数理解析研究所講究録 1312 巻 2003 年 27-30

DIFFERENCE IN PROJECTIONS

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ABSTRACT. Let P and Q be orthogonal projections. Then it is well known that

$$\|P-Q\|=\max\{\|PQ^{\perp}\|,\|P^{\perp}Q\|\}.$$

For this formula, we give more precise estimations by elementary methods. Among others, an operator inequality

$$-\|P^{\perp}Q\|\leq P-Q\leq\|PQ^{\perp}\|$$

is shown, in which the constants on both sides are optimal except the trivial cases. As a corollary, it is proved that $\|R+S\|=1+\|RS\|$ for orthogonal projections R and S.

1. INTRODUCTION

Throughout this note, an operator means a bounded linear operator acting on a Hilbert space $\mathcal{H}$ and $\sigma(T)$ denotes the spectrum of an operator T

The following result on the opening of two subspaces is well known:

(1) $$\|P-Q\|=\max\{\|P^{\perp}Q\|,\|Q^{\perp}P\|\}$$

for two (orthogonal) projections P and Q (see [1]), where $R^{\perp}=1-R$.

Izumino and Watatani [6] pointed out that (1) is assured by the following two facts:

(i) If A and B are positive operators with $AB=0$, then

$$\|A+B\|=\max\{\|A\|,\|B\|\}.$$

(ii) If P and Q are projections, then

$$(P-Q)^{2}=Q^{\perp}PQ^{\perp}+QP^{\perp}Q.$$

By the way, the formula (1) says that $\|P-Q\|\leq 1$. If $\|P-Q\|<1$ in particular, then P and Q are interchanged by a symmetry U (i.e., $U=U^{*}$ and $U^{2}=1$), which is given as follows (see [5]):

$$U=Q\{1-(P-Q)^{2}\}^{-1/2}P-Q^{\perp}\{1-(P-Q)^{2}\}^{-1/2}P^{\perp}.$$

Furthermore, Izumino and Watatani [6] proved that if P and Q are projections interchanged by a symmetry, then

(2) $$\|P-Q\|=\|PQ^{\perp}\|=\|P^{\perp}Q\|.$$

In this note, we shall give more precise descriptions for the formula (1). Among others, we present an operator inequality

$$-\|P^{\perp}Q\|\leq P-Q\leq\|PQ^{\perp}\|,$$

1991 *Mathematics Subject Classification.* 46A30, 47A66.
Key words and phrases. projection, norm, inequality.

in which the constants on both sides are optimal except the trivial cases. As an application, we have a result due to Duncan and Taylor [3] that $\|P+Q\| = 1+\|PQ\|$ for projections p and Q. In addition, we pose an elementary proof of the formula (2) under the assumption $\|P-Q\| < 1$.

2. Results

First of all, we prove that the following norm equalities hold for two projections.

Lemma 1. *Let P and Q be orthogonal projections on $\mathcal{H}$. Then the following statements hold*:

(i) *If $\|P-Q\| \in \sigma(P-Q)$, then $\|P-Q\| = \|PQ^\perp\|$.*

(ii) *If $-\|P-Q\| \in \sigma(P-Q)$, then $\|P-Q\| = \|P^\perp Q\|$.*

Proof. (i) If $a = \|P-Q\| \in \sigma(P-Q)$, then using the Berberian representation([2]) if necessary, we may assume that

$$(P-Q)x = (Q^\perp - P^\perp)x = ax \tag{3}$$

for some non-zero $x \in \mathcal{H}$. Hence we have $Q^\perp Px = aQ^\perp x$ and $PQ^\perp x = aPx$, and so

$$\|Q^\perp Px\| = a\|Q^\perp x\|, \quad (Q^\perp Px, x) = a\|Q^\perp x\|^2.$$

Also, it is easy to see $Px \neq 0$ because $a > 0$.

Moreover, since

$$(Q^\perp Px, x) = (x, PQ^\perp x) = a\|Px\|^2,$$

we have $\|Q^\perp x\| = \|Px\|$, and so $\|Q^\perp Px\| = a\|Q^\perp x\| = a\|Px\|$. That is, $a \leq \|Q^\perp P\| = \|P-Q\| = a$.

(ii) If $-\|P-Q\| \in \sigma(P-Q)$, then $\|Q-P\| \in \sigma(Q-P)$, so that (ii) is proved by (i). □

Remark. The following result is due to Izumino and Watatani [6]: *If $\|P-Q\| < 1$ for orthogonal projections P and Q on $\mathcal{H}$, then*

$$\|P-Q\| = \|PQ^\perp\| = \|P^\perp Q\|.$$

As an application of Lemma 1, we give it an elementary proof with no use of the existence of a symmetry. As a matter of fact, it is sufficient to consider the case $\|P-Q\| \in \sigma(P-Q)$. Then, Lemma 1 implies that $\|P-Q\| = \|PQ^\perp\|$ holds. On the other hand, it follows from (3) that $-QP^\perp x = \|P-Q\|Qx$, so that

$$\|QP^\perp x\|\|Qx\| \geq |(QP^\perp x, x)| = \|P-Q\|(Qx, x) = \|P-Q\|\|Qx\|^2.$$

Noting $Qx \neq 0$ as $\|P-Q\| < 1$, we have $\|P-Q\| = \|QP^\perp\|$.

Based on our consideration in Lemma 1, we have the following estimation of $P-Q$ itself.

Theorem 2. *Let P and Q be orthogonal projections on $\mathcal{H}$. Then*

$$-\|P^\perp Q\| \leq P-Q \leq \|PQ^\perp\|.$$

Moreover the constants $-\|P^\perp Q\|$ and $\|PQ^\perp\|$ are optimal except the trivial cases (a) *$P=1$ and $Q=0$, and* (b) *$P=0$ and $Q=1$, i.e., if $-c \leq P-Q \leq d$ for $c, d \geq 0$, then $c \geq \|P^\perp Q\|$ and $\|PQ^\perp\| \leq d$.*

Proof. If $P \geq Q$ or $P \leq Q$ holds, then the conclusion is clear.

Put $b = \sup\{\lambda; \lambda \in \sigma(P-Q)\} > 0$. Since we may assume that b is a positive eigenvalue of $P - Q$, we have $b \leq \|PQ^\perp\|$ as in the proof of Theorem 1 because (3) can be assumed for b. On the other hand, since $Q^\perp PQ^\perp = Q^\perp(P-Q)Q^\perp \leq bQ^\perp$, we have $\|PQ^\perp\|^2 \leq b$.

Therefore it suffices to show that $b \geq \|PQ^\perp\|$ under the case $\|PQ^\perp\| < 1$. First of all, if $\|P - Q\| < 1$, then the existence of a symmetry U with Q=UPU implies the symmetry of $\sigma(P - Q)$ with respect to the origin because

$$P - Q = P - UPU = U(UPU - P)U = U(Q - P)U = -U(P - Q)U.$$

Hence $\pm\|P - Q\| \in \sigma(P - Q)$ and so $b = \|P - Q\| \geq \|PQ^\perp\|$.

Next we suppose that $\|P - Q\| = 1$, and put $M = \{x \in H; Px = 0,\ Qx = x\}$. Then M is the eigenspace of -1 for $P - Q$, i.e., $M = \{x \in H; (P - Q)x = -x\}$. As a matter of fact, if $x \in M$, then $(P - Q)x = -x$ easily. Conversely, if $(P - Q)x = -x \neq 0$, then $Px + Q^\perp x = 0$, so that $Q^\perp Px = -Q^\perp x$ and $PQ^\perp x = -Px$. Hence it follows that

$$\|Q^\perp x\| = \|Q^\perp Px\| \leq \|Px\| = \|PQ^\perp x\| \leq \|Q^\perp x\|,$$

and so $\|PQ^\perp x\| = \|Q^\perp x\|$. If $Q^\perp x \neq 0$, then $\|PQ^\perp\| \geq 1$, which contradicts to the assumption $\|PQ^\perp\| < 1$. Namely we have $Q^\perp x = 0 = Px$.

So M is a (nontrivial) reducing subspace of both P and Q. We here put $P_1 = P|_{M^\perp}$ and $Q_1 = Q|_{M^\perp}$. Noting that $PQ^\perp|_M = 0$, we have $\|PQ^\perp\| = \|P_1Q_1^\perp\|(< 1)$, and $b_1 = \sup\{((P_1-Q_1)x, x); x \in M^\perp, \|x\| = 1\} = b$. Since $\|P_1Q_1^\perp\| < 1$, we have $P_1 \wedge Q_1^\perp = 0$, where $\wedge$ means the infimum for projections. Moreover we may assume that $P \wedge Q = P^\perp \wedge Q^\perp = 0$. Namely, P_1 and Q_1 are in generic position in the sense of Halmos [4]. So the structure theorem [4; Theorem 2] says that there exist commuting positive contractions S and C on some Hilbert space such that $S^2 + C^2 = 1$, $\ker S = \ker C = 0$ and that P_1 and Q_1 are unitarily equivalent to

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}.$$

Since it is easily checked that

$$EF^\perp E = \begin{pmatrix} S^2 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad (E - F)^2 = \begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix},$$

we have $\|E - F\| = \|S\| = \|EF^\perp\| < 1$. Applying P_1, Q_1 and b_1 to the first case in this proof, we have

$$b = \|P_1 - Q_1\| = \|P_1Q_1^\perp\| = \|PQ^\perp\|.$$

Finally the other part $-\|P^\perp Q\| \leq P - Q$ is equivalent to the inequality $Q - P \leq \|QP^\perp\|$. So there is nothing to do. □

As a direct consequence, we have the following result appeared in [3]:

Corollary 3. *If R and S are orthogonal projections, then*

$$\|R + S\| = 1 + \|RS\|.$$

Proof. We have

$$0 \leq 1 - \|R^\perp S^\perp\| \leq 1 + R - S^\perp \leq 1 + \|RS\|$$

by Theorem 2. As $R + S = 1 + R - S^\perp$ and the last inequality is optimal by Theorem 2 again, the conclusion $\|R + S\| = 1 + \|RS\|$ is obtained. □

Acknowledgement. The authors would like to express their thanks to Professor Araki and Professor Kosaki for their kind suggestion.

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