数理解析研究所講究録 1312 巻 2003 年 108-120

Specht ratio $S(1)$ can be expressed by generalized Kantorovich constant $K(p)$: $S(1)=e^{K'(1)}$ and its application to operator inequalities associated with $A\log A$

東京理科大学理学部　古田孝之 (Takayuki Furuta)

Abstract. An operator means a bounded linear operator on a Hilbert space H. We obtained the basic property between Specht ratio $S(1)$ and generalized Kantorovich constant $K(p)$ in [13], that is, Specht ratio $S(1)$ can be expressed by generalized Kantorovich constant $K(p)$: $S(1)=e^{K'(1)}$. We shall investigate several product type and difference type inequalities associated with $A\log A$ by applying this basic property to several Kantorovich type inequalities.

§1 Introduction.

An operator A is said to be positive operator (denoted by $T\geq 0$) if $(Ax,x)\geq 0$ for all x in H and also A is said to be strictly positive operator (denoted by $A>0$) if A is invertible positive operator.

Definition 1. Let $h>1$. $S(h,p)$ is defined by

(1.1) $$S(h,p)=\frac{h^{\frac{p}{h^p-1}}}{e\log h^{\frac{p}{h^p-1}}}\quad \text{for any real number } p$$

and $S(h,p)$ is denoted by $S(p)$ briefly. Especially $S(1)=S(h,1)=\dfrac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}$ is said to be *Specht ratio* and $S(1)>1$ is well known.

Let $h>1$. *The generalized Kantorovich constant* $K(h,p)$ is defined by

(1.2) $$K(h,p)=\frac{(h^p-h)}{(p-1)(h-1)}\left(\frac{(p-1)}{p}\frac{(h^p-1)}{(h^p-h)}\right)^p\quad \text{for any real number } p$$

and $K(h,p)$ is denoted by $K(p)$ briefly.

Basic Property [13]. *The following basic property among* $S(1)$, $K'(1)$ *and* $K'(0)$ *holds:*

(1.3) $$S(1)=e^{K'(1)}=e^{-K'(0)}\qquad (\text{i.e.,} S(1)=\exp[\lim_{p\to 1}K'(p)]=\exp[-\lim_{p\to 0}K'(p)]\ )$$

(1.4) $$K(0)=K(1)=1\qquad (\text{i.e.,}\ \lim_{p\to 0}K(p)=\lim_{p\to 1}K(p)=1\ )$$

(1.5) $$S(1)=\lim_{p\to 1}K(p)^{\frac{1}{p-1}}=\lim_{p\to 0}K(p)^{\frac{-1}{p}}.$$

We cite Figure 1 relation between $K(p)$ and $S(p)$ before the References.

In fact $K'(p)$ can be written as follows:

$$(*)\quad K'(p)=\frac{(\frac{(p-1)}{p}\frac{(h^p-1)}{(h^p-h)})^p}{(h-1)(h^p-1)}\left\{\frac{h^p(h^p-1+p-hp)\log h+(h^p-1)(h^p-h)\log\frac{(p-1)(h^p-1)}{p(h^p-h)}}{p-1}\right\}.$$

By using L.Hopital's theorem to (*), we have

$$\begin{aligned}\lim_{p\to 1}K'(p)&=\frac{h-1}{h\log h}\frac{1}{(h-1)^2}\Big\{h\log h(h\log h+1-h)+(h-1)h\log h\log\Big[\frac{h-1}{h\log h}\Big]\Big\}\\&=\frac{h}{h-1}\log h-1+\log\Big[\frac{h-1}{h\log h}\Big]\\&=\log\Big[\frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}\Big]\\&=\log S(1)\end{aligned}$$

so that we have $S(1)=e^{K'(1)}$ and also $S(1)=e^{-K'(0)}$ by the same way.

We remark that (1.5) is an immediate consequence of (1.3) by L'Hospital theorem. Another nice relation between $K(p)$ and $S(1)$ is in [26].

Let A be strictly positive operator satisfying $MI\geq A\geq mI>0$, where $M>m>0$. Put $h=\frac{M}{m}>1$. The celebrated Kantorovich inequality asserts that

$$(1.6)\qquad \frac{(1+h)^2}{4h}(Ax,x)^{-1}\geq(A^{-1}x,x)\geq(Ax,x)^{-1}$$

holds for every unit vector x and this inequality is just equivalent to the following one

$$(1.7)\qquad \frac{(1+h)^2}{4h}(Ax,x)^2\geq(A^2x,x)\geq(Ax,x)^2$$

holds for every unit vector x. We remark that $K(h,p)$ in (1.2) is an extension of $\frac{(1+h)^2}{4h}$ in (1.6) and (1.7) , in fact, $K(h,-1)=K(h,2)=\frac{(1+h)^2}{4h}$ holds.

Many papers on Kantorovich inequality have been published. Among others, there is a long research series by Mond-Pečarić, we cite [21][22] and [23] for examples.

We state the Jensen inequality as follows. (c.f. [Theorem 4, 1],[3, 4],[Theorem 2.1, 17])

Jensen inequality. Let f be an operator concave function on an interval I. If Φ is normalized positive linear map,then

$$f(\Phi(A))\geq\Phi(f(A))$$

for every self adjoint operator A on a Hilbert space H whose spectrum is contained in I.

On the other hand, the relative operator entropy $S(X|Y)$ for $X>0$ and $Y>0$ is defined in [7] as an extension of the operator entropy $S(X|I)=-X\log X$

(1.8) $$S(X|Y) = X^{\frac{1}{2}}[\log(X^{\frac{-1}{2}}YX^{\frac{-1}{2}})]X^{\frac{1}{2}}.$$

By using this $S(X|Y)$, we define $T(X|Y)$ for $X > 0$ and $Y > 0$;

(1.9) $$T(X|Y) = (X\sharp Y)X^{-1}S(X|Y)X^{-1}(X\sharp Y)$$

where $X\sharp Y = X^{\frac{1}{2}}(X^{\frac{-1}{2}}YX^{\frac{-1}{2}})^{\frac{1}{2}}X^{\frac{1}{2}}$. The power mean $X\sharp_p Y = X^{\frac{1}{2}}(X^{\frac{-1}{2}}YX^{\frac{-1}{2}})^p X^{\frac{1}{2}}$ for $p \in [0,1]$ is in [16] as an extension of $X\sharp Y$. We shall verify that $T(X|Y) = \lim_{p\to 1}(X\sharp_p Y)'$ in Proposition 3.2 and we remark that $S(X|Y) = \lim_{p\to 0}(X\sharp_p Y)'$ shown in [7].

Next we state the following several Kantorovich type inequalities.

Theorem A. *Let A be strictly positive operator on a Hilbert space H satisfying $MI \geq A \geq mI > 0$, where $M > m > 0$ and $h = \frac{M}{m} > 1$ and Φ be a normalized positive linear map on $B(H)$. Let $p \in (0,1)$. Then the following inequalities hold:*

(i) $$\Phi(A)^p \geq \Phi(A^p) \geq K(p)\Phi(A)^p$$

(ii) $$\Phi(A)^p \geq \Phi(A^p) \geq \Phi(A)^p - g(p)I$$

where $g(p) = m^p[\dfrac{h^p - h}{h-1} + (1-p)\Big(\dfrac{h^p-1}{p(h-1)}\Big)^{\frac{p}{p-1}}]$ *and $K(p)$ is defined in* (1.2).

The right hand side inequalities of (i) and (ii) in Theorem A follow by [Corollary 2.6, 18] and [23] and the left hand side one of (i) follows by Jensen inequality since $f(A) = A^p$ is operator concave for $p \in [0,1]$. More general forms than Theorem A are in [17] and related results to Theorem A are in [19][20].

Theorem B. *Let A and B be strictly positive operators on a Hilbert space H such that $M_1I \geq A \geq m_1I > 0$ and $M_2I \geq B \geq m_2I > 0$. Put $m = m_1m_2$, $M = M_1M_2$ and $h = \frac{M}{m} = \frac{M_1M_2}{m_1m_2} > 1$. Let $p \in (0,1)$. Then the following inequalities hold:*

(i) $$(A*B)^p \geq A^p * B^p \geq K(p)(A*B)^p$$

(ii) $$(A*B)^p \geq A^p * B^p \geq (A*B)^p - g(p)I$$

where $g(p) = m^p[\dfrac{h^p - h}{h-1} + (1-p)\Big(\dfrac{h^p-1}{p(h-1)}\Big)^{\frac{p}{p-1}}]$ *and $K(p)$ is defined in* (1.2).

The right hand side inequalities of (i) and (ii) follow by [Theorem 16, 25] and the left hand side one of (i) follows by [10] and [Theorem 1, 25].

Theorem C. *Let A,B, C and D be strictly positive operators on a Hilbert space H such that $M_1I \geq A \bigotimes B \geq m_1I > 0$ and $M_2I \geq C \bigotimes D \geq m_2I > 0$. Put $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$ and $h = \frac{M}{m} = \frac{M_1M_2}{m_1m_2} > 1$. Let $p \in (0,1)$. Then the following inequalities hold:*

(i) $(A*B)\sharp_p(C*D) \geq (A\sharp_p C)*(B\sharp_p D) \geq K(p)(A*B)\sharp_p(C*D)$

(ii) $(A*B)\sharp_p(C*D) \geq (A\sharp_p C)*(B\sharp_p D) \geq (A*B)\sharp_p(C*D) - g(p)I(A*B)$

where $g(p) = m^p[\frac{h^p-h}{h-1} + (1-p)\Big(\frac{h^p-1}{p(h-1)}\Big)^{\frac{p}{p-1}}]$ *and* $K(p)$ *is defined in* (1.2).

The right hand side inequalities of (i) and (ii) follow by [Corollary 4.4,18] and the left hand side inequality of (i) follows by [Theorem 4.1, 2] and also it follows by a corollary of [Theorem 5, 5].

Theorem D. *Let A and B be strictly positive operators on a Hilbert space H such that $M_1I \geq A \geq m_1I > 0$ and $M_2I \geq B \geq m_2I > 0$. Put $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$ and $h = \frac{M}{m} = \frac{M_1M_2}{m_1m_2} > 1$. Let $p \in (0,1)$ and also let Φ be normalized positive linear map on $B(H)$. Then the following inequalities hold:*

(i) $\Phi(A)\sharp_p\Phi(B) \geq \Phi(A\sharp_p B) \geq K(p)\Phi(A)\sharp_p\Phi(B)$

(ii) $\Phi(A)\sharp_p\Phi(B) \geq \Phi(A\sharp_p B) \geq \Phi(A)\sharp_p\Phi(B) - g(p)\Phi(A)$

where $g(p) = m^p[\frac{h^p-h}{h-1} + (1-p)\Big(\frac{h^p-1}{p(h-1)}\Big)^{\frac{p}{p-1}}]$ *and* $K(p)$ *is defined in* (1.2).

The right hand side inequalities of (i) and (ii) follow by [Corollary 3.5,18] and the left hand side one of (i) follows by [1] and [16].

The following result is contained in [Corollary 4.11, 18] together with [Corollary 8, 5].

Theorem E'. *Let A and B be strictly positive operators on a Hilbert space H such that $M_1I \geq A \geq m_1I > 0$ and $M_2I \geq B \geq m_2I > 0$. Let $p \in (0,1)$ and also $m = m_1^{\frac{1}{p}}M_2^{\frac{-1}{1-p}}$, $M = M_1^{\frac{1}{p}}m_2^{\frac{-1}{1-p}}$ and $h = \frac{M}{m} = (\frac{M_1}{m_1})^{\frac{1}{p}}(\frac{M_2}{m_2})^{\frac{1}{1-p}} > 1$. Then the following inequalities hold:*

(i) $$(A^{\frac{1}{p}}*I)^p(B^{\frac{1}{1-p}}*I)^{1-p} \geq A*B \geq K(p)(A^{\frac{1}{p}}*I)^p(B^{\frac{1}{1-p}}*I)^{1-p}$$

(i) $$(A^{\frac{1}{p}}*I)^p(B^{\frac{1}{1-p}}*I)^{1-p} \geq A*B \geq (A^{\frac{1}{p}}*I)^p(B^{\frac{1}{1-p}}*I)^{1-p} - g(p)(B*I)$$

where $g(p) = m^p[\frac{h^p-h}{h-1} + (1-p)\Big(\frac{h^p-1}{p(h-1)}\Big)^{\frac{p}{p-1}}]$ *and* $K(p)$ *is defined in* (1.2).

In fact put $A_3 = A^p$ and $B_3 = B^{1-p}$, then $M_1^pI \geq A_3 \geq m_1^pI > 0$ and $M_2^{1-p}I \geq B_3 \geq m_2^{1-p}I > 0$ under the hypotneses of Theorem E. By applying Theorem E' to A_3 and B_3, put $m_3 = m_1^{p\frac{1}{p}}M_2^{(1-p)\frac{-1}{1-p}} = \frac{m_1}{M_2}$, $M_3 = M_1^{p\frac{1}{p}}m_2^{(1-p)\frac{-1}{1-p}} = \frac{M_1}{m_2}$ and $h_3 = \frac{M_3}{m_3} = \frac{M_1M_2}{m_1m_2} > 1$, so we have the following result as a variation of Theorem E'.

Theorem E. *Let A and B be strictly positive operators on a Hilbert space H such that $M_1I \geq A \geq m_1I > 0$ and $M_2I \geq B \geq m_2I > 0$. Put $m = \frac{m_1}{M_2}$, $M = \frac{M_1}{m_2}$ and $h = \frac{M}{m} = \frac{M_1M_2}{m_1m_2} > 1$. Let $p \in (0,1)$. Then the following inequalities hold:*

(i) $$(A*I)^p(B*I)^{1-p} \geq A^p * B^{1-p} \geq K(p)(A*I)^p(B*I)^{1-p}$$

(ii) $$(A*I)^p(B*I)^{1-p} \geq A^p * B^{1-p} \geq (A*I)^p(B*I)^{1-p} - g(p)(B^{1-p}*I)$$

where $g(p) = m^p[\frac{h^p - h}{h-1} + (1-p)\left(\frac{h^p-1}{p(h-1)}\right)^{\frac{p}{p-1}}]$ *and* $K(p)$ *is defined in* (1.2).

We shall investigate several product type and difference type inequalities associated with $A \log A$ by applying the Basic Property to Theorem A, Theorem B, Theorem C, Theorem D and Theorem E which are Kantorovich type inequalities.

§2 Several product type and difference type inequalities associated with $A \log A$

In this §2 we shall state the following several product type and difference type inequalities associated with $A \log A$.

Theorem 2.1. *Let A be strictly positive operator on a Hilbert space H satisfying*

$MI \geq A \geq mI > 0$, where $M > m > 0$ and $h = \frac{M}{m} > 1$ and Φ be a normalized positive linear map on $B(H)$. Then the following inequalities hold:

(i) $$\begin{aligned}&[\log S(1)]\Phi(A) + \Phi(A)\log\Phi(A)\\ &\geq \Phi(A\log A)\\ &\geq \Phi(A)\log\Phi(A)\end{aligned}$$

(ii) $$\begin{aligned}&\frac{mh\log h}{h-1}(S(1)-1) + \Phi(A)\log\Phi(A)\\ &\geq \Phi(A\log A)\\ &\geq \Phi(A)\log\Phi(A).\end{aligned}$$

(iii) $$\log S(1) + \Phi(\log A) \geq \log\Phi(A) \geq \Phi(\log A),$$

where $S(1)$ *is defined in* (1.1).

We remark that the first inequality of (i) in Theorem 2.1 is the reverse inequality of the second one which is known by [Theorem 4, 1] and also the first inequality of (ii) is the reverse inequality of the second one , and the first inequality of (iii) in Theorem 2.1 is the reverse inequality of the second one which is known by Jensen inequality.

Theorem 2.2. *Let A and B be strictly positive operators on a Hilbert space H such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $m = m_1 m_2$, $M = M_1 M_2$ and $h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1$. Then the following inequalities hold:*

(i)
$$\begin{aligned}&[\log S(1)](A*B) + (A*B)\log(A*B)\\ &\geq A*(B\log B) + (A\log A)*B\\ &\geq (A*B)\log(A*B)\end{aligned}$$

(ii)
$$\begin{aligned}&\frac{mh\log h}{h-1}(S(1)-1) + (A*B)\log(A*B)\\ &\geq A*(B\log B) + (A\log A)*B\\ &\geq (A*B)\log(A*B)\end{aligned}$$

(iii)
$$\begin{aligned}&\log S(1) + (\log A)*I + I*(\log B)\\ &\geq \log(A*B)\\ &\geq (\log A)*I + I*(\log B)\end{aligned}$$

where $S(1)$ is defined in (1.1).

We remark that the first inequality of (i) in Theorem 2.2 is the reverse inequality of the second one and also the first inequality of (ii) is the reverse inequality of the second one, and the first inequality of (iii) in Theorem 2.2 is the reverse inequality of the second one.

Theorem 2.3. *Let A,B, C and D be strictly positive operators on a Hilbert space H such that $M_1 I \geq A \bigotimes B \geq m_1 I > 0$ and $M_2 I \geq C \bigotimes D \geq m_2 I > 0$. Put $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$ and $h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1$. Then the following inequalities hold:*

(i)
$$\begin{aligned}&[\log S(1)](C*D) + T(A*B|C*D)\\ &\geq T(A|C)*D + C*T(B|D)\\ &\geq T(A*B|C*D)\end{aligned}$$

(ii)
$$\begin{aligned}&\frac{mh\log h}{h-1}(S(1)-1)(A*B) + T(A*B|C*D)\\ &\geq T(A|C)*D + C*T(B|D)\\ &\geq T(A*B|C*D)\end{aligned}$$

(iii)
$$\begin{aligned}&[\log S(1)](A*B) + S(A|C)*B + A*S(B|D)\\ &\geq S(A*B|C*D)\end{aligned}$$

$$\geq S(A|C) * B + A * S(B|D)$$

where $S(X|Y)$ *and* $T(X|Y)$ *are defined in* (1.8) *and* (1.9) *and* $S(1)$ *is defined in* (1.1).

We remark that the first inequality of (i) in Theorem 2.3 is the reverse inequality of the second one and also the first inequality of (ii) is the reverse inequality of the second one, and the first inequality of (iii) in Theorem 2.3 is the reverse inequality of the second one.

Theorem 2.4. *Let A and B be strictly positive operators on a Hilbert space H such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$ and $h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1$. Let Φ be a normalized positive linear map on $B(H)$. Then the following inequalities hold:*

(i)
$$\begin{aligned}&[\log S(1)]\Phi(B) + T(\Phi(A)|\Phi(B))\\ &\geq \Phi(T(A|B))\\ &\geq T(\Phi(A)|\Phi(B))\end{aligned}$$

(ii)
$$\begin{aligned}&\frac{mh\log h}{h-1}(S(1)-1)\Phi(A) + T(\Phi(A)|\Phi(B))\\ &\geq \Phi(T(A|B))\\ &\geq T(\Phi(A)|\Phi(B))\end{aligned}$$

(iii)
$$\begin{aligned}&\log S(1)\Phi(A) + \Phi(S(A|B))\\ &\geq S(\Phi(A)|\Phi(B))\\ &\geq \Phi(S(A|B))\end{aligned}$$

where $S(X|Y)$ *and* $T(X|Y)$ *are defined in* (1.8) *and* (1.9) *and* $S(1)$ *is defined in* (1.1).

We remark that the first inequality of (i) in Theorem 2.4 is the reverse inequality of the second one and also the first inequality of (ii) is the reverse inequality of the second one, and the first inequality of (iii) in Theorem 2.4 is the reverse inequality of the second one in [Theorem 7, 7].

Theorem 2.5. *Let A and B be strictly positive operators on a Hilbert space H such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $m = \frac{m_1}{M_2}$, $M = \frac{M_1}{m_2}$ and $h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1$. Then the following inequalities hold:*

(i)
$$\begin{aligned}&[\log S(1)](A * I) + A * \log B + (A * I)\log(A * I)\\ &\geq (A\log A) * I + (A * I)\log(B * I)\end{aligned}$$

$$\geq A * \log B + (A * I)\log(A * I)$$

(ii) $$\frac{mh\log h}{h-1}(S(1)-1) + A * \log B + (A * I)\log(A * I)$$

$$\geq (A\log A) * I + (A * I)\log(B * I)$$

$$\geq A * \log B + (A * I)\log(A * I)$$

(iii) $$[\log S(1)](B * I) + (B * I)\log(B * I) + (\log A) * B$$

$$\geq I * (B\log B) + (\log(A * I))(B * I)$$

$$\geq (\log A) * B + (B * I)\log(B * I)$$

where $S(1)$ *is defined in* (1.1).

We remark that the first inequality of (i) in Theorem 2.5 is the reverse inequality of the second one and also the first inequality of (ii) is the reverse inequality of the second one, and the first inequality of (iii) is the reverse inequality of the second one.

We remark that Therem 2.3 is an extension of Theorem 2.2. In fact Theorem 2.3 when $A = B = I$ becomes Tgheorem 2.2. Also Therem 2.4 is an extension of Theorem 2.1. In fact Theorem 2.4 when $A = I$ becomes Theorem 2.1

§3 Parallel results to §2 and related remarks

We state an extension of Kantorovich inequality.

Theorem 3.1. *Let A be strictly positive operator satisfying $MI \geq A \geq mI > 0$, where $M > m > 0$. Put $h = \frac{M}{m} > 1$. Then the following inequalities* (i), (ii) *and* (iii) *hold for every unit vector x and follow from each other:*

(i) $$K(h,p)(Ax,x)^p \geq (A^p x,x) \geq (Ax,x)^p \quad \text{for any } p > 1.$$

(ii) $$(Ax,x)^p \geq (A^p x,x) \geq K(h,p)(Ax,x)^p \quad \text{for any } 1 > p > 0.$$

(iii) $$K(h,p)(Ax,x)^p \geq (A^p x,x) \geq (Ax,x)^p \quad \text{for any } p < 0.$$

We remark that the latter half inequality in (i) or (iii) of Theorem 3.1 and the former half one of (ii) are called *Hölder-McCarthy inequality* and the former one of (i) or (iii) and the latter half one of (ii) can be considered as *generalized Kantorovich inequality* and the reverse inequalities to *Hölder-McCarthy inequality.* (i) and (iii) are in [11] and the equivalence relation among (i),(ii) and (iii) is shown in [Theorem 3, 14] and several extensions of Theorem 3.1 are shown, for example,[Theorem 3.2, 17].

Related results to Theorem 3.1 and operator inequalities associated with Kantorovich type inequaloities are in Chapter III of [12].

In this section we sum up the following results which are obtained as applications of Basic Property and they are parallel results to §1 and §2 .

Theorem 3.2 [13]. *Let A be strictly positive operator satisfying $MI \geq A \geq mI > 0$, where $M > m > 0$. Put $h = \frac{M}{m} > 1$. Then the following inequalities hold for every unit vector x:*

(i)
$$\begin{aligned}&[\log S(1)](Ax,x) + (Ax,x)\log(Ax,x)\\ &\geq ((A\log A)x,x)\\ &\geq (Ax,x)\log(Ax,x).\end{aligned}$$

(ii)
$$\begin{aligned}&\frac{mh\log h}{h-1}(S(1)-1) + (Ax,x)\log(Ax,x)\\ &\geq ((A\log A)x,x)\\ &\geq (Ax,x)\log(Ax,x).\end{aligned}$$

(iii)
$$[\log S(1)] + ((\log A)x,x) \geq \log(Ax,x) \geq ((\log A)x,x).$$

Theorem 3.3 [15]. *Let A_j be strictly positive operator satisfying $MI \geq A_j \geq mI > 0$ for $j = 1,2,...,n$, where $M > m > 0$ and $h = \frac{M}{m} > 1$. Also λ_1, λ_2,...,λ_n be any positive numbers such that $\sum_{j=1}^n \lambda_j = 1$. Then the following inequalities hold:*

(i)
$$\begin{aligned}&[\log S(1)]\sum_{j=1}^n \lambda_j A_j + \Big(\sum_{j=1}^n \lambda_j A_j\Big)\log\Big(\sum_{j=1}^n \lambda_j A_j\Big)\\ &\geq \sum_{j=1}^n \lambda_j A_j \log A_j\\ &\geq \Big(\sum_{j=1}^n \lambda_j A_j\Big)\log\Big(\sum_{j=1}^n \lambda_j A_j\Big)\end{aligned}$$

(ii)
$$\begin{aligned}&\frac{mh\log h}{h-1}(S(1)-1) + \Big(\sum_{j=1}^n \lambda_j A_j\Big)\log\Big(\sum_{j=1}^n \lambda_j A_j\Big)\\ &\geq \sum_{j=1}^n \lambda_j A_j \log A_j\\ &\geq \Big(\sum_{j=1}^n \lambda_j A_j\Big)\log\Big(\sum_{j=1}^n \lambda_j A_j\Big).\end{aligned}$$

$$\text{(iii)} \qquad [\log S(1)] + \sum_{j=1}^{k} \lambda_j \log A_j \geq \log(\sum_{j=1}^{k} \lambda_j A_j) \geq \sum_{j=1}^{k} \lambda_j \log A_j.$$

We remark (iii) for $n = 2$ of Theorem 3.3 is shown in [9].

The following interesting result is shown in [6].

Theorem F. *Let A be strictly positive operator satisfying $MI \geq A \geq mI > 0$. Also let $h = \frac{M}{m} > 1$. Then the following inequality holds for every unit vector x:*

$$S(1)\Delta_x(A) \geq (Ax, x) \geq \Delta_x(A).$$

where $\Delta_x(A)$ for strictly positive operator A at a unit vector x is defined by $\Delta_x(A) = \exp\langle((\log A)x, x)\rangle$.

$\Delta_x(A)$ is defined in [8]. We remark that (ii) of Theorem 3.1 implies Theorem F via Basic Property. In fact (ii) of Theorem 3.1 ensures

$$(5.1) \qquad (Ax, x) \geq (A^p x, x)^{\frac{1}{p}} \geq K(h, p)^{\frac{1}{p}}(Ax, x) \quad \textit{for any } 1 > p > 0.$$

and is easily verified that $\lim_{p\to 0}(A^p x, x)^{\frac{1}{p}} = \Delta_x(A)$ and $\lim_{p\to 0} K(h, p)^{\frac{1}{p}} = \dfrac{1}{S(1)}$ by (1.5), so that (5.1) implies Theorem F.

Interesting closely related results to Theorem 3.2 and Theorem 3.3 are in [24].

This paper is based on my talk at "Structure of operators and related recent topics" which has been held at RIMS on January 23, 2003 and some results in this paper will appear elsewhere.

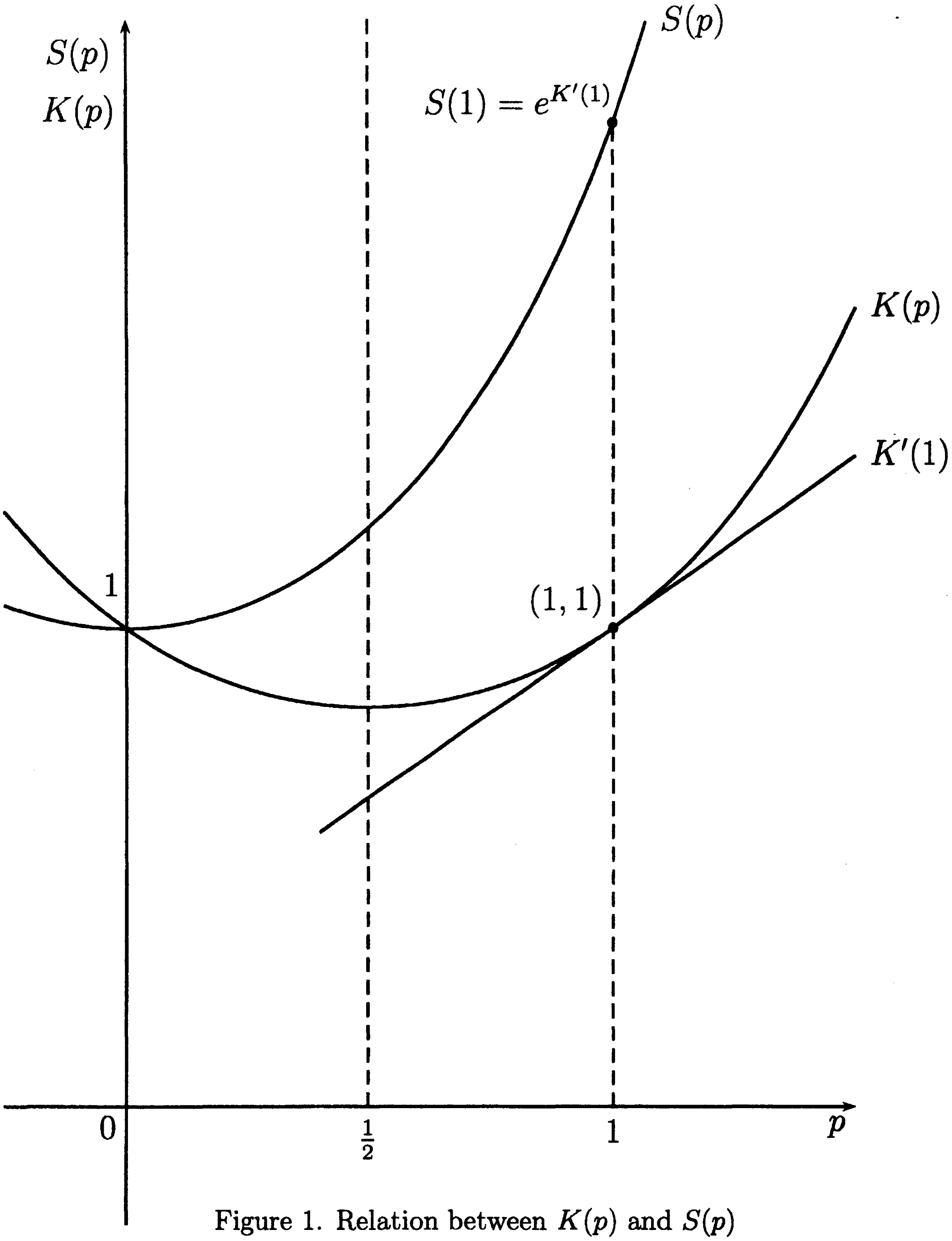

Figure 1. Relation between $K(p)$ and $S(p)$

References

[1] T.Ando, *Concavity of certain maps on positive definite matrices and applications to Hadamard products*, Linear Alg. and Appl., **26**(1979),203-241.

[2] J.S.Aujla and H.L.Vasudeva, *Inequalities involving Hadamard product and operator means*, Math. Japon., **42**(1995),265-272.

[3] M.D.Choi, *A Schwarz inequality for positive linear maps on C^*-algebras*, Illinois J. Math, **18**(1974),565-574.

[4] C.Davis, *A Schwarz inequality for convex operator functions*, Proc. Amer. Math. Soc., **8**(1957),42-44.

[5] J.I.Fujii, *The Marcus-Khan theorem for Hilbert space operators*, Mathematica Japonica, 41(1995),531-535.

[6] J.I.Fujii, S.Izumino and Y.Seo, *Determinant for positive operators and Specht's theorem*, Scientiae Mathematicae, **1**(1998), 307-310.

[7] J.I.Fujii and E.Kamei, *Relative operator entropy in noncommutative information theory*, Math. Japon, **34**(1989), 341-348.

[8] J.I.Fujii and Y.Seo, *Determinant for positive operators* , Scientiae Mathematicae, **1**(1998), 153-156.

[9] M.Fujii,Y.Seo and M.Tominaga, *Golden-Thompson type inequalities related to a geometric mean via Specht's ratio*, Math. Inequal. and Appl.**5**(2002),573-582.

[10] T.Furuta, *Hadamard product of positive operators*, circulated note in 1995.

[11] T.Furuta, *Operator inequalities associated with Hölder-McCarthy and Kantorovich inequalities,* J. Inequal. and Appl.,**2** (1998),137-148.

[12] T.Furuta, *Invitation to Linear Operators*, 2001, Taylor & Francis, London.

[13] T.Furuta, *Specht ratio $S(1)$ can be expressed by Kantorovich constant $K(p)$: $S(1) = \exp\langle[\frac{dK(p)}{dp}]_{p=1}\rangle$ and its application,* to appear in Math. Inequal. and Appl.

[14] T.Furuta, *Basic Property of generalizd Kantorovich constant $K(h,p) = \frac{(h^p-h)}{(p-1)(h-1)}(\frac{(p-1)}{p}\frac{h^p-1}{(h^p-h)})^p$ and its applications,* preprint.

[15] T.Furuta and J.Pečarić, *An operator inequality associated with the operator concavity of operator entropy $A \log A^{-1}$*, to appear in Math. Inequal. and Appl.

[16] F.Kubo and T.Ando, *Means of positive linear operators*, Math. Ann., **246**(1980), 205-224.

[17] Chi-Kwong Li and R.Mathias, *Matrix inequalities involving positive linear map*, Linear and Multilinear Alg., **41**(1996),221-231.

[18] J.Mićić, J.Pečarić and Y.Seo, *Complementary Inequalities to inequalities of Jensen and Ando based on the Mond-Pečarić method,* Linear Alg. and Appl., **318**(2000),87-107.

[19] J.Mićić, J.Pečarić, Y.Seo and M.Tominaga, *Inequalitiesfor positive linear maps on hermitian matrices*, Math. Inequal. and Appl, **4**(2000),559-591.

[20] J.Mićić, Y.Seo, S.Takahasi and M.Tominaga, *Inequalitiesfor of Furuta and Mond-Pečarić* , Math. Inequal. and Appl, **2**(1999),83-111.

[21] B.Mond and J.Pečarić, *Convex inequalities in Hilbert space,* Houston J.Math., **19**(1993),405-420.

[22] B.Mond and J.Pečarić, *A matrix version of Ky Fan Generalization of the Kantorovich inequality,* Linear and Multilinear Algebra, **36**(1994),217-221.

[23] B.Mond and J.Pečarić, Bound for Jensen's inequality for several operators, Houston J. Math., **20**(1994), 645-651.

[24] J.Pečarić and J.Mićić, *Chaotic order among means of positive operators,* Scienticae Mathematicae Online, **7**(2002),97-106.

[25] Y.Seo, S. Takahasi, J.Pečarić and J.Mićić, *Inequalities of Furuta and Mond-Pečarić on the Hadamard product,* J.Inequal. and Appl., **5**(2000),263-285.

[26] T.Yamazaki and M.Yanagida, *Characterization of chaotic order associated with Kantorovich inequality,* Scientiae Mathematicae, **2**(1999),37-50.

Department of Mathematical Information Science,
Faculty of Science,
Tokyo University of Science,
1-3 Kagurazaka, Shinjukuku,
Tokyo 162-8601, Japan
* e-mail: furuta@rs.kagu.tus.ac.jp