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Numerical Radius Norm for Module Maps
モジュール写像の数域半径ノルム

群馬大学・教育学部　　伊藤　隆 (Takashi ITOH)
Department of Mathematics, Faculty of Education,
Gunma University

In the present note we will explain a couple of results related to numerical radius norm for module maps on C^*-algebras.

(1) One is to extend the Ando-Okubo's theorem concerning Schur multipliers in the infinite dimensional setting.

(2) The other is to characterize a completely bounded module map.

This is joint with Masaru Nagisa (Chiba University).

1. Schur products and Schur multipliers

Let $M_n(\mathbb{C})$ be the $n \times n$ martix algebra over $\mathbb{C}$. For $a = [a_{ij}], b = [b_{ij}] \in M_n(\mathbb{C})$, ***the Schur product*** $\circ$ is defined by

$$a \circ b = [a_{ij} b_{ij}].$$

The Schur multiplier $S_a : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ for $a \in M_n(\mathbb{C})$ is defined by $S_a(x) = a \circ x$. ***The Schur norm*** for $a \in M_n(\mathbb{C})$ is defined by

$$\|a\|_s = \|S_a\| = \sup\{\|a \circ x\| \mid \|x\| = 1\}.$$

The following is due to Haagerup in which the Schur multiplier appears naturally in operator algebras.

Example [Haagerup, 5] Let G be a locally compact group, $C(G)$ the continuous functions and $R(G)$ the group von-Neumann algebra. For $\varphi \in C(G)$, if

$$M_\varphi : R(G) \ni \lambda_g \longmapsto \varphi(g)\lambda_g \in R(G)$$

is normal (σ-weak - σ-weak continuous), then

$$\|M_\varphi\|_{cb} = \sup\{\|[\varphi(g_j^{-1}g_i)]\|_s \mid g_i \in G, i \le n, n \in \mathbb{N}\}.$$

The next result is unpublished.

Theorem A[Haagerup, 5] Let $a = [a_{ij}] \in M_n(\mathbb{C})$. Then the following are equivalent:

1) $\|S_a\| \le 1$.

2) There are $0 \le r_1, r_2 \in M_n(\mathbb{C})$ such that

$$\begin{bmatrix} r_1 & a \\ a^* & r_2 \end{bmatrix} \ge 0, \quad r_1 \circ I \le I \text{ and } r_2 \circ I \le I.$$

3) a has a factorization $a = b^*c$ such that $b^*b \circ I \le I$, $c^*c \circ I \le I$.

4) There are vectors $\{\xi_i\}, \{\eta_i\} \subset \ell_n^2$, $(i = 1, \cdots, n)$ such that $\|\xi_i\|, \|\eta_i\| \le 1$. and $a_{ij} = (\xi_j | \eta_i)$.

We consider ***the numerical radius norm*** $w(\cdot)$ on $\mathbb{B}(\mathcal{H})$:

$$w(a) = \sup_{\xi \ne 0} \frac{|(a\xi \mid \xi)|}{\|\xi\|^2}.$$

It is easy to see that $w(a) \le \|a\| \le 2w(a)$.

We also consider the induced norm for S_a with respect to the numerical radius norm that will be denoted by $\|S_a\|_w$:

$$\|S_a\|_w \equiv \sup_{x \ne 0} \frac{w(a \circ x)}{w(x)}.$$

The following is due to Ando and Okubo which looks similar to Theorem A but each condisiton is finer than the above.

Theorem B[Ando-Okubo, 2] Let $a = [a_{ij}] \in M_n(\mathbb{C})$. Then the following are equivalent.

$1)_w$ $\|S_a\|_w \le 1$.

$2)_w$ There is a $0 \leq r \in M_n(\mathbb{C})$ such that

$$\begin{bmatrix} r & a \\ a^* & r \end{bmatrix} \geq 0, \quad \text{and } r \circ I \leq I.$$

$3)_w$ a has a factorization $a = b^*db$ such that $b^*b \circ I \leq I$, $d^*d \leq I$.

$4)_w$ There are vectors $\{\xi_i\} \subset \ell^2_n, (i = 1, \cdots, n)$ and a contraction $d \in M_n(\mathbb{C})$ such that $\|\xi_i\| \leq 1$ and $a_{ij} = (d\xi_j|\xi_i)$.

Remark The Haagerup's theorem is derived from the Ando-Okubo's theorem, because

$$\|S_a\| = \|S_{\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}}\|_w.$$

To show 1) $\Rightarrow$ 2), $\|S_a\| \leq 1$ implies

$$\|S_{\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}}\|_w \leq 1.$$

By the implication $1)_w \Rightarrow 2)_w$, there exists

$0 \leq r = \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix} \in M_n(\mathbb{C})$ such that

$$\begin{bmatrix} r_{11} & r_{21} & 0 & a \\ r_{12} & r_{22} & 0 & 0 \\ 0 & 0 & r_{11} & r_{21} \\ a^* & 0 & r_{12} & r_{22} \end{bmatrix} \geq 0.$$

This implies that $\begin{bmatrix} r_{11} & a \\ a^* & r_{22} \end{bmatrix} \geq 0.$

2. Module maps on operator systems

Let $\varphi : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$. Then the followings are equivalent:

1) There exists $a \in M_n(\mathbb{C})$ such that $\varphi = S_a$.

2) $\varphi(\lambda x \mu) = \lambda \varphi(x) \mu$ for $x \in M_n(\mathbb{C})$,

$$\lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \text{ and } \mu = \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix} \in M_n(\mathbb{C})$$

(i.e. ℓ_n^∞-module map)

Let A be a C^*-algebra and V an operator system in $\mathbb{B}(\mathcal{H})$, i.e., V is a self-adjoint subspace in $\mathbb{B}(\mathcal{H})$ with the identity. Let T be a bounded linear map from $(V, \|\cdot\|)$ to $(\mathbb{B}(\mathcal{H}), \|\cdot\|)$ We denote by $T \otimes \mathrm{id_n}$ the linear map

$$\mathbf{M}_n(V) \ni [x_{ij}] \longmapsto [T(x_{ij})] \in \mathbf{M}_n(\mathbb{B}(\mathcal{H})).$$

If $\sup_n \|T \otimes \mathrm{id_n}\|$ is bounded, then we say T is completely bounded and denote the supremum by $\|T\|_{cb}$. If $T \otimes \mathrm{id_n}$ is positive for all n, then we say T is completely positive.

We denote by $\|T\|_w$ the operator norm of T viewed as a bounded linear map from $(V, w(\cdot))$ to $(\mathbb{B}(\mathcal{H}), w(\cdot))$, i.e.,

$$\|T\|_w = \sup\{w(T(x)) \mid w(x) \leq 1,\ x \in V\}.$$

Every completely bounded map from an operator space to $\mathbb{B}(\mathcal{H})$ is also completely bounded with respect to numerical radius norm. We use the following notation:

$$\|T\|_{wcb} = \sup_{n \in \mathbb{N}} \|T \otimes \mathrm{id_n}\|_{\mathrm{w}}.$$

We call that an action of A on $\mathcal{H}$ is locally cyclic if, for any n and $\xi_1, \xi_2, \ldots, \xi_n \in \mathcal{H}$, there exists a vector $\eta \in \mathcal{H}$ such that

$$\xi_i \in \text{the norm closure of } \{a\eta \mid a \in A\}.$$

We remark that, for $x = (x_{ij}) \in \mathbf{M}_n(V) \subset \mathbb{B}(\mathcal{H}^n)$ and $a = (a_{kl}) \in M_{nm}(A)$, we can see

$$a^* \cdot x \cdot a = (\sum_{k,l} a_{ki}^* \cdot x_{kl} \cdot a_{lj}) \in \mathbf{M}_m(V) \subset \mathbb{B}(\mathcal{H}^m)$$

and we get

$$w(a^* \cdot x \cdot a) \leq \|a\|^2 w(x).$$

The condition(locally cyclic) implies that $\|\cdot\|_w = \|\cdot\|_{wcb}$.

Proposition 1 *Let A be a unital C^*-algebra, V an A-bimodule operator system and T a bounded A-bimodule map from V to $\mathbb{B}(\mathcal{H})$. If the action of A on $\mathcal{H}$ is locally cyclic, then we have*

$$\|T\|_w = \|T\|_{wcb}.$$

Let A be a unital C*-algebra and V an A-bimodule operator system in $\mathbb{B}(\mathcal{H})$. Set

$$N = \{\begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid x, w \in \mathbb{B}(\mathcal{H}), y, z \in V\}.$$

Then N is an A-bimodule operator system by the action

$$a \cdot \begin{pmatrix} x & y \\ z & w \end{pmatrix} \cdot b = \begin{pmatrix} axb & ayb \\ azb & awb \end{pmatrix} \quad \text{for } a, b \in A.$$

Theorem 2 *Let A be a unital C^*-algebra, V an A-bimodule operator system in $\mathbb{B}(\mathcal{H})$ and T a completely bounded A-bimodule map from V to $\mathbb{B}(\mathcal{H})$. Then we have*

$$\begin{aligned} &\|T\|_{wcb} \\ &= \inf\{\|S\| \mid \begin{pmatrix} S & T \\ T^* & S \end{pmatrix} : N \longrightarrow \mathbf{M}_2(\mathbb{B}(\mathcal{H})) \\ &\qquad\qquad \textit{A-bimodule completely positive}\ \} \\ &= \inf\{\|S\| \mid \begin{pmatrix} S & T \\ T^* & S \end{pmatrix} : N \longrightarrow \mathbf{M}_2(\mathbb{B}(\mathcal{H})) \\ &\qquad\qquad \textit{completely positive}\} \end{aligned}$$

where

$$\begin{pmatrix} S & T \\ T^* & S \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} S(x) & T(y) \\ T(z^*)^* & S(w) \end{pmatrix},$$

for $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in N = \{\begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid x, w \in \mathbb{B}(\mathcal{H}), y, z \in V\}.$

To show this, we need the Wittstock's Hahn Banach type theorem for completely bounded maps. The key operator space N_0 is that

$$N_0 = \left\{ \begin{bmatrix} a+x & y \\ z & a-x \end{bmatrix} \mid a \in A, x \in \mathbb{B}(\mathcal{H}), y, z \in V \right\}$$

and we consider

$$\varphi_0 \left(\begin{bmatrix} a+x & y \\ z & a-x \end{bmatrix} \right) = a + \frac{1}{2}(T(y) + T(z^*)^*).$$

We can see that φ_0 is a completely positive A-bimodule map. Using the Wittstock's Hahn-Banach theorem [13], we get the unital completely positive A-bimodule map φ from N to $\mathbb{B}(\mathcal{H})$ which is an extension of φ_0.

Set

$$S(x) = 2\varphi \left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \right)$$

Then we have a desired map.

Remark V. I. Paulsen and C. Y. Suen [9] introduced the norm $|||T|||$ for a completely bounded map T from a C*-algebra A to $\mathbb{B}(\mathcal{H})$ as follows:

$$|||T||| = \inf\{\|S\| \mid \begin{pmatrix} S & T \\ T^* & S \end{pmatrix} : \mathbf{M}_2(A) \longrightarrow \mathbf{M}_2(\mathbb{B}(\mathcal{H})) \text{ completely positive } \}.$$

By the injectivity of $\mathbb{B}(\mathcal{H})$, we can get

$$|||T||| = \|T\|_{wcb}.$$

We set $\mathcal{A}^{(*)} = \{x^* \in \mathbb{B}(\mathcal{H}) \mid x \in \mathcal{A}\}$.

Theorem 3 *Let $\mathcal{A}$ be a norm closed unital algebra on $\mathcal{H}$ and T a completely bounded left $\mathcal{A}^{(*)}$-, right $\mathcal{A}$-module map from $\mathbb{K}(\mathcal{H})$ to $\mathbb{B}(\mathcal{H})$. Then there exist $t=(t_{ij})\in\mathbb{B}(\ell^2(I))$ and $\{v_i \mid i\in I\}\subset\mathcal{A}'$ such that*

$$\|t\| = \|T\|_{wcb}, \quad \sum_{i\in I} v_i^* v_i \leq 1$$
$$T(x) = \sum_{i,j\in I} v_i^* t_{ij} x v_j \qquad (x\in\mathbb{K}(\mathcal{H})).$$

To see this, we may regard T as a normal completely bounded $\mathcal{A}^{(*)}-\mathcal{A}$-module map on $\mathbb{B}(\mathcal{H})$ Then there exist a $*$-representation π of $\mathbb{K}(\mathcal{H})$ on a Hilbert space $\mathcal{K}$, an isometry $w:\mathcal{H}\longrightarrow\mathcal{K}$ and an operator $s\in\pi(\mathbb{K}(\mathcal{H}))'$ such that

$$\|T\|_{wcb} = \|s\|, \quad T(\cdot) = w^* s\pi(\cdot)w.$$

Since all irreducible representations of $\mathbb{K}(\mathcal{H})$ are unitarily equivalent to the identity representation, we may assume that

$$\mathcal{K} = \mathcal{H}\otimes\ell^2(I),\ \pi(x) = x\otimes 1,\ s = (s_{ij}1_{\mathbb{B}(\mathcal{H})})_{i,j\in I}$$
$$(s_{ij}\in\mathbb{C}), \quad w = (w_i)_{i\in I}\in\mathbb{B}(\mathcal{H},\mathcal{H}\otimes\ell^2(I)),$$
$$T(x) = w^* s(x\otimes 1)w = \sum_{i,j\in I} w_i^* s_{i,j} x w_j \text{ for } x\in\mathbb{K}(\mathcal{H}).$$

We can replace $\{w_i\}$ by $\{v_i\}\subset\mathcal{A}'$.

Corollary 4 [Smith, 11] Let $\mathcal{A}$ and $\mathcal{B}$ be norm closed unital algebras on $\mathcal{H}$ and T a completely bounded left $\mathcal{A}$- right $\mathcal{B}$-module map from $\mathbb{K}(\mathcal{H})$ to $\mathbb{B}(\mathcal{H})$. Then there exist $\{a_i \mid i\in I\}\subset\mathcal{A}'$ and $\{b_i \mid i\in I\}\subset\mathcal{B}'$ such that

$$T(x) = \sum_{i\in I} a_i x b_i, \quad \|\sum_{i\in I} a_i a_i^*\|\|\sum_{i\in I} b_i^* b_i\| = \|T\|_{cb}^2$$

for $x\in\mathbb{K}(\mathcal{H})$.

We define the left action of $\mathcal{A}\oplus\mathcal{B}^{(*)}$ and the right action of $\mathcal{A}^{(*)}\oplus\mathcal{B}$ on $\mathbf{M}_{\not{k}}(\mathbb{K}(\mathcal{H}))$ and the left $\mathcal{A}\oplus\mathcal{B}^{(*)}$-, right $\mathcal{A}^{(*)}\oplus\mathcal{B}$-module completely bounded map $\tilde{T}$ from $\mathbb{M}_2(\mathbb{K}(\mathcal{H}))$ to $\mathbb{M}_2(\mathbb{B}(\mathcal{H}))$ as follows:

$$(a_1 \oplus b_1^*)\begin{pmatrix} x & y \\ z & w \end{pmatrix}(a_2^* \oplus b_2) = \begin{pmatrix} a_1 x a_2^* & a_1 y b_2 \\ b_1^* z a_2^* & b_1^* w b_2 \end{pmatrix}$$

$$\tilde{T}(\begin{pmatrix} x & y \\ z & w \end{pmatrix}) = \begin{pmatrix} 0 & T(y) \\ 0 & 0 \end{pmatrix}$$

where $x, y, z, w \in \mathbb{K}(\mathcal{H})$ and $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$. Then we show that $\|\tilde{T}\|_{wcb} = \|T\|_{cb}$. Apply the previous theorem for $\tilde{T}$. Then we have the desired form for T.

For a Hilbert space $\mathcal{H}$, we choose a completely orthonormal system $\{e_i \mid i \in I\}$. We denote by ℓ^∞ the maximal abelian subalgebra of $\mathbb{B}(\mathcal{H})$ generated by $\{e_i \otimes e_i \mid i \in I\}$, where $(e_i \otimes e_j)(\xi) = (\xi|e_j)e_i$ for $\xi \in \mathcal{H}$. Let T be a bounded ℓ^∞-bimodule map from $\mathbb{K}(\mathcal{H})$ to $\mathbb{B}(\mathcal{H})$. By the module property of T, we have the $I \times I$-matrix $a = (a_{ij})$ over $\mathbb{C}$ such that

$$T(e_i \otimes e_j) = a_{ij}(e_i \otimes e_j).$$

Since the set $\{a_{ij}\}_{i,j \in I}$ is bounded, we can define the bounded linear operator a_T from ℓ^1 to ℓ^∞ given by

$$a_T((\lambda_j)_{j \in I}) = (\sum_{j \in I} a_{ij}\lambda_j)_{i \in I} \quad \text{for } (\lambda_j)_{j \in I} \in \ell^1.$$

We will extend the Ando-Okubo's theorem.

Theorem 4 *Let T be an ℓ^∞-bimodule bounded linear map from $\mathbb{K}(\mathcal{H})$ to $\mathbb{B}(\mathcal{H})$. Then the following are equivalent:*

(1) $\|T\|_w \le 1$.

(2) $\|T\|_{wcb} \le 1$.

(3) *There exists a completely positive contraction S from $\mathbb{K}(\mathcal{H})$ to $\mathbb{B}(\mathcal{H})$ such that $\begin{pmatrix} S & T \\ T^* & S \end{pmatrix} : \mathbb{M}_2(\mathbb{K}(\mathcal{H})) \longrightarrow \mathbb{M}_2(\mathbb{B}(\mathcal{H}))$ is ℓ^∞-bimodule completely positive.*

(4) *There exsit a bounded linear operator v from ℓ^1 to ℓ^2 and $b \in \mathbb{B}(\ell^2)$ such that $a_T = v^* b v$ and $\|v\|^2\|b\| \le 1$.*

(5) *There exist $\{\xi_i \mid i \in I\} \subset \mathcal{H}$ and $b \in \mathbb{B}(\mathcal{H})$ such that*

$$\|\xi_i\| \le 1,\ \|b\| \le 1,\ a_{ij} = (b\xi_j|\xi_i).$$

Note added in proof.

C.-Y. Suen has already shown Theorem 2 in a similar setting in the paper: Induced completely bounded norms and inflated Schur product, Acta Sci. Math.(Szeged) 66, (2000), 273-286.

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