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# KANTOROVICH TYPE OPERATOR INEQUALITIES VIA THE SPECHT RATIO II

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ABSTRACT. Yamazaki [14] showed new order preserving operator inequalities on the usual order and the chaotic order by estimating the lower bound of the difference. Mond and Shisha [7, 8] gave an estimate of the difference of the arithmetic mean to the geometric one, as a converse of the arithmetic-geometric mean inequality. In this report, we shall present other order preserving operator inequalities on the usual order and the chaotic one via the Mond-Shisha difference. Among others, as an application of the Furuta inequality, we show that if $A$ and $B$ are positive operators on a Hilbert space $H$ and $k \geq B \geq 1/k$ for some $k \geq 1$, then for a given $\delta \in [0,1]$, $A^\delta \geq B^\delta$ implies

$$A^p + 2k^{p-2\delta} L(1, k^{2p-2\delta}) \log M_{k^2}(p-\delta) I \geq B^p \quad \text{holds for all } p \geq 2\delta,$$

where the case $\delta = 0$ means the chaotic order and the Specht ratio $M_k(r)$ is defined for each $r > 0$ as

$$M_k(r) = \frac{(k^r-1)k^{\frac{r}{k^r-1}}}{re\log k} \quad (k>0, k \neq 1) \quad \text{and} \quad M_1(r) = 1.$$

## 1. INTRODUCTION

We shall consider a bounded linear operator on a complex Hilbert space $H$. An operator $A$ is said to be positive ( in symbol: $A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in H$. The Löwner-Heinz theorem asserts that $A \geq B \geq 0$ ensures $A^p \geq B^p$ for all $p \in [0,1]$. However $A \geq B$ does not always ensure $A^p \geq B^p$ for $p > 1$ in general. Yamazaki [13] showed that $t^2$ is order preserving in the following sense:

$$(1) \qquad A \geq B \geq 0 \quad \text{and} \quad M \geq B \geq m > 0 \quad \text{imply} \quad A^2 + \frac{(M-m)^2}{4} I \geq B^2.$$

Moreover, he showed the following order preserving operator inequality as an extension of (1):

**Theorem A .** *Let $A$ and $B$ be positive operators on a Hilbert space $H$ satisfying $M \geq B \geq m > 0$. If $A \geq B > 0$, then*

$$A^p + M(M^{p-1} - m^{p-1})I \geq A^p + C(m, M, p)I \geq B^p \qquad \textit{for all } p \geq 1,$$

*where*

$$C(m, M, p) \doteq \frac{mM^p - Mm^p}{M-m} + (p-1)\left(\frac{M^p - m^p}{p(M-m)}\right)^{\frac{p}{p-1}}.$$

For positive invertible operators $A$ and $B$ on a Hilbert space $H$, the order defined by $\log A \geq \log B$ is called *the chaotic order.* Since $\log t$ is an operator monotone function, the chaotic order is weaker than the usual one $A \geq B$. J.I.Fujii and the author [1] showed the following order preserving operator inequalities on the chaotic order which is parallel to Theorem A.

**Theorem B .** *Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$ satisfying $M \geq B \geq m > 0$. If $\log A \geq \log B$, then*

$$A^p + \frac{M}{m}(M^p - m^p)I \geq A^p + \frac{1}{m}C(m, M, p+1)I \geq B^p \qquad \textit{for all } p \geq 0,$$

In fact, $\log A \geq \log B$ does not always ensure $A \geq B$ in general. However, by Theorem B, it follows that

$$\log A \geq \log B \quad \text{and} \quad M \geq B \geq m > 0 \quad \text{imply} \quad A + \frac{(M-m)^2}{4m}I \geq B.$$

On the other hand, Specht [9] estimated the upper bound of the arithmetic mean by the geometric one for positive numbers: For $x_1, \cdots, x_n \in [m, M]$ with $M \geq m > 0$,

$$M_h(1)\sqrt[n]{x_1 \cdots x_n} \geq \frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n},$$

where $h = \frac{M}{m} (\geq 1)$ is a generalized condition number in the sense of Turing [12] and the Specht ratio $M_h(1)$ is defined for $h \geq 1$ as

$$(2) \qquad M_h(1) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h} \quad (h > 1) \quad \text{and} \quad M_1(1) = 1.$$

Yamazaki [14] showed a new characterization of chaotic order as follows:

**Theorem C .** *Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$ satisfying $M \geq B \geq m > 0$. Then $\log A \geq \log B$ is equivalent to*

$$A^p + L(m^p, M^p)\log M_h(p)I \geq B^p \quad \textit{holds for all } p > 0,$$

*where $h = \frac{M}{m} > 1$, the logarithmic mean $L(m, M) = \frac{M-m}{\log M - \log m}$ and a generalized Specht ratio $M_h(p)$ is defined as*

$$(3) \qquad M_h(p) = \frac{(h^p - 1)h^{\frac{p}{h^p-1}}}{pe \log h} \quad (h > 0, h \neq 1) \quad \textit{and} \quad M_1(p) = 1.$$

What is the meaning of the constant $L(m^p, M^p)\log M_h(p)$ in Theorem C ? Mond and Shisha [7, 8] made an estimate of the difference between the arithmetic mean and the geometric one: For $x_1, \cdots, x_n \in [m, M]$ with $M \geq m > 0$,

$$\sqrt[n]{x_1 \cdots x_n} + D(m, M) \geq \frac{x_1 + \cdots + x_n}{n},$$

where $h = \frac{M}{m} (\geq 1)$ and

$$D(m, M) = \theta M + (1-\theta)m - M^\theta m^{1-\theta} \quad \text{and} \quad \theta = \log\left(\frac{h-1}{\log h}\right)\frac{1}{\log h}$$

which we call *the Mond-Shisha difference.* As a matter of fact, J.I.Fujii and the author [1] showed that the Mond-Shisha difference exactly coincides with the constant in Theorem C via the Specht ratio: If $M > m > 0$, then

$$D(m^p, M^p) = L(m^p, M^p)\log M_h(p)$$

where $h = \frac{M}{m}$.

Comparing Theorem A with Theorem B, we observe the difference between $p$ and $p-1$ in the power of the constant. Hence one might expect that the following result holds under the usual order as a parallel result to Theorem C via the Mond-Shisha difference: Let $A$ and $B$ be positive invertible operators satisfying $M \geq B \geq m > 0$. Then

$$A \geq B \quad \text{implies} \quad A^p + mL(m^{p-1}, M^{p-1}) \log M_h(p-1)I \geq B^p \quad \text{for all } p \geq 2,$$

where $h = \frac{M}{m} \geq 1$. However, we have a counterexample to this conjecture. Put

$$A = \begin{pmatrix} 3 & 1 \\ 1 & \frac{3}{2} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

then $A \geq B \geq 0$ and $2I \geq B \geq \frac{1}{2}I$. Then we have $mL(m^1, M^1) \log M_h(1) = 0.126638$. On the other hand, $A^2 + \alpha I \geq B^2$ holds if and only if $\alpha \geq \frac{-35+\sqrt{1465}}{8} = 0.409415$. Therefore $A^2 + mL(m^1, M^1) \log M_h(1)I \not\geq B^2$.

We collect the difference between the usual order and the chaotic one in the following table.

TABLE 1. The difference between the usual order and the chaotic order

| $A \geq B$ | $\log A \geq \log B$ |
|---|---|
| $A^p + M(M^{p-1} - m^{p-1})I \geq B^p$ for $p \geq 1$ | $A^p + \frac{M}{m}(M^p - m^p)I \geq B^p$ for $p > 0$ |
| $A^p + C(m, M, p)I \geq B^p$ for $p \geq 1$ | $A^p + \frac{1}{m}C(m, M, p+1)I \geq B^p$ for $p > 0$ |
| $A^p + \frac{1}{m^{r-1}}C(m^r, M^r, 1 + \frac{p-1}{r})I \geq B^p$ for $p, r \geq 1$ | $A^p + \frac{1}{m^r}C(m^r, M^r, 1 + \frac{p}{r})I \geq B^p$ for $p, r > 0$ |
| $A^p + \dfrac{(M^{p-1} - m^{p-1})^2}{4m^{p-2}}I \geq B^p$ for $p \geq 2$ | $A^p + \dfrac{(M^p - m^p)^2}{4m^p}I \geq B^p$ for $p > 0$ |
| ??? | $A^p + L(m^p, M^p) \log M_h(p)I \geq B^p$ for $p > 0$ |

In this report, we shall present order preserving operator inequalities on the usual order and the chaotic one in terms of the Mond-Shisha difference. As an application of the Furuta inequality, we show that if $A$ and $B$ are positive operators and $k \geq B \geq 1/k$ for some $k \geq 1$, then for a given $\delta \in [0,1]$ $A^\delta \geq B^\delta$ implies

$$A^p + 2k^{p-2\delta} L(1, k^{2p-2\delta}) \log M_{k^2}(p-\delta)I \geq B^p \quad \text{holds for all } p \geq 2\delta.$$

## 2. Main results

First of all, we present other characterizations of the chaotic order via the Mond-Shisha difference.

**Theorem 1.** *Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$ satisfying $k \geq B \geq \frac{1}{k}$ for some $k \geq 1$. Then the following are mutually equivalent:*

(i) $\log A \geq \log B$

(ii) $(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^s + qk^r L(1, k^{\frac{(p+t)s+r}{q}}) \log M_k((p+t)s+r)I \geq B^{(p+t)s+r}$

*holds for $p \geq 0$, $t \geq 0$, $s \geq 0$, $q \geq 1$ with $(t+r)q \geq (p+t)s+r$.*

(iii) $(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^s + 2k^{(p+t)s-2t} L(1, k^{2(p+t)s-2t}) \log M_k(2(p+t)s-2t)I \geq B^{(p+t)s}$

*holds for $p \geq 0$, $t \geq 0$, $s \geq 0$ with $(p+t)s \geq 2t$.*

(iv) $A^p + 2k^p L(1, k^{2p}) \log M_k(2p)I \geq B^p$ *holds for $p > 0$,*

*where $M_k(r)$ is defined as (3).*

Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$. We consider an order $A^\delta \geq B^\delta$ for $\delta \in (0,1]$ which interpolates the usual order $A \geq B$ and the chaotic one $\log A \geq \log B$ continuously, where the case of $\delta = 0$ means the chaotic order. By virtue of the Furuta inequality, we show the following order preserving operator inequality associated with the Mond-Shisha difference.

**Theorem 2.** *Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$ satisfying $k \geq B \geq \frac{1}{k} > 0$. If $A^\delta \geq B^\delta$ for some $\delta \in [0,1]$, then*

$$A^p + 2k^{p-2\delta} L(1, k^{2p-2\delta}) \log M_{k^2}(p-\delta)I \geq B^p \quad \text{holds for all } p \geq 2\delta,$$

*where $M_k(r)$ is defined as (3).*

If we put $\delta = 1$ in Theorem 2, then we have a usual order version via the Mond-Shisha difference.

**Theorem 3.** *Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$ satisfying $k \geq B \geq \frac{1}{k} > 0$. If $A \geq B$, then*

$$A^p + 2k^{p-2} L(1, k^{2p-2}) \log M_{k^2}(p-1)I \geq B^p \quad \text{holds for all } p \geq 2,$$

*where $M_k(r)$ is defined as (3).*

**Remark 4.** *Theorem 2 interpolates Theorems 1 ( iv) and Theorem 3 by means of the Mond-Shisha diffrence. Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$ satisfying $k \geq B \geq \frac{1}{k} > 0$. Then the following assertions hold:*

( i ) $A \geq B$ *implies* $A^p + 2k^{p-2} L(1, k^{2p-2}) \log M_{k^2}(p-1)I \geq B^p$ *for all* $p \geq 2$.

( ii ) $A^\delta \geq B^\delta$ *implies* $A^p + 2k^{p-2\delta} L(1, k^{2p-2\delta}) \log M_{k^2}(p-\delta)I \geq B^p$ *for all* $p \geq 2\delta$.

( iii ) $\log A \geq \log B$ *implies* $A^p + 2k^p L(1, k^{2p}) \log M_{k^2}(p)I \geq B^p$ *for all* $p > 0$.

*It follows that the Mond-Shisha difference of ( ii ) interpolates the scalars of ( i ) and ( iii ) continuously. In fact, if we put $\delta = 1$ in ( ii ), then we have ( i ), also if we put $\delta \to 0$ in ( ii ), then we have ( iii ).*

TABLE 2. Kantorovich constant

| | |
|---|---|
| $A \geq B$ and $M \geq B \geq m$ | $\log A \geq \log B$ and $M \geq B \geq m$ |
| $A^2 + \frac{(M-m)^2}{4} I \geq B^2$ | $A + \frac{(M-m)^2}{4m} I \geq B$ |

TABLE 3. Mond-Shisha difference

| | |
|---|---|
| $A \geq B$ and $k \geq B \geq 1/k$ | $\log A \geq \log B$ and $k \geq B \geq 1/k$ |
| $A^2 + L(1,k^2) \log M_{k^2}(1)^2 I \geq B^2$ | $A + kL(1,k^2) \log M_{k^2}(1)^2 I \geq B$ |

## 3. PROOF OF RESULTS

To prove our results, we collect several properties of the Specht ratio, see [11, 15]:

**Lemma 5.** (i) $M_k(r) = M_{k^r}(1)$ *for* $k > 0$ *and* $r > 0$.
(ii) $k \to M_k(1)$ *is increasing for* $k > 1$ *and decreasing for* $1 > k > 0$.
(iii) $M_k(1) = M_{k^{-1}}(1)$ *for* $k > 0$.
(iv) *For* $k > 1$, $M_k(p)^{1/p} \to 1$ *as* $p \to 0$.

**Lemma 6.** *Let* $A$ *and* $B$ *be positive operators on a Hilbert space* $H$ *satisfying* $k \geq B \geq \frac{1}{k}$ *for some* $k \geq 1$. *If* $A^p \geq B^p$ *for some* $p \in (0,1]$, *then*

$$A + \frac{1}{p} L(1,k) \log M_k(1) I \geq B,$$

*where* $M_k(1)$ *is defined as* (2).

*Proof.* The following reverse inequality of Young's one is shown in [11]: For a positive operator $A$ satisfying $k \geq A \geq \frac{1}{k}$ for some $k \geq 1$,

$$A^p + L(1,k) \log M_k(1) I \geq pA + (1-p)I \tag{4}$$

holds for all $1 > p > 0$. Then we have

$$\begin{aligned} L(1,k) \log M_k(1) I + pA + (1-p)I & \\ \geq L(1,k) \log M_k(1) I + A^p & \quad \text{by the Young inequality and } 1 > p > 0 \\ \geq L(1,k) \log M_k(1) I + B^p & \quad \text{by } A^p \geq B^p \\ \geq pB + (1-p)I & \quad \text{by (4) and } k \geq B \geq 1/k > 0. \end{aligned}$$

□

The following order preserving operator inequality is our key lemma in this report.

**Lemma 7.** *Let $A$ and $B$ be positive operators on a Hilbert space $H$ satisfying $k \geq B \geq \frac{1}{k} > 0$ for some $k \geq 1$. If $A \geq B$, then*

$$A^p + pL(1, k^p) \log M_k(p) I \geq B^p \quad \textit{for all } p \geq 1,$$

*where $M_k(1)$ is defined as (2).*

*Proof.* Since $(A^p)^{1/p} \geq (B^p)^{1/p}$ for $0 < \frac{1}{p} \leq 1$ and $k^p \geq B^p \geq k^{-p}$, it follows from Lemma 6 that

$$A^p + pL(1, k^p) \log M_k(p) I \geq B^p \quad \text{for all } p \geq 1.$$

□

To prove Theorem 1, we need the following result [3, Proposition 7]:

**Theorem D .** *Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$. If $\log A \geq \log B$, then*

$$\{B^{\frac{r}{2}} \left(B^{\frac{t}{2}} A^p B^{\frac{t}{2}}\right)^s B^{\frac{r}{2}}\}^{\frac{1}{q}} \geq B^{\frac{(p+t)s+r}{q}}$$

*holds for $p, t, s, r \geq 0$ and $q \geq 1$ with $(t+r)q \geq (p+t)s + r$.*

*Proof of Theorem 1.*
(i) $\Longrightarrow$ (ii): By Theorem D, (i) ensures

$$\{B^{\frac{r}{2}} \left(B^{\frac{t}{2}} A^p B^{\frac{t}{2}}\right)^s B^{\frac{r}{2}}\}^{\frac{1}{q}} \geq B^{\frac{(p+t)s+r}{q}} \tag{5}$$

holds for $p, t, s, r \geq 0$ and $q \geq 1$ with

$$(t+r)q \geq (p+t)s + r. \tag{6}$$

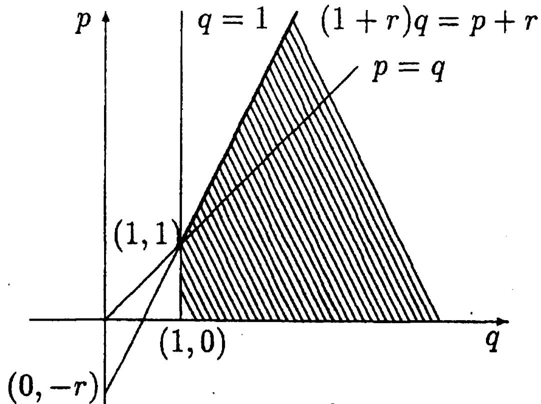
Put $A_1 = A^{\frac{(p+t)s+r}{q}}$ and $B_1 = (A^{\frac{r}{2}} \left(A^{\frac{t}{2}} B^p A^{\frac{t}{2}}\right)^s A^{\frac{r}{2}})^{1/q}$, then $A_1 \geq B_1$ by (5) and $k \geq A \geq 1/k > 0$ assures $k^{\frac{(p+t)s+r}{q}} \geq A^{\frac{(p+t)s+r}{q}} \geq k^{-\frac{(p+t)s+r}{q}}$. By applying Lemma 7 to $A_1$ and $B_1$, we have

$$A_1^q + qL(1, k^{(p+t)s+r} \log M_{k^{(p+t)s+r}}(1) I \geq B_1^q.$$

Multipying $B^{-\frac{r}{2}}$ on both sides, we have (ii).
(ii) $\Longrightarrow$ (iii): Put $r = (p+t)s - 2t \geq 0$ and $q = 2$ in (ii). Then the condition (6) is satisfied and $(p+t)s \geq 2t$, so we have (iii).
(iii) $\Longrightarrow$ (iv): If we put $t = 0$ and $s = 1$ in (iii), then we have (iv).
(iv) $\Longrightarrow$ (i): If we put $p \to 0$ in (iv), then we have (i) by (iv) of Lemma 5. □

Related to the extension of the Löwner-Heinz theorem, Furuta [4] established the following ingenious order preserving inequality which is now called the Furuta inequality.

**Theorem F (Furuta inequality)**
If $A \geq B \geq 0$, then for each $r \geq 0$
(i) $\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(B^{\frac{r}{2}}B^pB^{\frac{r}{2}}\right)^{\frac{1}{q}}$
and
(ii) $\left(A^{\frac{r}{2}}A^pA^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}$
hold for $p \geq 0$ and $q \geq 1$ with
$(1+r)q \geq p+r$.




**Figure**

Alternative proofs of Theorem F have been given in [2], [6], and one-page proof in [5]. The domain drawn for $p, q$ and $r$ in Figure is the best possible one [10] for Theorem F.

To prove Theorem 2, we need the following Furuta inequality:

**Theorem F'** . *Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$ and $\delta \in [0,1]$. Then the following properties are mutually equivalent:*
(i) $A^\delta \geq B^\delta$
(ii) $\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{\delta+r}{p+r}} \geq B^{\delta+r}$ *for $p \geq \delta$ and $r \geq 0$.*

*Proof of Theorem 2.*
Suppose that $A^\delta \geq B^\delta$ for some $\delta \in [0,1]$. By the Furuta inequlity, we have

$$\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{\delta+r}{p+r}} \geq B^{\delta+r} \qquad \text{for } p \geq \delta \text{ and } r \geq 0.$$

and $k^{\delta+r} \geq B^{\delta+r} \geq k^{-\delta-r}$.
By Lemma 7 and $\frac{p+r}{\delta+r} \geq 1$, it follows that

$$B^{\frac{r}{2}}A^pB^{\frac{r}{2}} + \frac{p+r}{\delta+r}L(1,k^{p+r})\log M_k(p+r)I \geq B^{p+r}.$$

Hence we have

$$A^p + \frac{p+r}{\delta+r}k^rL(1,k^{p+r})\log M_k(p+r)I \geq B^p$$

for $p \geq \delta$ and $r \geq 0$. Put $r = p - 2\delta (\geq 0)$, then

$$A^p + 2k^{p-2\delta}L(1,k^{2p-2\delta})\log M_k(2p-2\delta)I \geq B^p$$

for all $p \geq 2\delta$. □

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