal{H}_{R,m}(q^2)$ generated by $\{ X_i \mid f < 1 < m \}$ and $\mathcal{H}_{R,n-f}(q^2)$ with the subalgebra of $\mathcal{H}_{R,n}(q^2)$ generated by $\{ X_i \mid f < 1 < n \}$, we define an R-module homomorphism $\iota : \mathcal{H}_{R,m-f}(q^2) \otimes \mathcal{H}_{R,n-f}(q^2) \to B^f$ as

$$\iota : X_w \otimes X_v \mapsto \prod_{i=1}^{f} E_{i,i} T_w \hat{T}_v.$$

The map ι will allow us to produce a cellular structure on B^f/B^{f+1} corresponding to a cellular structure on $\mathcal{H}_{R,m-f}(q^2) \otimes \mathcal{H}_{R,n-f}(q^2)$. The cellular structure on B^f/B^{f+1} will be used to refine the filtration (12) and so obtain a cellular basis for $\mathcal{A}_{m,n}(r,q)$.

Now fix, for each integer $0 \le f \le \min\{m,n\}$, a cellular basis $(\mathscr{C}_f, \Lambda_f)$ for the algebra $\mathcal{H}_{R,m-f}(q^2) \otimes \mathcal{H}_{R,n-f}(q^2)$; that is for each f, the collection

$$\mathscr{C}_f = \{ c^\lambda_{\mathfrak{v}\mathfrak{u}} \mid \mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\lambda), \lambda \in \Lambda_f \}$$

is a free R basis for $\mathcal{H}_{R,m-f}(q^2) \otimes \mathcal{H}_{R,n-f}(q^2)$ satisfying the Definition 2.1; it will be necessary to assume that the anti-involution $c^\lambda_{\mathfrak{v}\mathfrak{u}} \to c^\lambda_{\mathfrak{u}\mathfrak{v}}$ coincides with the anti-involution defined by $X_w \otimes X_v \mapsto X_{w^{-1}} \otimes X_{v^{-1}}$. For $\lambda \in \Lambda_f$ we let A^λ denote the R-submodule of $\mathcal{H}_{R,m-f}(q^2) \otimes \mathcal{H}_{R,n-f}(q^2)$ generated by the elements

$$\{ c^\mu_{\mathfrak{v}\mathfrak{u}} \mid \mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\mu) \text{ and } \mu \ge \lambda \}$$

so that $\check{A}^\lambda = \sum_{\mu > \lambda} A^\mu$. For each $c_{\mathfrak{v}\mathfrak{u}} = c^\lambda_{\mathfrak{v}\mathfrak{u}}$ we define the element $b_{\mathfrak{v}\mathfrak{u}} \in B^f/B^{f+1}$ to be

$$b_{\mathfrak{v}\mathfrak{u}} = \iota(c_{\mathfrak{v}\mathfrak{u}}) + B^{f+1}$$

and let $B^\lambda \subseteq B^f/B^{f+1}$ denote the $\mathcal{A}_{m,n}(r,q)$-bimodule generated by the elements

$$\{ b_{\mathfrak{v}\mathfrak{u}} \mid \mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\lambda) \}.$$

We set $\check{B}^\lambda \subseteq B^\lambda$ to be the $\mathcal{A}_{m,n}(r,q)$-bimodule generated by

$$\{ b_{\mathfrak{v}\mathfrak{u}} \mid \mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\mu) \text{ for } \mu > \lambda \}$$

and define the right cell module $C^\lambda_{\mathfrak{v}}$ to be the right $\mathcal{A}_{m,n}(r,q)$-submodule of $B^\lambda/\check{B}^\lambda$ generated by the elements

$$\{ \check{B}^\lambda + b_{\mathfrak{v}\mathfrak{u}} \mid \mathfrak{u} \in \mathcal{I}_f(\lambda) \}.$$

Our purpose is to construct a free R-basis for each of the B^λ, $\check{B}^\lambda$ and $C^\lambda_{\mathfrak{v}}$ and to show that $C^\lambda_{\mathfrak{v}}$ is a cell module for $\mathcal{A}_{m,n}(r,q)$ in the sense of Graham and Lehrer.

The next statement is an immediate consequence of the above definitions.

Proposition 6.2. *Let* $0 \le f \le \min\{m,n\}$ *and* $\lambda \in \Lambda_f$. *Then*

(1) $B^f/B^{f+1} = \sum_{\lambda\in\Lambda_f} B^\lambda$;
(2) $\check{B}^\lambda \subseteq B^\lambda$;
(3) $\iota(A^\lambda) \subseteq B^\lambda$ *and* $\iota(\check{A}^\lambda) \subseteq \check{B}^\lambda$.

We now set about constructing bases for the quotients B^f/B^{f+1} and hence for $\mathcal{A}_{m,n}(r,q)$. In each case the basis will be expressed in terms of $\mathscr{C}_f$ and $\mathscr{D}_\nu$ where ν is the multi-partition with $\nu^{(2)} = (m-f)$ and $\nu^{(4)} = (n-f)$.

Given $(w,v) \in \mathfrak{S}_m \times \mathfrak{S}_n$, it will be convenient to write $T_w^\sharp$ for $(-q^2)^{l(w)}T_{w^{-1}}^{-1}$ and $\hat{T}_v^\sharp$ for $(-q^2)^{l(v)}\hat{T}_{v^{-1}}^{-1}$. Note that we have not defined $\sharp$ to be a map of $\mathcal{A}_{m,n}(r,q)$.

Proposition 6.3. *Let* $1 \le j < f \le \min\{m,n\}$ *and* $(v,w) \in \mathfrak{S}_m \times \mathfrak{S}_n$. *If* $(j+1)v < (j)v \le f$, *then*

$$\prod_{i=1}^f E_{i,i}T_v\hat{T}_w^\sharp = \begin{cases} -q^2\prod_{i=1}^f E_{i,i}T_{vs_j}\hat{T}_{s_jw}^\sharp & \text{if } l(s_jw) < l(w) \\ \prod_{i=1}^f E_{i,i}T_{vs_j}((q^2-1)\hat{T}_w^\sharp - \hat{T}_{s_jw}^\sharp) & \text{if } l(w) < l(s_jw). \end{cases}$$

Proof. If $(j+1)v < (j)v$, we have $\prod_{i=1}^f E_{i,i}T_v\hat{T}_w^\sharp = \prod_{i=1}^f E_{i,i}T_jT_{s_jv}\hat{T}_w^\sharp$. Now

$$E_{j,j}E_{j+1,j+1}T_jT_{s_jv}\hat{T}_w^\sharp = E_{j,j}E_{j+1,j+1}\hat{T}_jT_{s_jv}\hat{T}_w^\sharp = E_{j,j}E_{j+1,j+1}T_{s_jv}\hat{T}_j\hat{T}_w^\sharp$$

so, in case $l(s_jw) < l(w)$, we have

$$\prod_{i=1}^f E_{i,i}T_v\hat{T}_w^\sharp = \prod_{i=1}^f E_{i,i}T_{s_jv}\hat{T}_j\hat{T}_j^\sharp\hat{T}_{s_jw}^\sharp = -q^2\prod_{i=1}^f E_{i,i}T_{s_jv}\hat{T}_{s_jw}^\sharp.$$

If $l(w) < l(s_jw)$ we argue similarly, using the fact that $\hat{T}_j = (q^2-1) - \hat{T}_j^\sharp$, to complete the proof. □

Corollary 6.4. *Let* $1 < f \le \min\{m,n\}$ *and* $\lambda \in \Lambda_f$. *If* $(w,v) \in \mathfrak{S}_m \times \mathfrak{S}_n$ *and* $\mathfrak{u} \in \mathcal{I}_f(\lambda)$ *then there exist* $a_{(u,t)}$, *for* $(u,t) \in \mathfrak{S}_m \times \mathfrak{S}_n$, *such that* $(i)u < (i+1)u$ *for* $1 \le i < f$ *and*

$$b_{\mathfrak{v}\mathfrak{u}}T_w\hat{T}_v^\sharp \equiv \sum_{(u,t)} a_{(u,t)}b_{\mathfrak{v}\mathfrak{u}}T_u\hat{T}_t^\sharp \mod \check{B}^\lambda.$$

Moreover, in this expression, the (u,t) *and* $a_{(u,t)}$ *do not depend on* $\mathfrak{v}$ *or* $\mathfrak{u}$.

Proof. If $(j+1)w < (j)w$ for some $1 \le j < f$ then by Proposition 6.3, we can rewrite $b_{\mathfrak{v}\mathfrak{u}}T_w\hat{T}_v^\sharp$ as a linear combination of $b_{\mathfrak{v}\mathfrak{u}}T_{s_jw}\hat{T}_v^\sharp$ and $b_{\mathfrak{v}\mathfrak{u}}T_{s_jw}\hat{T}_{s_jv}^\sharp$. Since $l(s_jw) < l(w)$ and the statement holds true in case $l(w) = 0$, we are done by induction. □

Our next observation is that straightening laws in $\mathcal{H}_{R,m-f}(q^2)\otimes\mathcal{H}_{R,n-f}(q^2)$ are inherited by B^f/B^{f+1}.

Lemma 6.5. *Let* $1 < f \le \min\{m,n\}$ *and* $\lambda \in \Lambda_f$. *If* $(w,v) \in \mathfrak{S}_m \times \mathfrak{S}_n$ *and* $\mathfrak{u} \in \mathcal{I}_f(\lambda)$ *then there exist* $a_{(u,t)}, a_\mathfrak{s} \in R$, *for* $(u,t) \in \mathfrak{S}_m \times \mathfrak{S}_n$ *and* $\mathfrak{s} \in \mathcal{I}_f(\lambda)$, *such that* $(i)u < (i+1)u$ *whenever* $f < i < m$ *and* $(i)t < (i+1)t$ *whenever* $f < i < n$, *and*

$$b_{\mathfrak{v}\mathfrak{u}}T_w\hat{T}_v^\sharp \equiv \sum_{(u,t)} a_{(u,t)} \sum_{\mathfrak{s}\in\mathcal{I}_f(\lambda)} a_\mathfrak{s} b_{\mathfrak{v}\mathfrak{s}}T_u\hat{T}_t^\sharp$$

for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$.

Proof. By Corollary 6.4 we may assume that $(j)w < (j+1)w$ whenever $1 \le j < f$. Now suppose that $f < n$ and that $(j+1)w < (j)w$ for some $f < j < n$. Then $l(s_jw) < l(w)$ and, by definition of the map ι,

$$b_{\mathfrak{v}\mathfrak{u}}T_w\hat{T}_v^\sharp = \iota(c_{\mathfrak{v}\mathfrak{u}})T_w\hat{T}_v^\sharp = \iota(c_{\mathfrak{v}\mathfrak{u}})T_jT_{s_jw}\hat{T}_v^\sharp = \iota(c_{\mathfrak{v}\mathfrak{u}}X_j\otimes 1)T_{s_jw}\hat{T}_v^\sharp.$$

Now there exist $a_{\mathfrak{s}} \in R$, for $\mathfrak{s} \in \mathcal{I}_f(\lambda)$, such that for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$,

$$c_{\mathfrak{v}\mathfrak{u}} X_j \otimes 1 \equiv \sum_{\mathfrak{s} \in \mathcal{I}_f(\lambda)} a_{\mathfrak{s}} c_{\mathfrak{v}\mathfrak{s}} \mod \check{A}^\lambda,$$

and since $\iota(\check{A}^\lambda) \subseteq \check{B}^\lambda$, it follows that

$$\begin{aligned}\iota(c_{\mathfrak{v}\mathfrak{u}} X_j \otimes 1) T_{s_j w} \hat{T}_v^\sharp &\equiv \sum_{\mathfrak{s} \in \mathcal{I}_f(\lambda)} a_{\mathfrak{s}} \iota(c_{\mathfrak{v}\mathfrak{s}}) T_{s_j w} \hat{T}_v^\sharp \mod \check{B}^\lambda \\ &\equiv \sum_{\mathfrak{s} \in \mathcal{I}_f(\lambda)} a_{\mathfrak{s}} b_{\mathfrak{v}\mathfrak{s}} T_{s_j w} \hat{T}_v^\sharp \mod \check{B}^\lambda.\end{aligned}$$

By induction on $l(w)$ we may therefore suppose that

$$b_{\mathfrak{v}\mathfrak{u}} T_w \hat{T}_v^\sharp \equiv \sum_{\mathfrak{s} \in \mathcal{I}_f(\lambda)} a_{\mathfrak{s}} b_{\mathfrak{v}\mathfrak{s}} T_u \hat{T}_v^\sharp \mod \check{B}^\lambda,$$

where $(i)u < (i+1)u$ whenever $f < i < m$. Applying a similar argument to v completes the proof of the Lemma. $\square$

From Corollary 6.4 and Lemma 6.5 we obtain the following.

Lemma 6.6. *Let $0 \le f \le \min\{m,n\}$, $\lambda \in \Lambda_f$ and ν be the multi-partition with $\nu^{(2)} = (m-f)$ and $\nu^{(4)} = (n-f)$. If $\mathfrak{u} \in \mathcal{I}_f(\lambda)$ and $(w,v) \in \mathfrak{S}_m \times \mathfrak{S}_n$ then there exist $a_{(u,t)}, a_{\mathfrak{s}} \in R$, for $(u,t) \in \mathscr{D}_\nu$, and $\mathfrak{s} \in \mathcal{I}_f(\lambda)$, such that*

$$b_{\mathfrak{v}\mathfrak{u}} T_w \hat{T}_v^\sharp \equiv \sum_{(u,t)} a_{(u,t)} \sum_{\mathfrak{s}} a_{\mathfrak{s}} b_{\mathfrak{v}\mathfrak{s}} T_u \hat{T}_t^\sharp \mod \check{B}^\lambda$$

for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$.

The next few Lemmas show that we have the required multiplicative properties for a cellular basis.

Lemma 6.7. *Let $0 < f \le \min\{m,n\}$ and $\lambda \in \Lambda_f$. If $(w,v) \in \mathfrak{S}_m \times \mathfrak{S}_n$ and $\mathfrak{u} \in \mathcal{I}_f(\lambda)$ then there exist $a_{(u,t)}, a_{\mathfrak{s}} \in R$, for $(u,t) \in \mathscr{D}_\nu$, $\mathfrak{s} \in \mathcal{I}_f(\lambda)$, such that*

$$b_{\mathfrak{v}\mathfrak{u}} T_w \hat{T}_v^\sharp T_i \hat{T}_j \equiv \sum_{(u,t)} a_{(u,t)} \sum_{\mathfrak{s}} a_{\mathfrak{s}} b_{\mathfrak{v}\mathfrak{s}} \hat{T}_u \hat{T}_t^\sharp \mod \check{B}^\lambda$$

for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$.

Proof. Note that

$$b_{\mathfrak{v}\mathfrak{u}} T_w \hat{T}_v^\sharp T_i \hat{T}_j = \begin{cases} b_{\mathfrak{v}\mathfrak{u}} T_{ws_i} \hat{T}_v^\sharp \hat{T}_j & \text{if } l(w) < l(ws_i), \\ q^2 b_{\mathfrak{v}\mathfrak{u}} T_{ws_i} \hat{T}_v^\sharp \hat{T}_j + (q^2-1) b_{\mathfrak{v}\mathfrak{u}} T_w \hat{T}_v^\sharp \hat{T}_j & \text{if } l(ws_i) < l(w). \end{cases}$$

Similarly, by writing $\hat{T}_j = (q^2-1) - \hat{T}_j^\sharp$, we may eliminate the term $\hat{T}_j$ from either of the above expressions so that

$$b_{\mathfrak{v}\mathfrak{u}} T_u \hat{T}_v^\sharp \hat{T}_j = \begin{cases} -q^2 b_{\mathfrak{v}\mathfrak{u}} T_u \hat{T}_{vs_j}^\sharp & \text{if } l(vs_j) < l(v), \\ (q^2-1) b_{\mathfrak{v}\mathfrak{u}} T_u \hat{T}_v^\sharp - b_{\mathfrak{v}\mathfrak{u}} T_u \hat{T}_{vs_j}^\sharp & \text{if } l(v) < l(ws_j) \end{cases}$$

where $u = w$ or $u = ws_i$. Now use Lemma 6.6 to rewrite each of the resulting summands in the required form. $\square$

Lemma 6.8. *Let $(w,v) \in \mathfrak{S}_m \times \mathfrak{S}_n$ and $\lambda \in \Lambda_f$. Then, for all $\mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\lambda)$,*

(1) *if $(1)w = (1)v = 1$ then $b_{\mathfrak{v}\mathfrak{u}} T_w \hat{T}_v^\sharp E = z b_{\mathfrak{v}\mathfrak{u}} T_w \hat{T}_v^\sharp$;*
(2) *if $f < (1)w^{-1}$ and $f < (1)v^{-1}$, then $b_{\mathfrak{v}\mathfrak{u}} T_w \hat{T}_v^\sharp E \equiv 0 \mod B^{f+1}$.*

Proof. We claim that if $(1)w = 1$ then $T_wE = ET_w$; if $l(w) = 0$ there is nothing to show, so suppose that $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression for w. Since $s_{i_k} \neq s_1$,

$$T_wE = T_{ws_{i_k}}T_{i_k}E = T_{ws_{i_k}}ET_{i_k} = ET_{ws_{i_k}}T_{i_k}$$

where, since $l(ws_{i_k}) < l(w)$, the last equality follows by induction on $l(w)$. This proves the claim. An identical argument shows that under the hypotheses of the first item $\hat{T}_v^\sharp E = E\hat{T}_v^\sharp$. Therefore, under the same hypotheses, $b_{\mathfrak{vu}}T_w\hat{T}_v^\sharp E = b_{\mathfrak{vu}}ET_w\hat{T}_v^\sharp = zb_{\mathfrak{vu}}T_w\hat{T}_v^\sharp$ from which we obtain the first item.

Now for the second item. By Lemma 6.7, there is no harm in supposing that $(w,v) \in \mathscr{D}_\nu$ where ν is the multi-partition $\nu = ((1^f),(m-f),(1^f),(m-f))$. In this case, $(f+1)w = 1$ and $(f+1)v = 1$. Therefore $w = s_fs_{f-1}\cdots s_1u$ and $v = s_fs_{f-1}\cdots s_1t$ with $l(w) = l(u) + f$ and $l(v) = l(t) + f$. Moreover, since $(1)u = 1$ and $(1)t = 1$, we have

$$\begin{aligned}\prod_{i=1}^{f} E_{i,i}T_w\hat{T}_v^\sharp E_{1,1} &= \prod_{i=1}^{f} E_{i,i}T_fT_{f-1}\ldots T_1\hat{T}_f^\sharp\hat{T}_{f-1}^\sharp\ldots\hat{T}_1^\sharp T_u\hat{T}_t^\sharp E_{1,1}\\ &= -q^2\prod_{i=2}^{f} E_{i,i}T_fT_{f-1}\ldots T_2\hat{T}_f^\sharp\hat{T}_{f-1}^\sharp\ldots\hat{T}_2^\sharp E_{1,1}T_1\hat{T}_1^{-1}E_{1,1}T_u\hat{T}_t^\sharp\\ &= -q^2\prod_{i=2}^{f} E_{i,i}T_fT_{f-1}\ldots T_2\hat{T}_f^\sharp\hat{T}_{f-1}^\sharp\ldots\hat{T}_2^\sharp E_{2,2}E_{1,1}T_u\hat{T}_t^\sharp\end{aligned}$$

where we have used the fact that $E_{1,1}T_1\hat{T}_1^{-1}E_{1,1} = E_{1,1}E_{2,2}$. Now, we repeat the process, using successively the relations $E_{i,i}T_i\hat{T}_i^{-1}E_{i,i} = E_{i,i}E_{i+1,i+1}$ to eliminate, for $2 \le i \le f$, the terms $T_i\hat{T}_i^\sharp$ from the above expression, finally obtaining

$$\prod_{i=1}^{f} E_{i,i}T_w\hat{T}_v^\sharp E_{1,1} = (-q^2)^f\prod_{i=1}^{f+1} E_{i,i}T_u\hat{T}_t^\sharp$$

which completes the proof of the second item. □

Lemma 6.9. *Let $0 < f \le \min\{m,n\}$ and $\lambda \in \Lambda_f$. If $(w,v) \in \mathfrak{S}_m \times \mathfrak{S}_n$ and $\mathfrak{u} \in \mathcal{I}_f(\lambda)$ then there exist $a_{(u,t)}, a_{\mathfrak{s}} \in R$, for $(u,t) \in \mathscr{D}_\nu$ and $\mathfrak{s} \in \mathcal{I}_f(\lambda)$, such that*

$$b_{\mathfrak{vu}}T_w\hat{T}_v^\sharp E \equiv \sum_{(u,t)} a_{(u,t)}\sum_{\mathfrak{s}} a_{\mathfrak{s}}b_{\mathfrak{vs}}\hat{T}_u\hat{T}_t^\sharp \mod \check{B}^\lambda$$

for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$.

Proof. By the preceding Lemmas we may suppose that $(w,v) \in \mathscr{D}_\nu$ where ν is the multi-partition $\nu = ((1^f),(m-f),(1^f),(n-f))$. We now have four minor cases to consider individually. Firstly, if $f < (1)w^{-1}$ and $f < (1)v^{-1}$ then $b_{\mathfrak{vu}}T_w\hat{T}_v^\sharp E \equiv 0 \mod \check{B}^\lambda$ by Lemma 6.8. Next, if $1 = (1)w^{-1}$ and $1 = (1)v^{-1}$ then $b_{\mathfrak{vu}}T_w\hat{T}_v^\sharp E = zb_{\mathfrak{vu}}T_w\hat{T}_v^\sharp$, also by Lemma 6.8. Now, if $1 = (1)w^{-1}$ and $f < (1)v^{-1}$ then

$$\prod_{i=1}^{f} E_{i,i}T_w\hat{T}_v^\sharp E_{1,1} = \prod_{i=2}^{f} E_{i,i}E_{1,1}T_w\hat{T}_v^\sharp E_{1,1} = \prod_{i=2}^{f} E_{i,i}T_wE_{1,1}\hat{T}_v^\sharp E_{1,1} \tag{13}$$

Since $(w,v) \in \mathscr{D}_\nu$ and $f < (1)v^{-1}$, we must have $(1)v = f+1$. Therefore $v = s_fs_{f-1}\cdots s_1v'$ where $l(v) = l(v') + f$ and $(1)v' = 1$. It now follows that $T_{v'}^\sharp E_{1,1} = E_{1,1}T_{v'}^\sharp$ so,

$$\begin{aligned}E_{1,1}\hat{T}_v{}^\sharp E_{1,1} &= E_{1,1}\hat{T}_f^\sharp\hat{T}_{f-1}^\sharp\cdots\hat{T}_1^\sharp\hat{T}_{v'}^\sharp E_{1,1} = -q^2\hat{T}_f^\sharp\hat{T}_{f-1}^\sharp\cdots\hat{T}_2^\sharp E_{1,1}\hat{T}_1^{-1}E_{1,1}\hat{T}_{v'}^\sharp\\ &= -qr^{-1}E_{1,1}\hat{T}_f^{-1}\hat{T}_{f-1}^{-1}\cdots\hat{T}_2^{-1}\hat{T}_{v'}^\sharp,\end{aligned}$$

by $E_{1,1}T_1^{-1}E_{1,1} = (qr)^{-1}E_{1,1}$. Substituting the above expression into (13),

$$\prod_{i=1}^{f} E_{i,i}T_w\hat{T}_v^\sharp E_{1,1} = -qr^{-1}\prod_{i=1}^{f} E_{i,i}T_w\hat{T}_f^\sharp\hat{T}_{f-1}^\sharp\cdots\hat{T}_2^\sharp\hat{T}_{v'}^\sharp$$

whence

$$b_{\mathfrak{v}\mathfrak{u}}T_w\hat{T}_v^\sharp E = -qr^{-1}b_{\mathfrak{v}\mathfrak{u}}T_w\hat{T}_f^\sharp\hat{T}_{f-1}^\sharp\cdots\hat{T}_2^\sharp\hat{T}_{v'}^\sharp.$$

The last expression can now be rewritten, using Lemma 6.7, as an R-linear combination of terms in the required form. Since a similar argument applies to the case where $f < (1)w^{-1}$ and $1 = (1)v^{-1}$, the proof of the Lemma is now complete. □

Corollary 6.10. *Let $0 \le f \le \min\{m,n\}$ and ν be the multi-partition with $\nu^{(2)} = (m-f)$ and $\nu^{(4)} = (n-f)$ and suppose that $\lambda \in \Lambda_f$, $(w,v) \in \mathscr{D}_\nu$.*

(1) *If $b \in \mathcal{A}_{m,n}(r,q)$ and $\mathfrak{u} \in \mathcal{I}_f(\lambda)$ then there exist $(u,t) \in \mathscr{D}_\nu$, $\mathfrak{s} \in \mathcal{I}_f(\lambda)$ and $a_{(u,t)}, a_\mathfrak{s} \in R$ depending on $\mathfrak{u}$ and (w,v), such that*

$$b_{\mathfrak{v}\mathfrak{u}}T_wT_v^\sharp b \equiv \sum_{(u,t)\in\mathscr{D}_\nu} a_{(u,t)} \sum_{\mathfrak{s}\in\mathcal{I}_f(\lambda)} a_\mathfrak{s} b_{\mathfrak{v}\mathfrak{s}}T_uT_t^\sharp \mod \check{B}^\lambda$$

for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$.

(2) *The collection*

$$\left\{ b_{\mathfrak{v}\mathfrak{u}}T_w\hat{T}_v^\sharp + \check{B}^\lambda \mid (w,v) \in \mathscr{D}_\nu \text{ and } \mathfrak{u} \in \mathcal{I}_f(\lambda) \right\}$$

generates $C_\mathfrak{v}^\lambda$ as an R-module.

(3) *If $\mathfrak{t}, \mathfrak{v} \in \mathcal{I}_f(\lambda)$ then $C_\mathfrak{t}^\lambda$ and $C_\mathfrak{v}^\lambda$ are isomorphic as right $\mathcal{A}_{m,n}(r,q)$-modules.*

Proof. Since $\mathcal{A}_{m,n}(r,q)$ is generated by the $T_i, \hat{T}_j$ and E, the first item is an immediate consequence of Lemmas 6.7 and 6.9. The second and third items of the Lemma follow directly from the first statement. □

Lemma 6.11. *Let $0 \le f \le \min\{m,n\}$ and ν be the multi-partition with $\nu^{(2)} = (m-f)$ and $\nu^{(4)} = (n-f)$. Then the set*

$$\left\{ (T_u\hat{T}_t^\sharp)^* b_{\mathfrak{v}\mathfrak{u}}T_w\hat{T}_v^\sharp + \check{B}^\lambda \mid (u,t),(w,v) \in \mathscr{D}_\nu \text{ and } \mathfrak{v},\mathfrak{u} \in \mathcal{I}_f(\lambda) \right\}$$

generates $B^\lambda/\check{B}^\lambda$ as an R-module.

Proof. We argue by induction on $\le$. Let λ be a minimal element in $(\Lambda, \le)$ so that $\check{B}^\lambda = \{0\}$ and pick $\mathfrak{v} \in \mathcal{I}_f(\lambda)$. Since

$$\left\{ b_{\mathfrak{v}\mathfrak{u}}T_w\hat{T}_v^\sharp + \check{B}^\lambda \mid (w,v) \in \mathscr{D}_\nu \text{ and } \mathfrak{u} \in \mathcal{I}_f(\lambda) \right\}$$

generates $C_\mathfrak{v}^\lambda$ as a left R-module, whenever $b \in \mathcal{A}_{m,n}(r,q)$ we have $(w',v') \in \mathscr{D}_\nu$ and $\mathfrak{s} \in \mathcal{I}_f(\lambda)$ such that

$$\begin{aligned}(b\,(T_u\hat{T}_t^\sharp)^*\, b_{\mathfrak{v}\mathfrak{u}}T_w\hat{T}_v^\sharp)^* &= (T_w\hat{T}_v^\sharp)^*\, b_{\mathfrak{v}\mathfrak{u}}T_u\hat{T}_t^\sharp\, b^* \\ &\equiv \sum_{(w',v')\in\mathscr{D}_\nu} a_{(w',v')} \sum_{\mathfrak{s}\in\mathcal{I}_f(\lambda)} a_\mathfrak{s}(T_w\hat{T}_v^\sharp)^*\, b_{\mathfrak{v}\mathfrak{u}}T_{w'}\hat{T}_{v'}^\sharp \mod \check{B}^\lambda.\end{aligned}$$

Since $\check{B}^\lambda = \{0\}$, applying the anti-involution $*$ once more shows that we have a generating set for B^λ as an R-module. If $\lambda < \mu$ then proceed by induction on $\le$ to obtain a generating set for B^μ as an R-module. □

Proposition 6.12. *Let* $0 \le f \le \min\{m,n\}$ *and* $\nu = ((1)^f,(m-f),(1^f),(n-f))$. *Then the collection*

$$\left\{(T_u\hat{T}_t^\sharp)^* b_{\mathfrak{v}\mathfrak{u}} T_w \hat{T}_v^\sharp \mid (u,t),(w,v) \in \mathscr{D}_\nu, \mathfrak{v},\mathfrak{u} \in \mathcal{I}_f(\lambda) \text{ and } \lambda \in \Lambda_f\right\} \tag{14}$$

is a free R*-basis for* B^f/B^{f+1}.

Proof. That (14) generates B^f/B^{f+1} as an R-module follows from Lemma 6.11, so we show that the collection (14) is linearly independent over R and we do this by constructing a corresponding R-basis for $\mathcal{A}_{m,n}(r,q)$.

For $\lambda \in \Lambda_f$ and $\mathfrak{v},\mathfrak{u} \in \mathcal{I}_f(\lambda)$, let $a_{\mathfrak{v}\mathfrak{u}}^{(w,v)} \in R$, for $(w,v) \in \mathfrak{S}_{m-2f} \times \mathfrak{S}_{n-2f}$, denote elements satisfying

$$c_{\mathfrak{v}\mathfrak{u}}^\lambda = \sum_{(w,v)\in\mathfrak{S}_{m-2f}\times\mathfrak{S}_{n-2f}} a_{\mathfrak{v}\mathfrak{u}}^{(w,v)} X_w \otimes X_v.$$

Then the element $\hat{b}_{\mathfrak{v}\mathfrak{u}} \in B^f$ defined by

$$\hat{b}_{\mathfrak{v}\mathfrak{u}} = \prod_{i=1}^f E_{i,i} \cdot \sum_{(w,v)\in\mathfrak{S}_{m-2f}\times\mathfrak{S}_{n-2f}} a_{\mathfrak{v}\mathfrak{u}}^{(w,v)} T_w \check{T}_v \tag{15}$$

will be a coset representative for $b_{\mathfrak{v}\mathfrak{u}}$ in B^f.

Now recall that, since $B^f/B^{f+1} = \sum_{\lambda\in\Lambda_f} B^\lambda$, the collection

$$\left\{(T_u\hat{T}_t^\sharp)^* b_{\mathfrak{v}\mathfrak{u}} T_w \hat{T}_v^\sharp \mid (u,t),(w,v) \in \mathscr{D}_\nu, \mathfrak{v},\mathfrak{u} \in \mathcal{I}_f(\lambda) \text{ and } \lambda \in \Lambda_f\right\}$$

generates B^f/B^{f+1} as an R-module. Therefore, the collection

$$\mathscr{C} = \bigcup_{f=0}^{\min\{m,n\}} \left\{(T_u\hat{T}_t^\sharp)^* \hat{b}_{\mathfrak{v}\mathfrak{u}} T_w \hat{T}_v^\sharp \mid (u,t),(w,v) \in \mathscr{D}_\nu, \mathfrak{v},\mathfrak{u} \in \mathcal{I}_f(\lambda) \text{ and } \lambda \in \Lambda_f\right\}$$

generates $\mathcal{A}_{m,n}(r,q)$ as an R-module and to prove the Proposition, it suffices to show the linear independence of $\mathscr{C}$. To this end,

$$\begin{aligned}|\mathscr{C}| &= \sum_{f=0}^{\min\{m,n\}} |\mathscr{D}_\nu|^2 \sum_{\lambda\in\Lambda_f} |\mathcal{I}_f(\lambda)|^2 \\ &= \sum_{f=0}^{\min\{m,n\}} \left[\binom{m}{f}\binom{n}{f} f!\right]^2 (m-f)!(n-f)!\end{aligned}$$

where, for $0 \le f \le \min\{m,n\}$, ν is the multi-partition $\nu = ((1^f),(m-f),(1^f),(n-f))$. Now each summand in the above expression evaluates the number of walled diagrams with f horizontal bars in the algebra $\mathcal{B}_{m,n}(y)$. From Theorem 5.1, it follows that $|\mathscr{C}| = \dim_R(\mathcal{A}_{m,n}(r,q))$. This completes the proof of the Proposition. □

We are now in a position to show that $\mathcal{A}_{m,n}(r,q)$ is cellular. Define

$$\Lambda = \bigcup_{f=0}^{\min\{m,n\}} \Lambda_f$$

and give Λ a partial order, writing $\lambda \le \mu$ if either (i) $\lambda \in \Lambda_f$ and $\mu \in \Lambda_g$ where $f \le g$ or, (ii) $\lambda,\mu \in \Lambda_f$ and $\lambda \le \mu$ in $(\Lambda_f, \le)$. Set, for each $\lambda \in \Lambda_f$,

$$\mathcal{I}(\lambda) = \{\, (\mathfrak{t},(w,v)) \mid \mathfrak{t} \in \mathcal{I}_f(\lambda) \text{ and } (w,v) \in \mathscr{D}_\nu \,\}$$

where ν is the multi-partition $\nu = ((1^f),(m-f),(1^f),(n-f))$.

For $(\mathfrak{t},(w,v)),(\mathfrak{s},(u,t))$ in $\mathcal{I}(\lambda)$ we define

$$\hat{b}_{(\mathfrak{t},(w,v))(\mathfrak{s},(u,t))} := (T_u\hat{T}_t^\sharp)^*\hat{b}_{\mathfrak{t}\mathfrak{s}}T_w\hat{T}_v^\sharp$$

and let $\check{\mathcal{A}}^\lambda$ be the R-module generated by

$$\left\{ \hat{b}_{(\mathfrak{s},(w,v))\,(\mathfrak{t},(u,t))} \,\middle|\, \mu > \lambda \text{ and } (\mathfrak{s},(w,v))\,(\mathfrak{t},(u,t)) \in \mathcal{I}(\mu) \right\}.$$

Theorem 6.13. *For $0 \le f \le \min\{m,n\}$, let $(\mathscr{C}_f,\Lambda_f)$ be a cellular basis for $\mathcal{H}_{m-f}(q^2)\otimes\mathcal{H}_{n-f}(q^2)$. Then the collection*

$$\mathscr{C} = \left\{ \hat{b}_{(\mathfrak{s},(w,v))\,(\mathfrak{t},(u,t))} \,\middle|\, (\mathfrak{s},(w,v)),(\mathfrak{t},(u,t)) \in \mathcal{I}(\lambda) \text{ and } \lambda \in \Lambda \right\}$$

is a free $\mathcal{R}$-basis for $\mathcal{A}_{m,n}(r,q)$. Furthermore, the following hold.

(1) *The R-linear map determined by*

$$\hat{b}_{(\mathfrak{s},(w,v))\,(\mathfrak{t},(u,t))} \mapsto \hat{b}_{(\mathfrak{t},(u,t))\,(\mathfrak{s},(w,v))}$$

for all $\hat{b}_{(\mathfrak{s},(w,v))(\mathfrak{t},(u,t))} \in \mathscr{C}$ is an anti-involution of $\mathcal{A}_{m,n}(r,q)$.

(2) *If $\lambda \in \Lambda$, $(\mathfrak{t},(u,t)) \in \mathcal{I}(\lambda)$ and $b \in \mathcal{A}_{m,n}(r,q)$ then there exist $a_{(\mathfrak{u},(u',t'))}$, for $(\mathfrak{u},(u',t')) \in \mathcal{I}(\lambda)$, such that*

$$(\,\hat{b}_{(\mathfrak{s},(w,v))\,(\mathfrak{t},(u,t))}\,)\,b \equiv \sum_{(\mathfrak{u},(t',u'))\in\mathcal{I}(\lambda)} a_{(\mathfrak{u},(u',t'))}\;\hat{b}_{(\mathfrak{s},(w,v))\,(\mathfrak{u},(t',u'))} \mod \check{\mathcal{A}}^\lambda$$

for all $(\mathfrak{s},(w,v)) \in \mathcal{I}(\lambda)$.

Consequently $(\mathscr{C},\Lambda)$ is a cellular basis for for $\mathcal{A}_{m,n}(r,q)$.

Proof. By Proposition 6.12, the collection of elements $\hat{b}_{(\mathfrak{t},(w,v))(\mathfrak{s},(u,t))}$ forms a free R-basis for $\mathcal{A}_{m,n}(r,q)$. Since $\hat{b}_{(\mathfrak{t},(w,v))(\mathfrak{s},(u,t))} = (T_w\hat{T}_v^\sharp)^*\hat{b}_{\mathfrak{t}\mathfrak{s}}T_u\hat{T}_t^\sharp$, we observe from the definition of $\hat{b}_{\mathfrak{t}\mathfrak{s}}$ given in (15), that the map defined on generators by $E \mapsto E$, $T_w \mapsto T_{w^{-1}}$ and $\hat{T}_v \mapsto \hat{T}_{v^{-1}}$ is an algebra anti-involution of $\mathcal{A}_{m,n}(r,q)$ which, applied to the basis $\mathscr{C}$, sends $\hat{b}_{(\mathfrak{t},(w,v))(\mathfrak{s},(u,t))} \mapsto \hat{b}_{(\mathfrak{s},(u,t))(\mathfrak{t},(w,v))}$. □

7. A Murphy Basis for $\mathcal{A}_{m,n}(r,q)$

Recall that $\mathcal{H}_{R,m-f}(q^2)\otimes\mathcal{H}_{R,n-f}(q^2) \subseteq \mathcal{H}_{R,m}(q^2)\otimes\mathcal{H}_{R,n}(q^2)$ was identified with the subalgebra generated by the elements $\{\, X_i\otimes 1, 1\otimes X_j \mid f<i<m, f<j<n \,\}$. A Murphy basis for $\mathcal{H}_{R,m-f}(q^2)\otimes\mathcal{H}_{R,n-f}(q^2)$ can be given using Proposition 2.10. Let Λ_f denote the set of multi-partitions

$$\Lambda_f = \{(\lambda^{(1)},\dots,\lambda^{(4)}) \mid (\lambda^{(2)},\lambda^{(4)}) \in \Gamma_f\}.$$

The set Λ_f is partially ordered by $\lambda \trianglelefteq \mu$ if

$$\sum_{i=1}^{j}\lambda_i^{(2)} \le \sum_{i=1}^{j}\mu_i^{(2)} \quad\text{and}\quad \sum_{i=1}^{k}\lambda_i^{(4)} \le \sum_{i=1}^{k}\mu_i^{(4)} \qquad \text{for all } j,k \ge 1.$$

To each multi-partition $\lambda \in \Lambda_f$, associate the element

$$m_\lambda = \sum_{v\in\mathfrak{S}_{\lambda^{(2)}}} X_v \otimes \sum_{w\in\mathfrak{S}_{\lambda^{(4)}}} X_w,$$

and to each pair $\mathfrak{v},\mathfrak{u}$ of standard λ-multi-tableaux we assign the element

$$m_{\mathfrak{v}\mathfrak{u}} = (X^*_{d(\mathfrak{v}^{(2)})}\otimes X^*_{d(\mathfrak{v}^{(4)})})m_\lambda(X_{d(\mathfrak{u}^{(2)})}\otimes X_{d(\mathfrak{u}^{(4)})}) \tag{16}$$

and let $\check{A}^\lambda$ be the R-submodule of $\mathcal{H}_{R,m-f}(q^2)\otimes\mathcal{H}_{R,n-f}(q^2)$ generated by the elements

$$\{m_{\mathfrak{v}\mathfrak{u}} \mid \mathfrak{v},\mathfrak{u} \in \mathrm{Std}(\mu) \text{ and } \mu \triangleright \lambda\}.$$

The statements below now follow from Proposition 2.10.

(1) The set
$$\mathscr{C}_f = \{ m_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ and } \lambda \in \Lambda_f \}$$
is an R-basis for $\mathcal{H}_{R,m-f}(q^2) \otimes \mathcal{H}_{R,n-f}(q^2)$.

(2) The map $* : \mathcal{H}_{R,m-f}(q^2)\otimes\mathcal{H}_{R,n-f}(q^2) \to \mathcal{H}_{R,m-f}(q^2)\otimes\mathcal{H}_{R,n-f}(q^2)$ defined on generators by $* : X_v \otimes X_w \mapsto X_{v^{-1}} \otimes X_{w^{-1}}$, coincides with the map $m_{\mathfrak{st}} \mapsto m_{\mathfrak{ts}}$ for all $\mathfrak{s}, \mathfrak{t} \in \mathscr{C}_f$ and extends linearly to an R-linear algebra anti-involution of $\mathcal{H}_{R,m-f}(q^2) \otimes \mathcal{H}_{R,n-f}(q^2)$;

(3) Let $\lambda \in \Lambda_f$. If $\mathfrak{u} \in \mathrm{Std}(\lambda)$ and $h \in \mathcal{H}_{R,m-f}(q^2) \otimes \mathcal{H}_{R,n-f}(q^2)$, then there exist $a_{\mathfrak{s}} \in R$, for $\mathfrak{s} \in \mathrm{Std}(\lambda)$, such that
$$m_{\mathfrak{vu}} h \equiv \sum_{\mathfrak{s}} a_{\mathfrak{s}} m_{\mathfrak{vs}} \mod \check{A}^\lambda$$
for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$.

In particular, we have a cellular basis for $\mathcal{H}_{m-f}(q^2) \otimes \mathcal{H}_{n-f}(q^2)$.

Now let
$$\Lambda = \bigcup_{f=0}^{\min\{m,n\}} \Lambda_f$$
and extend the order $\trianglelefteq$ to Λ by writing $\lambda \trianglelefteq \mu$ if either (i) $\lambda \in \Lambda_f$ and $\mu \in \Lambda_g$ where $f < g$ or, (ii) $\lambda, \mu \in \Lambda_f$ and $\lambda \trianglelefteq \mu$ in $(\Lambda_f, \trianglelefteq)$.

By Theorem 6.13 the cellular basis for $\mathcal{A}_{m,n}(r,q)$ will be indexed by the ordered pairs
$$\mathcal{I}(\lambda) = \bigcup_{f=0}^{\min\{m,n\}} \{ (\mathfrak{s}, (w,v)) \mid \mathfrak{s} \in \mathrm{Std}(\lambda), \lambda \in \Lambda_f \text{ and } (w,v) \in \mathscr{D}_\nu \}$$
where, for each f, ν is the multi-partition with $\nu^{(2)} = (m-f)$ and $\nu^{(4)} = (n-f)$. Each pair $(\mathfrak{s}, (w,v)) \in \mathcal{I}(\lambda)$ corresponds to a unique λ-multi-tableau

(17)
$$(\mathfrak{s}, (w,v)) \leftrightarrow \mathfrak{v} = \mathfrak{t}^\lambda(d(\mathfrak{s}^{(2)}), d(\mathfrak{s}^{(4)}))(v,w).$$

There is no harm therefore in identifying $\mathcal{I}(\lambda)$ with the multi-tableaux
$$\mathcal{I}(\lambda) = \{\mathfrak{t} \mid \mathfrak{t} = \mathfrak{t}^\lambda(d(\mathfrak{s}^{(2)}), d(\mathfrak{s}^{(4)}))(v,w) \text{ where } \mathfrak{s} \in \mathrm{Std}(\lambda) \text{ and } (v,w) \in \mathscr{D}_\nu \}.$$

We now define for each multi-partition λ the element
$$\hat{b}_\lambda := \prod_{i=1}^{f} E_{i,i} \sum_{w \in \mathfrak{S}_{\lambda^{(2)}}} T_w \sum_{v \in \mathfrak{S}_{\lambda^{(4)}}} T_v$$
and, for $(\mathfrak{s}, (w,v)), (\mathfrak{t}, (u,t)) \in \mathcal{I}(\lambda)$, set
$$\hat{b}_{(\mathfrak{s},(w,v)),(\mathfrak{t},(u,t))} = (T_{d(\mathfrak{s}^{(2)})} T_w \check{T}_{d(\mathfrak{s}^{(4)})} \check{T}_v^\sharp)^* \, \hat{b}_\lambda (T_{d(\mathfrak{t}^{(2)})} T_u \check{T}_{d(\mathfrak{t}^{(4)})} \check{T}_t^\sharp).$$

In light of (17) we may write this more compactly as
$$\hat{b}_{\mathfrak{vu}} = \hat{b}_{(\mathfrak{s},(w,v)),(\mathfrak{t},(u,t))}$$
where $\mathfrak{v} = \mathfrak{t}^\lambda(d(\mathfrak{s}^{(2)}), d(\mathfrak{s}^{(4)}))(v,w)$ and $\mathfrak{u} = \mathfrak{t}^\lambda(d(\mathfrak{t}^{(2)}), d(\mathfrak{t}^{(4)}))(u,t)$.

Let $\check{A}^\lambda$ be the R-submodule of $\mathcal{A}_{m,n}(r,q)$ generated by
$$\{ \hat{b}_{\mathfrak{vu}} \mid \mathfrak{v}, \mathfrak{u} \in \mathcal{I}(\mu) \text{ and } \mu > \lambda \}.$$

Theorem 7.1. *The collection*
$$\mathscr{M} = \left\{ \hat{b}_{\mathfrak{vu}} \,\middle|\, \mathfrak{u}, \mathfrak{v} \in \mathcal{I}(\lambda) \text{ and } \lambda \in \Lambda \right\}$$
defined above is a free R-basis for $\mathcal{A}_{m,n}(r,q)$. Furthermore, the following hold.

(1) *The R-linear map determined by*

$$\hat{b}_{\mathfrak{v}\mathfrak{u}} \mapsto \hat{b}_{\mathfrak{u}\mathfrak{v}}$$

for all $\hat{b}_{\mathfrak{v}\mathfrak{u}} \in \mathscr{M}$ *is an algebra anti-involution of* $\mathcal{A}_{m,n}(r,q)$.

(2) *If* $\lambda \in \Lambda$, $\mathfrak{u} \in I(\lambda)$ *and* $b \in \mathcal{A}_{m,n}(r,q)$ *then there exist* $a_{\mathfrak{s}}$, *for* $\mathfrak{s} \in I(\lambda)$, *such that*

$$\hat{b}_{\mathfrak{v}\mathfrak{u}}\, b \equiv \sum_{\mathfrak{s}} a_{\mathfrak{s}}\, \hat{b}_{\mathfrak{v}\mathfrak{s}} \mod \check{\mathcal{A}}^{\lambda}$$

for all $\mathfrak{v} \in \mathcal{I}(\lambda)$.

Consequently $(\mathscr{M}, \Lambda)$ *is a cellular basis for for* $\mathcal{A}_{m,n}(r,q)$.

8. Specht Modules for $A_{m,n}(\hat{r}, \hat{q})$

Let $A_{m,n}(\hat{r},\hat{q}) = \mathcal{A}_{m,n}(r,q) \otimes_R \kappa$ denote the specialization of $\mathcal{A}_{m,n}(r,q)$ to the field $\kappa = \mathbb{C}(\hat{r},\hat{q})$ and

$$\mathscr{M} = \left\{ \hat{b}_{\mathfrak{v}\mathfrak{u}} \,\middle|\, \mathfrak{v},\mathfrak{u} \in \mathcal{I}(\lambda) \text{ and } \lambda \in \Lambda \right\}$$

be the specialization to κ of the Murphy basis for $\mathcal{A}_{m,n}(r,q)$ given by Theorem 7.1. Then $\mathscr{M}$ will be a basis for $A_{m,n}(\hat{r},\hat{q})$. For $\lambda \in \Lambda$, let N^{λ} be the κ-module with basis

$$\{\, \hat{b}_{\mathfrak{v}\mathfrak{u}} \mid \mathfrak{v},\mathfrak{u} \in \mathcal{I}(\mu) \text{ and } \mu \trianglerighteq \lambda \,\}$$

and $\check{N}^{\lambda} = \sum_{\mu \triangleright \lambda} N^{\mu}$. Define S^{λ} to be the right $A_{m,n}(\hat{r},\hat{q})$-submodule of $N^{\lambda}/\check{N}^{\lambda}$ generated by $\check{N}^{\lambda} + \hat{b}_{\lambda}$. Being isomorphic to a right cell module, S^{λ} has a κ-basis

$$\{\, \check{N}^{\lambda} + \hat{b}_{\mathfrak{t}^{\lambda}\mathfrak{v}} \mid \mathfrak{v} \in \mathcal{I}(\lambda) \}.$$

For $\mathfrak{v} \in \mathcal{I}(\lambda)$, let $\hat{b}_{\mathfrak{v}}$ denote the element $\check{N}^{\lambda} + \hat{b}_{\mathfrak{t}^{\lambda}\mathfrak{v}}$ in S^{λ}. As in Lemma 2.5, there is a symmetric bilinear form $\langle\ ,\ \rangle : S^{\lambda} \times S^{\lambda} \to \kappa$ defined by

$$\langle \hat{b}_{\mathfrak{v}}, \hat{b}_{\mathfrak{u}} \rangle \hat{b}_{\lambda} \equiv \hat{b}_{\mathfrak{t}^{\lambda}\mathfrak{v}} \hat{b}_{\mathfrak{u}\mathfrak{t}^{\lambda}} \mod \check{N}^{\lambda}$$

for all multi-tableaux $\mathfrak{v},\mathfrak{u} \in \mathcal{I}(\lambda)$. Since $\langle\ ,\ \rangle$ is associative,

$$\operatorname{rad} S^{\lambda} = \{\, b \in S^{\lambda} \mid \langle b, b' \rangle = 0 \text{ for all } b' \in S^{\lambda} \,\}$$

is a $A_{m,n}(\hat{r},\hat{q})$-submodule of S^{λ}. Naturally, we define D^{λ} to be the right $A_{m,n}(\hat{r},\hat{q})$-module $S^{\lambda}/\operatorname{rad} S^{\lambda}$. We now have the following consequences of Theorem 2.8 Theorem 2.9 respectively.

Theorem 8.1. *The set*

$$\{\, D^{\lambda} \mid \lambda \in \Lambda_f \textit{ such that } D^{\lambda} \neq 0 \,\}$$

is a complete set of non-isomorphic absolutely irreducible $A_{m,n}(\hat{r},\hat{q})$*-modules.*

Theorem 8.2. *The algebra* $A_{m,n}(\hat{r},\hat{q})$ *is semisimple if and only if* $D^{\lambda} = S^{\lambda}$ *for all* $\lambda \in \Lambda$.

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