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Some Operator Functions Implying Order Preserving Inequalities

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This paper is a resume based on my talk at "Structure of operators and related recent topics" which has been held at RIMS on January 24, 2003 and also this is early announcement of [9].

As an application of our previous result [Theorem 1, 11], we show a simple proof of the following result:

If $A\geq B\geq C\geq 0$ with $A>0$ and $B>0$, then for each $t\in[0,1]$, and $p\geq t$, the following (i) *and* (ii) *hold for a fixed real number q and they are mutually equivalent*:

(i) *if $q\geq 0$, then*

$$G_{p,q,t}(A,B,C,r,s)=A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^{p}B^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing function for $r\geq t$ and $s\geq 1$ such that $(p-t)s\geq q-t$.

(ii) *if $p\geq q$, then*

$$G_{p,q,t}(A,B,C,r,s)=A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^{p}B^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing function for $s\geq 1$ and $r\geq\max\{t,t-q\}$.

This result is further extension of our previous paper [Theorem 2, 11]. On the other hand, M.Uchiyama [17] shows the following interesting result

(iii) *If $A\geq B\geq C\geq 0$ with $B>0$, then for each $t\in[0,1]$ and $p\geq 1$,*

$$A^{1-t+r}\geq\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^{p}B^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}\quad \textit{holds for } r\geq t \textit{ and } s\geq 1.$$

We show that (i) is equivalent to (iii), that is, follows from each other and also as an application of our previous result [Theorem 1, 11], we give a simple proof of M.Uchiyama's result [Theorem 3.4, 17].

1 Introduction.

A capital letter means a bounded linear operator on a Hilbert space.

Theorem L-H (Löwner-Heinz inequality) [13][15].

$$A \ge B \ge 0 \text{ ensures } A^{\alpha} \ge B^{\alpha} \text{ for all } \alpha \in [0,1].$$

Theorem L-H is very useful, but the condition " $\alpha \in [0,1]$ " is too restrictive to calculate operator inequalities, the following result has been obtained from this point of view.

Theorem F (Furuta inequality) [4].

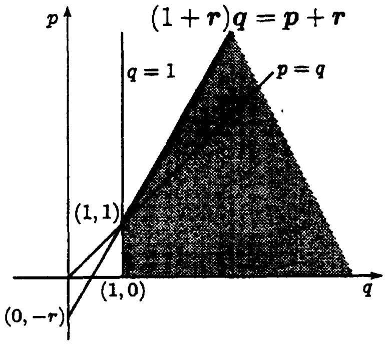
If $A \ge B \ge 0$, then for each $r \ge 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \ge (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \ge (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \ge 0$ and $q \ge 1$ with $(1+r)q \ge p+r$.



FIGURE

Alternative proofs are in [14][1] and one page proof in [5]. It is proved in [16] that The domain drawn for p,q and r in Figure is the best possible one for Theorem F. The following Theorem G is an extension of Theorem F.

Theorem G [6][2]. *If $A \ge B \ge 0$ with $A > 0$, then for $t \in [0,1]$ and $p \ge 1$*

$$A^{1-t+r} \ge \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \quad \text{holds for } s \ge 1 \text{ and } r \ge t.$$

Very recently M.Uchiyama shows the following interesting extension of Theorem G.

Theorem U [17]. *If $A \ge B \ge C \ge 0$ with $B > 0$, then for $t \in [0,1]$ and $p \ge 1$*

$$A^{1-t+r} \ge \{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \quad \text{holds for } s \ge 1 \text{ and } r \ge t.$$

We show that Theorem U is equivalent to (i) of Theorem 1 under below, that is, follows from each other and also as an application of our previous result [Theorem 1, 11], we give a simple proof of M.Uchiyama's result [Theorem 3.4, 17].

Operator Functions Implying Theorem U.

Theorem 1. *If $A \geq B \geq C \geq 0$ with $A > 0$ and $B > 0$, then for each $t \in [0,1]$, and $p \geq t$, the following* (i) *and* (ii) *hold for a fixed real number q and they are mutually equivalent:*

(i) *if $q \geq 0$, then*

$$G_{p,q,t}(A,B,C,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^{p}B^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing function for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q-t$.

(ii) *if $p \geq q$, then*

$$G_{p,q,t}(A,B,C,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^{p}B^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing function for $s \geq 1$ and $r \geq \max\{t, t-q\}$.

We need the following results to prove Theorem 1.

Theorem A [11]. *Let A and B be positive invertible operators on a Hilbert space satisfying*

$$A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \text{ for fixed } \alpha_0 \geq 0 \text{ and } \beta_0 \geq 0 \text{ with } \alpha_0+\beta_0 > 0.$$

Then the following (i) *and* (ii) *hold and they are mutually equivalent:*

(i) *for any fixed $\delta \geq -\beta_0$,*

$$f(\lambda,\mu) = A^{\frac{-\mu}{2}}(A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}}A^{\frac{-\mu}{2}}$$

is decreasing function for $\mu \geq 1$ and $\lambda \geq 1$ such that $\alpha_0\lambda \geq \delta$.

(ii) *for any fixed $\delta \leq \alpha_0$,*

$$f(\lambda,\mu) = A^{\frac{-\mu}{2}}(A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}}A^{\frac{-\mu}{2}}$$

is decreasing function for $\lambda \geq 1$ and $\mu \geq 1$ such that $\beta_0\mu \geq -\delta$.

Lemma B [6]. *Let X be a positive invertible operator and Y be an invertible operator. For any real number λ,*

$$(YXY^*)^{\lambda} = YX^{\frac{1}{2}}(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}})^{\lambda-1}X^{\frac{1}{2}}Y^*.$$

3 Equivalence Relation Associated with Theorem 1.

We show the following equivalence relation between Theorem 1 and related operator inequalities.

Theorem 2. *The following* (i),(ii),(iii) *and* (iv) *hold and follow from each other.*

(i) *If* $A \geq B \geq C \geq 0$ *with* $A > 0$ *and* $B > 0$, *then for each* $t \in [0,1]$ *and* $p \geq 1$,

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \quad \textit{holds for } r \geq t \textit{ and } s \geq 1.$$

(ii) *If* $A \geq B \geq C \geq 0$ *with* $A > 0$ *and* $B > 0$, *then for each* $1 \geq q \geq t \geq 0$ *and* $p \geq q$,

$$A^{q-t+r} \geq \{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} \quad \textit{holds for } r \geq t \textit{ and } s \geq 1.$$

(iii) *If* $A \geq B \geq C \geq 0$ *with* $A > 0$ *and* $B > 0$, *then for each* $t \in [0,1]$ *and* $p \geq 1$,

$$F_{p,t}(A,B,C,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing function for $r \geq t$ *and* $s \geq 1$.

(iv) *If* $A \geq B \geq C \geq 0$ *with* $A > 0$ *and* $B > 0$, *then for each* $t \in [0,1]$, $q \geq 0$ *and* $p \geq t$,

$$G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing function for $r \geq t$ *and* $s \geq 1$ *such that* $(p-t)s \geq q-t$.

We remark that Theorem 2 is an extension of [10], a proof of (i) of Theorem 2 is in [Proposition 4.1, 17], one page proof of (i) by using Theorem G itself is in [8], and also mean theoretic proof of (i) is in [3].

4 Satellite Inequalities.

As simple applications of Theorem 1 and Theorem 2, we show the following satellite inequalities.

Theorem 3. *If $A \geq B \geq C > 0$, then the following inequalities* (i) *and* (ii) *hold for each $t \in [0,1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$:*

(i)
$$B^{\frac{t}{2}}C^{\frac{-r}{2}}\{C^{\frac{r}{2}}(B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^sC^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}C^{\frac{-r}{2}}B^{\frac{t}{2}}$$
$$\geq B^{\frac{t}{2}}C^{\frac{-t}{2}}\{C^{\frac{t}{2}}(B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^sC^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}C^{\frac{-t}{2}}B^{\frac{t}{2}}$$
$$\geq A \geq B \geq C$$
$$\geq B^{\frac{t}{2}}A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}A^{\frac{-t}{2}}B^{\frac{t}{2}}$$
$$\geq B^{\frac{t}{2}}A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}A^{\frac{-r}{2}}B^{\frac{t}{2}}.$$

(ii)
$$B^{\frac{t}{2}}C^{\frac{-r}{2}}\{C^{\frac{r}{2}}(B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^sC^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}C^{\frac{-r}{2}}B^{\frac{t}{2}}$$
$$\geq B^{\frac{t}{2}}C^{\frac{-r}{2}}(C^{\frac{r}{2}}B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}}C^{\frac{r}{2}})^{\frac{1+r-t}{p+r-t}}C^{\frac{-r}{2}}B^{\frac{t}{2}}$$
$$\geq A \geq B \geq C$$
$$\geq B^{\frac{t}{2}}A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}}A^{\frac{r}{2}})^{\frac{1+r-t}{p+r-t}}A^{\frac{-r}{2}}B^{\frac{t}{2}}$$
$$\geq B^{\frac{t}{2}}A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}A^{\frac{-r}{2}}B^{\frac{t}{2}}.$$

Corollary 4. *If $A \geq B > 0$, then the following inequalities* (i) *and* (ii) *hold for each $t \in [0,1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$:*

(i)
$$B^{\frac{-(r-t)}{2}}\{B^{\frac{r}{2}}(B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^sB^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}B^{\frac{-(r-t)}{2}}$$
$$\geq \{B^{\frac{t}{2}}(B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^sB^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+r}}$$
$$\geq A \geq B$$
$$\geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+r}}$$
$$\geq A^{\frac{-(r-t)}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}A^{\frac{-(r-t)}{2}}$$

(ii)
$$B^{\frac{-(r-t)}{2}}\{B^{\frac{r}{2}}(B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^sB^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}B^{\frac{-(r-t)}{2}}$$
$$\geq B^{\frac{-(r-t)}{2}}(B^{\frac{r-t}{2}}A^pB^{\frac{r-t}{2}})^{\frac{1+r-t}{p+r-t}}B^{\frac{-(r-t)}{2}}$$
$$\geq A \geq B$$
$$\geq A^{\frac{-(r-t)}{2}}(A^{\frac{r-t}{2}}B^pA^{\frac{r-t}{2}})^{\frac{1+r-t}{p+r-t}}A^{\frac{-(r-t)}{2}}$$
$$\geq A^{\frac{-(r-t)}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}A^{\frac{-(r-t)}{2}}.$$

Corollary 5. *If $A \geq B > 0$, then the following inequality holds for $p \geq 1$ and $r \geq 0$*

$$B^{\frac{-r}{2}}(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1+r}{p+r}}B^{\frac{-r}{2}} \geq A \geq B \geq A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}A^{\frac{-r}{2}}.$$

5 M.Uchiyama's Result via Theorem A.

The following result is contained in Theorem 3.4 of [17].

Theorem V. *Let A and B be both positive invertible operators. Also let a, b, and c be positive real numbers and d a real number. Define $F(r,s)$ and $G(r,s)$ by*

$$F(r,s) = A^{\frac{r}{2}}(A^{\frac{-r}{2}}B^sA^{\frac{-r}{2}})^{\frac{r}{r+sc}}A^{\frac{r}{2}} \quad \text{for } r > 0,\ s > 0$$

and

$$G(r,s) = A^{\frac{r}{2}}(A^{\frac{-r}{2}}B^sA^{\frac{-r}{2}})^{\frac{r+d}{r+sc}}A^{\frac{r}{2}} \quad \text{for } r > 0,\ s > 0 \text{ with } 0 \leq \tfrac{r+d}{r+sc} \leq 1.$$

Let $a > 0$, $b > 0$ and $-a \leq d \leq bc$. Then for $r_2 \geq r_1 \geq a$ and $s_2 \geq s_1 \geq b$ the following hold:

(a) *if $F(a,b) \leq 1$, then $G(r_2,s_2) \leq G(r_1,s_1)$*

(b) *if $F(a,b) \geq 1$, then $G(r_2,s_2) \geq G(r_1,s_1)$.*

On the other hand, in Theorem A replacing A by A^{β_0} and B by B^{α_0}, then we have the following result in [12].

Corollary C. *Let A and B be positive invertible operators on a Hilbert space satisfying*

$$A^{\beta_0} \geq (A^{\frac{\beta_0}{2}}B^{\alpha_0}A^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \text{ for fixed } \alpha_0 > 0 \text{ and } \beta_0 > 0\ .$$

Then for any fixed $\delta \geq -\beta_0$,

$$f(\alpha,\beta) = A^{\frac{-\beta}{2}}(A^{\frac{\beta}{2}}B^{\alpha}A^{\frac{\beta}{2}})^{\frac{\delta+\beta}{\alpha+\beta}}A^{\frac{-\beta}{2}}$$

is decreasing function of α and β such that $\alpha \geq \max\{\delta, \alpha_0\}$ and $\beta \geq \beta_0$.

We can give a proof of Theorem V via Corollary C.

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