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Elementary operator のスペクトルの解析とその作用素方程式への応用

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Abstract

In this talk, we study the structure of the approximate spectra of analytic elementary operators and characterize the solvability of several types of operator equation. Moreover, we prove the spectral mapping theorems for the approximate spectra of bounded linear operators on Banach spaces.

It has been a problem of essential importance to study the spectral properties of elementary operators.

Let $\mathfrak{X}$ be a complex Banach space and $\mathcal{L}(\mathfrak{X})$ the Banach algebra of all bounded linear operators on $\mathfrak{X}$. We denote the spectrum of an operator $T \in \mathcal{L}(\mathfrak{X})$ by $\sigma(T)$, that is, the set of all complex numbers λ such that $\lambda I - T$ fails to be invertible, where I stands for the identity operator on $\mathfrak{X}$.

An elementary operator on $\mathcal{L}(\mathfrak{X})$ is defined by

$$\Phi_{\mathbf{A},\mathbf{B}}(X) := \sum_{j=1}^{n} A_j X B_j \quad (X \in \mathcal{L}(\mathfrak{X})),$$

where $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = (B_1, \ldots, B_n)$ are both n-tuples of mutually commuting operators in $\mathcal{L}(\mathfrak{X})$. $\Phi_{\mathbf{A},\mathbf{B}}$ is a bounded linear operator on $\mathcal{L}(\mathfrak{X})$ (i.e., $\Phi_{\mathbf{A},\mathbf{B}} \in \mathcal{L}(\mathcal{L}(\mathfrak{X}))$) and this operation was first introduced in order to solve the following type of operator equation:

$$A_1 X B_1 + A_2 X B_2 + \cdots + A_n X B_n = Y. \tag{0.1}$$

Here, Y is any fixed operator in $\mathcal{L}(\mathfrak{X})$ and the problem is exactly to find solutions $X \in \mathcal{L}(\mathfrak{X})$ to (0.1). However, since $\mathcal{L}(\mathfrak{X})$ is a non-commutative algebra whenever $\dim \mathfrak{X} \geq 2$, this problem in its full generality is far from tractable even though $\mathfrak{X}$ is finite dimensional. Therefore it has been a matter of significance to characterize the solvability of the equation (0.1). The most desirable case must be the case where (0.1) has a unique solution X for each given Y, and this statement is equivalent to the statement $0 \notin \sigma(\Phi_{\mathbf{A},\mathbf{B}})$ (i.e., $\Phi_{\mathbf{A},\mathbf{B}}$ is an invertible operator on $\mathcal{L}(\mathfrak{X})$). For such reasons, the solvability problem of the equation (0.1) comes back to the analysis of the spectrum of the corresponding elementary operator.

Incidentally, the most important operator equation in terms of application is of the form $AX - XB = Y$ and hence the corresponding elementary operator $\delta_{A,B}(X) := AX - XB$ has been much studied by many mathematicians. In particular, the next theorem by Rosenblum and Kleinecke is famous.

Theorem 1 ([12, Corollary 3.3], [11, Theorem 10]).

$$\sigma(\delta_{A,B}) = \{\, \alpha - \beta \mid \alpha \in \sigma(A),\ \beta \in \sigma(B) \,\}.$$

This Rosenblum-Kleinecke theorem tells us the fact that the operator equation $AX - XB = Y$ has a unique solution X for each given Y if and only if $\sigma(A) \cap \sigma(B) = \emptyset$. This simple characterization of the solvability of $AX - XB = Y$ is useful in connection with many topics in operator theory, including the similarity problem of 2×2 operator matrices, the commutativity properties of operators, and so forth. See [1].

In 1959, Lumer and Rosenblum [11] succeeded in extending Theorem 1 to the case of analytic elementary operators.

For every $T \in \mathcal{L}(\mathfrak{X})$, $\mathcal{A}(\sigma(T))$ stands for the algebra of all complex-valued functions f analytic on $\sigma(T)$, and $f(T)$ means the standard analytic functional calculation of T by f.

An elementary operator $\Phi_{\mathbf{A},\mathbf{B}}$ is said to be analytic if there exist operators $A, B \in \mathcal{L}(\mathfrak{X})$ and $\mathbf{A} = (A_1, \ldots, A_n)$ (resp. $\mathbf{B} = (B_1, \ldots, B_n)$) is generated by A (resp. B) in the following sense:

$$A_j = f_j(A) \text{ for some } f_j \in \mathcal{A}(\sigma(A))$$

and

$$B_j = g_j(B) \text{ for some } g_j \in \mathcal{A}(\sigma(B))$$

for $j = 1, \dots, n$. By the definition, this operation is of the form

$$\Psi(X) = \sum_{j=1}^{n} f_j(A) X g_j(B) \quad (X \in \mathcal{L}(\mathfrak{X}))$$

and in this case, A and B are said to be the generating operators of Ψ. Lumer and Rosenblum completely determined the spectrum of Ψ in terms of the spectra of the generating operators A and B.

Theorem 2 ([11, Theorem 10]).

$$\sigma(\Psi) = \left\{ \sum_{j=1}^{n} f_j(\alpha) g_j(\beta) \;\middle|\; \alpha \in \sigma(A), \beta \in \sigma(B) \right\}. \tag{0.2}$$

Theorem 2 is a considerable extension of Theorem 1 and the formula (0.2) claims that the following analytic type operator equation

$$f_1(A) X g_1(B) + f_2(A) X g_2(B) + \cdots + f_n(A) X g_n(B) = Y \tag{0.3}$$

has a unique solution X for each given Y if and only if the complex-valued function H of two variables of the form $H(z, w) = f_1(z) g_1(w) + \cdots + f_n(z) g_n(w)$ has no zeros on the Cartesian product $\sigma(A) \times \sigma(B)$.

In this article, we analyze the structure of certain parts of the spectrum of an analytic elementary operator Ψ, in order to apply those structures to the solvability problem of the operator equation (0.3).

For every $T \in \mathcal{L}(\mathfrak{X})$,

$$\sigma_{\mathrm{ap}}(T) := \{ \lambda \in \mathbb{C} \mid \lambda I - T \text{ is not bounded below} \}$$

is called the approximate point spectrum of T. (Here, $S \in \mathcal{L}(\mathfrak{X})$ is said to be bounded below if there exists a constant $c > 0$ such that $\|Sx\| \geq c\|x\|$ for all $x \in \mathfrak{X}$.) On the other hand,

$$\sigma_{\mathrm{ad}}(T) := \{ \lambda \in \mathbb{C} \mid \lambda I - T \text{ is not surjective} \}$$

is called the approximate defect spectrum of T. σ_{ap} and σ_{ad} are referred to as the approximate spectra of operators. The main topic of this article is the analysis of the structure of these spectra of analytic elementary operators.

Our first result is the following.

Theorem 3 ([9, Theorem 1]).

$$\sigma_{\mathrm{ap}}(\Psi) \supseteq \left\{ \sum_{j=1}^{n} f_j(\alpha) g_j(\beta) \;\middle|\; \alpha \in \sigma_{\mathrm{ap}}(A),\ \beta \in \sigma_{\mathrm{ad}}(B) \right\}$$

and

$$\sigma_{\mathrm{ad}}(\Psi) \supseteq \left\{ \sum_{j=1}^{n} f_j(\alpha) g_j(\beta) \;\middle|\; \alpha \in \sigma_{\mathrm{ad}}(A),\ \beta \in \sigma_{\mathrm{ap}}(B) \right\}.$$

An operator $T \in \mathcal{L}(\mathfrak{X})$ is said to satisfy the condition $(\boldsymbol{\alpha})$ if $\sigma(T) = \sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{ad}}(T)$ holds. As a direct consequence of Theorem 3 and the Lumer-Rosenblum theorem (Theorem 2), we can obtain the following characterizations for the solvability of the operator equation (0.3).

Theorem 4 (([10, Theorem 3.2 and Corollary 3.3])). Suppose that $A, B \in \mathcal{L}(\mathfrak{X})$ both satisfy the condition $(\boldsymbol{\alpha})$. Then the following three statements on the operator equation (0.3) are mutually equivalent:

(i) There exists a unique solution X to (0.3) for each Y;

(ii) There exists at least one solution X to (0.3) for each Y;

(iii) There exists a constant $c > 0$ such that, if X_1 (resp. X_2) is a solution to (0.3) for Y_1 (resp. Y_2) then $\|Y_1 - Y_2\| \geq c\|X_1 - X_2\|$.

Moreover, in [10], we succeeded in showing the following spectral mapping theorems for the approximate spectra of bounded linear operators on Banach spaces, by means of the semi-continuity properties of those spectra and Runge's theorem.

Theorem 5 (([10, Theorem 1.2])). For any $T \in \mathcal{L}(\mathfrak{X})$ and any $f \in \mathcal{A}(\sigma(T))$, the following equations hold.

$$\sigma_{\mathrm{ap}}(f(T)) = f(\sigma_{\mathrm{ap}}(T))$$

and

$$\sigma_{\mathrm{ad}}(f(T)) = f(\sigma_{\mathrm{ad}}(T)).$$

As a corollary of Theorem 5, we can obtain the following inclusion relations for the approximate spectra of an analytic elementary operator Ψ. This is a slight progress for getting the condition for that the inclusion relations in Theorem 3 hold with equality.

Corollary 6 (([10, Theorem 3.4])).

$$\sigma_{\text{ap}}(\Psi) \subseteq \left\{ \sum_{j=1}^{n} f_j(\alpha_j) g_j(\beta_j) \;\middle|\; \alpha_j \in \sigma_{\text{ap}}(A),\ \beta_j \in \sigma_{\text{ad}}(B) \right\}.$$

If $\mathfrak{X}$ is a Hilbert space, then

$$\sigma_{\text{ad}}(\Psi) \subseteq \left\{ \sum_{j=1}^{n} f_j(\alpha_j) g_j(\beta_j) \;\middle|\; \alpha_j \in \sigma_{\text{ad}}(A),\ \beta_j \in \sigma_{\text{ap}}(B) \right\}.$$

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