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On generalized numerical range of the Aluthge transformation

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Abstract

In this report the authors show that the Aluthge transformation $\tilde{T}$ of a matrix T and a polynomial f satisfy the inclusion relation $W_C(f(\tilde{T})) \subset W_C(f(T))$ for the generalized numerical range if C is a Hermitian matrix or a rank-one matrix.

1. The Aluthge transformation

In the development of operator theory, Aluthge [1] introduced a transformation $\tilde{T}$ for a bounded linear operator T on a complex Hilbert space H with the help of the polar decomposition $T = V|T|$ as follows:

Definition 1 (Aluthge transformation [1]). *Let $T = V|T|$ be the polar decomposition of a bounded linear operator T. Then the Aluthge transformation $\widetilde{T}$ of T is defined by*

$$\tilde{T} = |T|^{\frac{1}{2}} V |T|^{\frac{1}{2}}.$$

We remark that $\widetilde{T}$ is defined by using a partial isometry V and $|T|$ with $T = V|T|$ and $N(V) = N(|T|)$. But in fact, $\widetilde{T}$ does not depend on the choice of V (see [19]), for example, if $T = U|T|$ is a matrix with unitary U, then $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$.

As properties of $\tilde{T}$, the following assertions are well known:

(i) $\sigma(T) = \sigma(\tilde{T})$, where $\sigma(T)$ means the spectrum of an operator T.

(ii) $\|T\| \geq \|\tilde{T}\|$.

(i) has been shown in [9], and we can obtain (ii) easily as follows:

$$\|\tilde{T}\| \leq \||T|^{\frac{1}{2}}\| \cdot \|V\| \cdot \||T|^{\frac{1}{2}}\| \leq \|T\|.$$

Recently, many authors discuss the nth iterated Aluthge transformation which is denoted by $\widetilde{T_n}$, i.e.,

$$\widetilde{T_n} = \widetilde{(\widetilde{T_{n-1}})} \qquad \text{and} \qquad \widetilde{T_0} = T,$$

and the following interesting property is shown in [20].

$$\lim_{n\to\infty} \|\widetilde{T_n}\| = r(T),$$

where $r(T)$ is the spectral radius of T.

2. Numerical range

In this section, we shall introduce the numerical range and a result on that of the Aluthge transformation.

Definition 2 (Numerical range). *For an operator T, the numercal range $W(T)$ of T is the subset of the complex numbers $\mathbb{C}$, given by,*

$$W(T) = \{\langle Tx, x\rangle \ : \ x \in H, \|x\| = 1\}.$$

The following properties of the numerical range are well known.

(i) $W(T)$ is a convex set (Hausdorff-Toeplitz).

(ii) $\sigma(T) \subset \overline{W(T)}$.

As a result on the numerical range of $\tilde{T}$, the following result has been shown.

Theorem 2.A ([18]). *Let T be a bounded linear operator, then the following inclusion relation holds.*

$$\overline{W(\tilde{T})} \subset \overline{W(T)}. \tag{2.1}$$

Theorem 2.A was firstly shown in [10] in case T is a 2×2 matrix (in this case, $W(\tilde{T})$ and $W(T)$ are closed subsets of the complex number $\mathbb{C}$). Then one of the authors [19] proved that (2.1) holds if T admits a decomposition $T = U|T|$ for an isometry operator U. This condition is always satisfied if T is an $n \times n$ matrix, or H is finite dimensional. In [19], the relation (2.1) is shown by using the property of the numerical range

$$\overline{W(\tilde{T})} = \bigcap_{\lambda\in\mathbb{C}} \{z \in \mathbb{C} : |z - \lambda| \leq w(T - \lambda I)\}, \tag{2.2}$$

where $w(T)$ is the *numerical radius* of T, that is,

$$w(T) = \sup\{|z| : z \in W(T)\}$$

and the following characterization of $w(T) \le 1$ by Berger and Stampfli [3]:

$$\text{(2.3)} \qquad w(T) \le 1 \text{ if and only if } \|T - zI\| \le 1 + \sqrt{1+|z|^2} \text{ for all } z \in \mathbb{C}.$$

In a recent paper [18], Wu showed that the inclusion (2.1) holds for every bounded linear operator T on a Hilbert space H. He showed this result by using the previous result shown in [19] and some properties of numerical range and Aluthge transformation, so this proof is not easy. In this report, we shall obtain a simplified proof of Theorem 2.A in Section 4.

3. C-NUMERICAL RANGE

As a generalization of the numerical range, for $n \times n$ matrices C and T, the C-numerical range of T is defined in [7] as follows:

Definition 3 (C-numerical range [7]). *For $n \times n$ matrices C and T, the C-numerical range $W_C(T)$ of T is the compact subset of complex number $\mathbb{C}$, given by,*

$$W_C(T) = \{\mathrm{tr}(CU^*TU) : U \text{ is a unitary matrix}\}.$$

Put $C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$, then $W_C(T) = W(T)$, so we can regard $W_C(T)$ as a generalization of $W(T)$. But $W_C(T)$ is not always convex as follows:

Example ([17]). Let

$$T = C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

Put unitary matrices U_1 and U_2 as follows:

$$U_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then we have

$$CU_1^*TU_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad CU_2^*TU_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix},$$

that is, $1, 2i \in W_C(T)$. But put a unitary matrix U_3 as follows:

$$U_3 = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}.$$

$$\operatorname{tr}(CU_3^*TU_3) = |u_{22}|^2 - |u_{33}|^2 + i(|u_{23}|^2 + |u_{32}|^2).$$

Assume that $\frac{1+2i}{2} \in W_C(T)$. Then the following relations hold:

$$\begin{cases} |u_{22}|^2 - |u_{33}|^2 & = \frac{1}{2}, \\ |u_{23}|^2 + |u_{32}|^2 & = 1. \end{cases}$$

So U_3 can not be unitary, and it is a contradiction. Hence $\frac{1+2i}{2} \notin W_C(T)$.

In fact, it is known that $W_C(T)$ is star-shaped as follows:

Theorem 3.A ([4]). *For any $n \times n$ matrices C and T, the range $W_C(T)$ is star-shaped with star center at $y = \frac{1}{n}\operatorname{tr}(C)\operatorname{tr}(T)$, i.e., if $x \in W_C(T)$, then*

$$\lambda x + (1-\lambda)y \in W_C(T) \qquad \text{for all } \lambda \in [0,1].$$

Especially, when C is a Hermitian matrix or a rank-one matrix, the range $W_C(T)$ is a convex set (cf. [17] and [16]). In these cases, we can rephrase them in the following ways:

The case that C is a Hermitian matrix. We assume that the spectrum of the Hermitian matrix C is the set

$$c = (c_1, c_2, \dots, c_n).$$

Since C is a Hermitian matrix, there is a unitary matrix U such that

$$U^*CU = \begin{pmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n \end{pmatrix}.$$

Hence the set $W_C(T)$ can be rewritten as follows:

$$W_C(T) = \left\{ \sum_{j=1}^{n} c_j \langle Tx_j, x_j \rangle : \{x_1, x_2, \dots, x_n\} \text{ is an orthonormal basis of } \mathbb{C}^n \right\},$$

which is denoted by $W_c(T)$ and we call $W_c(T)$ the c-numerical range of T. Poon [14] gave an alternative proof of the convexity of $W_c(T)$ using some type of majorization property (cf. [8, page 87-88]).

The case that C is a nonzero $n \times n$ matrix of rank one. We assume that the operator norm of C is 1. Then there exsits a unitary matrix U such that

$$U^*CU = \begin{pmatrix} q & \sqrt{1-|q|^2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where q is an eigenvalue of C with $|q| \leq 1$. Hence the set $W_C(T)$ can be rewritten as follows:

$$W_C(T) = \{\langle Tx, y\rangle : x, y \in \mathbb{C}^n, \|x\| = \|y\| = 1, \langle x, y\rangle = q\},$$

which is denoted by $W_q(T)$ and we call $W_q(T)$ the q-numerical range of T.

In this report, firstly, we shall obtain the direct proof of Theorem 2.A without using (2.2) and (2.3). Secondly, we shall generalize this result to c-numerical range in Section 5 as follows:

$$W_c(f(\widetilde{T})) \subset W_c(f(T)) \tag{3.1}$$

holds for all polynomial f. Lastly, we shall show the same relation (3.1) holds for q-numerical range.

4. Simplified proof of Theorem 2.A

In this section, we shall obtain a direct proof of Theorem 2.A without using (2.2) and (2.3). To prove the this result, we prepare an obvious lemma.

Lemma 4.A ([9]). *Let A be a self-adjoint operator and B be an operator. Then AB is invertible if and only if BA is invertible. Hence $\sigma(AB) = \sigma(BA)$.*

We denote the real part of an operator A by $\Re(A) = \frac{A+A^*}{2}$.

Simplified proof of Theorem 2.A. Let $T = V|T|$ be a polar decomposition of T. Since

$$\begin{aligned}\Re(|T|V) &= \frac{|T|V + V^*|T|}{2}\\ &= \frac{V^*V|T|V + V^*|T|V^*V}{2}\\ &= V^*\frac{V|T| + |T|V^*}{2}V\\ &= V^*\Re(T)V,\end{aligned}$$

we have

$$\begin{aligned}\left\langle \Re(|T|V)x, x\right\rangle &= \left\langle V^*\Re(T)Vx, x\right\rangle\\ &= \left\langle \Re(T)Vx, Vx\right\rangle\\ &= \left\langle \Re(T)\frac{Vx}{\|Vx\|}, \frac{Vx}{\|Vx\|}\right\rangle \langle Vx, Vx\rangle.\end{aligned}$$

Hence

$$W\Big(\Re(|T|V)\Big) \subset W\Big(\Re(T)\Big)W(V^*V). \tag{4.1}$$

If $0 \in W(V^*V)$, then $0 \in W\Big(\Re(T)\Big)$. By $W(V^*V) = [0,1]$ and Hausdorff-Toeplitz Theorem, we obtain

$$
\begin{aligned}
W\Big(\Re(|T|V)\Big) &\subset W\Big(\Re(T)\Big)W(V^*V) \quad \text{by (4.1)} \\
&= \Big\{\alpha\Big\langle \Re(T)x, x\Big\rangle \;:\; \|x\| = 1, \alpha \in [0,1]\Big\} \\
&= W\Big(\Re(T)\Big).
\end{aligned}
\tag{4.2}
$$

If $0 \notin W(V^*V)$, then V is an isometry, so $W\Big(\Re(|T|V)\Big) \subset W\Big(\Re(T)\Big)$ holds by (4.1).

On the other hand, for any two operators H and K, the following relation is easily obtained:

$$
\Re\{\Re(H)K\} = \frac{1}{2}\{\Re(HK) + \Re(K^*H)\}. \tag{4.3}
$$

Therefore we have

$$
\begin{aligned}
\overline{W\Big(\Re(\widetilde{T})\Big)} &= \overline{W\Big(|T|^{\frac{1}{2}}\Re(V)|T|^{\frac{1}{2}}\Big)} \\
&= \mathrm{conv}\sigma\Big(|T|^{\frac{1}{2}}\Re(V)|T|^{\frac{1}{2}}\Big) \\
&= \Re\mathrm{conv}\sigma\Big(|T|^{\frac{1}{2}}\Re(V)|T|^{\frac{1}{2}}\Big) \\
&= \Re\mathrm{conv}\sigma\Big(\Re(V)|T|\Big) \quad \text{by Lemma 4.A} \\
&\subset \Re\overline{W\Big(\Re(V)|T|\Big)} \\
&= \overline{W\Big(\Re\{\Re(V)|T|\}\Big)} \\
&= \frac{1}{2}\overline{W\Big(\Re(V|T|) + \Re(|T|V)\Big)} \quad \text{by (4.3)} \\
&\subset \frac{1}{2}\Big\{\overline{W\Big(\Re(T)\Big)} + \overline{W\Big(\Re(|T|V)\Big)}\Big\} \\
&\subset \frac{1}{2}\Big\{\overline{W\Big(\Re(T)\Big)} + \overline{W\Big(\Re(T)\Big)}\Big\} \quad \text{by (4.2)} \\
&= \overline{W\Big(\Re(T)\Big)},
\end{aligned}
$$

where $\mathrm{conv}\sigma(T)$ means the convex hull of $\sigma(T)$.

Since $\widetilde{(e^{i\theta}T)} = e^{i\theta}\widetilde{T}$ holds for each $\theta \in [0, 2\pi)$, we have

$$
\overline{W\Big(\Re\{e^{i\theta}\widetilde{T}\}\Big)} \subset \overline{W\Big(\Re\{e^{i\theta}T\}\Big)} \quad \text{for all } \theta \in [0, 2\pi),
$$

so that we obtain (2.1). □

Remark. In our proof of Theorem 2.A, the equation (4.3) plays an important role. (4.3) is also useful to extend the relation (2.1) to c-numerical range or q-numerical range.

5. c-Numerical range of the Aluthge transformation of a matrix

In this section, we shall generalize Theorem 2.A to c-numerical range and T to $f(T)$ where f is a polynomial.

Theorem 5.1. *Let T be an $n \times n$ matrix, f be a polynomial and $c = (c_1, c_2, \ldots, c_n)$ be a finite real sequence. Then the following inclusion holds:*

$$W_c(f(\widetilde{T})) \subset W_c(f(T)).$$

In this result, we may assume that $c = (c_1, c_2, \ldots, c_n)$ is a finite real sequence arranged in the decreasing order $c_1 \geq c_2 \geq \ldots \geq c_n$ by the definition of $W_c(T)$.

To prove Theorem 5.1, we shall prepare the following results:

Theorem 5.A ([12]). *Let T be an $n \times n$ matrix and $c = (c_1, c_2, \ldots, c_n)$ is a finite real sequence arranged in the decreasing order $c_1 \geq c_2 \geq \ldots \geq c_n$. Then*

$$\max\left\{\Re(z\,e^{i\theta}) : z \in W_c(T)\right\} = \sum_{j=1}^{n} c_j\, \lambda_j\left(\Re(e^{i\theta}T)\right), \tag{5.1}$$

holds for every $0 \leq \theta \leq 2\pi$, where $\lambda_j(S)$ means the jth eigenvalue of an $n \times n$ Hermitian matrix S:

$$\lambda_1(S) \geq \lambda_2(S) \geq \ldots \geq \lambda_n(S).$$

Lemma 5.B ([6], [13, page 237])). *Suppose that T is an $n \times n$ complex matrix and $\{\Re\lambda_1(T), \Re\lambda_2(T), \ldots, \Re\lambda_n(T)\}$ denotes the set of real parts of eigenvalues of T arranged in the decreasing order. Then the inequality*

$$\sum_{j=1}^{k} \Re\lambda_j(T) \leq \sum_{j=1}^{k} \lambda_j\left(\Re(T)\right)$$

holds for every $1 \leq k \leq n-1$.

Lemma 5.C ([5], [13, page 241])). *Suppose that G and H are $n \times n$ Hermitian matrices. Then the inequality*

$$\sum_{j=1}^{k} \lambda_j\,(G+H) \leq \sum_{j=1}^{k} \left\{\lambda_j(G) + \lambda_j(H)\right\}$$

holds for every $1 \leq k \leq n-1$.

Lemma 5.2. *Let A be a positive invertible matrix and X be an arbitrary matrix. Then for each polynomial f and real θ, there exists a matrix S such that*

$$e^{i\theta} f(A^{\frac{1}{2}} X A^{\frac{1}{2}}) = A^{\frac{1}{2}} S A^{\frac{1}{2}},$$
$$e^{i\theta} f(XA) = SA,$$
$$e^{i\theta} f(AX) = AS.$$

Proof. Let $f(z) = f(0) + g(z)z$, where $g(z)$ is also a polynomial. By using the equation

$$(A^{\frac{1}{2}} X A^{\frac{1}{2}})^n = A^{\frac{1}{2}} (XA)^{n-1} X A^{\frac{1}{2}},$$

we obtain the following equation:

$$\begin{aligned} f(A^{\frac{1}{2}}XA^{\frac{1}{2}}) &= f(0)I + g(A^{\frac{1}{2}}XA^{\frac{1}{2}})A^{\frac{1}{2}}XA^{\frac{1}{2}} \\ &= f(0)I + A^{\frac{1}{2}}g(XA)XA^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}\{f(0)A^{-1} + g(XA)X\}A^{\frac{1}{2}}. \end{aligned} \tag{5.2}$$

By setting

$$S = e^{i\theta}\{f(0)A^{-1} + g(XA)X\},$$

we have

$$\begin{aligned} e^{i\theta}f(A^{\frac{1}{2}}XA^{\frac{1}{2}}) &= e^{i\theta}A^{\frac{1}{2}}\{f(0)A^{-1} + g(XA)X\}A^{\frac{1}{2}} \quad \text{by (5.2)} \\ &= A^{\frac{1}{2}}SA^{\frac{1}{2}}, \\ SA &= e^{i\theta}\{f(0)A^{-1} + g(XA)X\}A \\ &= e^{i\theta}\{f(0)I + g(XA)XA\} \\ &= e^{i\theta}f(XA) \end{aligned}$$

and

$$\begin{aligned} AS &= e^{i\theta}A\{f(0)A^{-1} + g(XA)X\} \\ &= e^{i\theta}\{f(0)I + Ag(XA)X\} \\ &= e^{i\theta}f(AX) \qquad \text{by } A(XA)^nX = (AX)^{n+1}. \end{aligned}$$

Hence the proof is complete. □

Proof of Theorem 5.1. We use a polar decomposition $T = U|T|$ where U is a unitary matrix. Put $A = |T| \geq 0$ and $X = U$. By perturbing A to $A + \varepsilon I$ for small $\varepsilon > 0$, we need only to prove Theorem 5.1 for a positive invertible A. By Theorem 5.A, we shall show the following inequality

$$\sum_{j=1}^{n} c_j \lambda_j \left(\Re\{e^{i\theta} f(\tilde{T})\}\right) \leq \sum_{j=1}^{n} c_j \lambda_j \left(\Re\{e^{i\theta} f(T)\}\right) \tag{5.3}$$

for every $0 \leq \theta \leq 2\pi$. Moreover by the following equations

$$
\begin{aligned}
\sum_{j=1}^{n} c_j \lambda_j\Big(\Re\{e^{i\theta} f(\tilde{T})\}\Big) &= \sum_{j=1}^{n-1}(c_j - c_{j+1}) \sum_{k=1}^{j} \lambda_k\Big(\Re\{e^{i\theta} f(\tilde{T})\}\Big) + c_n \sum_{k=1}^{n} \lambda_k\Big(\Re\{e^{i\theta} f(\tilde{T})\}\Big), \\
\sum_{j=1}^{n} c_j \lambda_j\Big(\Re\{e^{i\theta} f(T)\}\Big) &= \sum_{j=1}^{n-1}(c_j - c_{j+1}) \sum_{k=1}^{j} \lambda_k\Big(\Re\{e^{i\theta} f(T)\}\Big) + c_n \sum_{k=1}^{n} \lambda_k\Big(\Re\{e^{i\theta} f(T)\}\Big), \\
\sum_{k=1}^{n} \lambda_k\Big(\Re\{e^{i\theta} f(\tilde{T})\}\Big) &= \sum_{k=1}^{n} \lambda_k\Big(\Re\{e^{i\theta} f(A^{\frac{1}{2}} X A^{\frac{1}{2}}\}\Big) \\
&= \sum_{k=1}^{n} \lambda_k\Big(\Re(A^{\frac{1}{2}} S A^{\frac{1}{2}})\Big) \quad \text{by Lemma 5.2} \\
&= \operatorname{tr}\Big(\Re(A^{\frac{1}{2}} S A^{\frac{1}{2}})\Big) = \Re\Big\{\operatorname{tr}(A^{\frac{1}{2}} S A^{\frac{1}{2}})\Big\} = \Re\Big(\operatorname{tr}(SA)\Big) \\
&= \operatorname{tr}\Big(\Re(SA)\Big) = \sum_{k=1}^{n} \lambda_k\Big(\Re\{e^{i\theta} f(XA)\}\Big) \quad \text{by Lemma 5.2} \\
&= \sum_{k=1}^{n} \lambda_k\Big(\Re\{e^{i\theta} f(T)\}\Big),
\end{aligned}
$$

it is sufficient to prove the inequality

$$
\sum_{j=1}^{k} \lambda_j\Big(\Re\{e^{i\theta} f(\widetilde{T})\}\Big) \leq \sum_{j=1}^{k} \lambda_j\Big(\Re\{e^{i\theta} f(T)\}\Big)
$$

holds for $0 \leq \theta \leq 2\pi$ and every $k = 1, 2, \ldots, n-1$.

By using Lemma 5.2 and Fan's two inequalities, we have

$$\sum_{j=1}^{k} \lambda_j\Big(\Re\{e^{i\theta}f(\widetilde{T})\}\Big)$$

$$=\sum_{j=1}^{k} \lambda_j\Big(\Re\{e^{i\theta}f(A^{\frac{1}{2}}XA^{\frac{1}{2}})\}\Big)$$

$$=\sum_{j=1}^{k} \lambda_j\Big(\Re(A^{\frac{1}{2}}SA^{\frac{1}{2}})\Big) \quad \text{by Lemma 5.2}$$

$$=\sum_{j=1}^{k} \lambda_j\Big(A^{\frac{1}{2}}\Re(S)A^{\frac{1}{2}}\Big)$$

$$=\sum_{j=1}^{k} \Re\lambda_j\Big(A^{\frac{1}{2}}\Re(S)A^{\frac{1}{2}}\Big)$$

$$=\sum_{j=1}^{k} \Re\lambda_j\Big(\Re(S)A\Big) \quad \text{by Lemma 4.A}$$

$$\leq\sum_{j=1}^{k} \lambda_j\Big(\Re\{\Re(S)A\}\Big) \quad \text{by Lemma 5.B}$$

$$=\frac{1}{2}\sum_{j=1}^{k} \lambda_j\Big(\Re(SA)+\Re(AS)\Big) \quad \text{by (4.3)}$$

$$\leq\frac{1}{2}\left\{\sum_{j=1}^{k} \lambda_j\Big(\Re(SA)\Big)+\sum_{j=1}^{k} \lambda_j\Big(\Re(AS)\Big)\right\} \quad \text{by Lemma 5.C}$$

$$=\frac{1}{2}\left\{\sum_{j=1}^{k} \lambda_j\Big(\Re\{e^{i\theta}f(XA)\}\Big)+\sum_{j=1}^{k} \lambda_j\Big(\Re\{e^{i\theta}f(AX)\}\Big)\right\} \quad \text{by Lemma 5.2}$$

$$=\frac{1}{2}\left\{\sum_{j=1}^{k} \lambda_j\Big(\Re\{e^{i\theta}f(T)\}\Big)+\sum_{j=1}^{k} \lambda_j\Big(U^*\Re\{e^{i\theta}f(T)\}U\Big)\right\}$$

$$=\sum_{j=1}^{k} \lambda_j\Big(\Re\{e^{i\theta}f(T)\}\Big).$$

Hence the proof of Theorem 5.1 is complete. □

The case $f(z)=z$, we obtain the following corollary.

Corollary 5.3. *Let T be an $n\times n$ matrix and $c=(c_1,c_2,\dots,c_n)$ be a finite real sequence. Then the following inclusion holds:*

$$W_c(\widetilde{T})\subset W_c(T).$$

6. q-Numerical range of the Aluthge transformation of a matrix

It is known that there is a close relationship between the family of q-numerical ranges $W_q(T)$ $(0 \le q \le 1)$ of a matrix T and the Davis-Wielandt shell $W(T, T^*T)$ of T. The latter is defined by

$$W(T, T^*T) = \{(\langle Tx, x\rangle, \langle T^*Tx, x\rangle) \in \mathbb{C} \times \mathbb{R} : x \in \mathbb{C}^n, \|x\| = 1\}.$$

It is shown that the range $W(T, T^*T)$ is convex if T is an $n \times n$ matrix for $n \ge 3$ in [2]. In the case T is a 2×2 matrix, the range $W(T, T^*T)$ is convex if its affine hull is 2-dimensional, and the range $W(T, T^*T)$ is the boundary of a convex set if its affine hull is 3-dimensional. The following lemma provides a tool to compare the q-numerical ranges of two matrices.

Lemma 6.A ([11, page 389, Theorem 2.1]). *Suppose that A is an $n \times n$ matrix and B is an $m \times m$ matrix. Then the following two conditions are mutually equivalent:*

(i) *The inclusion $W_q(B) \subset W_q(A)$ holds for every $0 \le q \le 1$.*
(ii) *The inclusion $W(B) \subset W(A)$ and the inequality*

$$\max\{h : (z, h) \in W(B, B^*B)\} \le \max\{h : (z, h) \in W(A, A^*A)\}$$

hold for every $z \in W(B)$.

In this section, we shall prove the following theorem.

Theorem 6.1. *Suppose that T is an $n \times n$ matrix and $f(z)$ is a polynomial in z. Then the inclusion*

$$W_q(f(\tilde{T})) \subset W_q(f(T)) \tag{6.1}$$

holds for every complex number q with $|q| \le 1$.

To prove Theorem 6.1, we have an alternative condition of (ii) in the above Lemma 6.A.

Lemma 6.2. *Suppose that A is an $n \times n$ matrix and B is an $m \times m$ matrix. Then the following two conditions are mutually equivalent:*

(i): *The inclusion $W_q(B) \subset W_q(A)$ holds for every $0 \le q \le 1$.*
(iii): *The inequality*

$$\lambda_1(B^*B + k\,\Re(e^{i\theta}B)) \le \lambda_1(A^*A + k\,\Re(e^{i\theta}A))$$

holds for every $0 \le \theta \le 2\pi$ and $k \ge 0$.

Proof. We prove the equivalence of condition (ii) of Lemma 6.A and condition (iii) of Lemma 6.2. We compare the following two compact convex sets:

$$A_0 = \{(z,t) \in \mathbb{C} \times \mathbb{R} : z \in W(A), 0 \le t \le \max\{h : (z,h) \in W(A, A^*A)\}\}$$

and

$$B_0 = \{(z,t) \in \mathbb{C} \times \mathbb{R} : z \in W(B), 0 \le t \le \max\{h : (z,h) \in W(B, B^*B)\}\}.$$

For every $0 \le \theta \le 2\pi$, we consider the projection $\Pi = \Pi_\theta$ given by

$$(z,t) = (\Re(z), \Im(z), t) \to \Re(e^{i\theta}z) + i\,t = (\cos\theta\Re(z) - \sin\theta\Im(z)) + i\,t.$$

Then (ii) of Lemma 6.A holds if and only if the condition $B_0 \subset A_0$ holds, and also this condition is equivalent to

$$\Pi_\theta(B_0) \subset \Pi_\theta(A_0) \tag{6.2}$$

for every $0 \le \theta \le 2\pi$, where the compact convex sets $\Pi_\theta(A_0)$ and $\Pi_\theta(B_0)$ are characterized by

$$\Pi_\theta(A_0) = \mathrm{conv}(W(\Re(e^{i\theta}A)), W(\Re(e^{i\theta}A) + i\,A^*A))$$

and

$$\Pi_\theta(B_0) = \mathrm{conv}(W(\Re(e^{i\theta}B)), W(\Re(e^{i\theta}B) + i\,B^*B)).$$

Each of these sets contains its projection onto the real line. These sets are contained in the closed upper half plane $\Im(z) \ge 0$. Thus, for each $0 \le \theta \le 2\pi$, the inclusion relation (6.2) is equivalent to the inequality

$$\begin{aligned} &\max\{\Im(z) + k\,\Re(z) : z \in W(\Re(e^{i\theta}B) + i\,B^*B)\} \\ &\le \max\{\Im(z) + k\,\Re(z) : z \in W(\Re(e^{i\theta}A) + i\,A^*A)\} \end{aligned} \tag{6.3}$$

for every $k \in \mathbf{R}$ (cf. [15, page 81, Theorem A]). By basic properties of the numerical range, we have

$$\begin{aligned} &\max\{\Im(z) + k\,\Re(z) : z \in W(\Re(e^{i\theta}A) + i\,A^*A)\} \\ &= \max W(A^*A + k\,\Re(e^{i\theta}A)) \\ &= \lambda_1(A^*A + k\,\Re(e^{i\theta}A)) \end{aligned}$$

and

$$\begin{aligned} &\max\{\Im(z) + k\,\Re(z) : z \in W(\Re(e^{i\theta}B) + i\,B^*B)\} \\ &= \lambda_1(B^*B + k\,\Re(e^{i\theta}B)) \end{aligned}$$

(cf. [8, page 9-11]), so that (6.3) is equivalent to

$$\lambda_1(B^*B + k\,\Re(e^{i\theta}B)) \le \lambda_1(A^*A + k\,\Re(e^{i\theta}A))$$

for every $0 \le \theta \le 2\pi$ and $k \in \mathbf{R}$. By replacing θ by $\theta + \pi$, we may restrict the range of k as $k \ge 0$. Thus the condition (ii) of Lemma 6.A and the condition (iii) of Lemma 6.2 are equivalent. □

Proof of Theorem 6.1. Since the equation

$$W_{cq}(S) = c\,W_q(S)$$

holds for any complex numbers c, q with $|c| = 1$ and $|q| \le 1$, it is sufficient to prove (6.1) for $0 \le q \le 1$. Therefore we have only to prove the inequality

$$\lambda_1\Big(f(\tilde{T})^* f(\tilde{T}) + k\Re\{e^{i\theta} f(\tilde{T})\}\Big) \le \lambda_1\Big(f(T)^* f(T) + k\Re\{e^{i\theta} f(T)\}\Big) \tag{6.4}$$

for every $0 \le \theta \le 2\pi$ and $k \ge 0$ by Lemma 6.2.

To prove the inequality (6.4), we shall prove that the following inequality holds for a positive matrix A and an arbitrary X:

$$\begin{aligned} &\lambda_1\Big(f(A^{\frac{1}{2}}XA^{\frac{1}{2}})^* f(A^{\frac{1}{2}}XA^{\frac{1}{2}}) + k\Re\{e^{i\theta} f(A^{\frac{1}{2}}X A^{\frac{1}{2}})\}\Big) \\ &\le \frac{1}{2}\lambda_1\Big(f(XA)^* f(XA) + k\Re\{e^{i\theta} f(XA)\}\Big) \\ &\qquad + \frac{1}{2}\lambda_1\Big(f(AX)^* f(AX) + k\Re\{e^{i\theta} f(AX)\}\Big). \end{aligned} \tag{6.5}$$

By perturbing A to $A + \varepsilon I$ for small $\varepsilon > 0$, it suffices to prove (6.5) for a positive invertible matrix A.

using Lemma 5.2, the Cauchy-Schwarz inequality and the Arithmetic-Geo
inequality, we have

$$
\begin{aligned}
&\lambda_1\Big(f(A^{\frac12}XA^{\frac12})^*f(A^{\frac12}XA^{\frac12}) + k\Re\{e^{i\theta}f(A^{\frac12}XA^{\frac12})\}\Big)\\
&= \lambda_1\Big(A^{\frac12}S^*ASA^{\frac12} + k\Re\{A^{\frac12}SA^{\frac12}\}\Big) \quad \text{by Lemma 5.2}\\
&= \lambda_1\Big(A^{\frac12}S^*ASA^{\frac12} + k\{A^{\frac12}\Re(S)A^{\frac12}\}\Big)\\
&= \Re\lambda_1\Big(S^*ASA + k\Re(S)A\Big) \quad \text{by Lemma 4.A}\\
&\le \lambda_1\Big(\Re\{S^*ASA + k\Re(S)A\}\Big) \quad \text{by Lemma 5.B}\\
&= \max_{x\in\mathbf{C}^n,\|x\|=1}\left[\Re\Big\langle S^*ASAx, x\Big\rangle + \frac{k}{2}\Big\langle\{\Re(SA)+\Re(AS)\}x, x\Big\rangle\right] \quad \text{by (4.3)}\\
&= \max_{x\in\mathbf{C}^n,\|x\|=1}\left[\Re\Big\langle SAx, ASx\Big\rangle + \frac{k}{2}\Big\langle\{\Re(SA)+\Re(AS)\}x, x\Big\rangle\right]\\
&\le \max_{x\in\mathbf{C}^n,\|x\|=1}\left[\Big\langle AS^*SAx, x\Big\rangle^{\frac12}\Big\langle S^*A^2Sx, x\Big\rangle^{\frac12} + \frac{k}{2}\Big\langle\{\Re(SA)+\Re(AS)\}x, x\Big\rangle\right]\\
&\le \max_{x\in\mathbf{C}^n,\|x\|=1}\left[\frac12\Big\langle f(XA)^*f(XA)x, x\Big\rangle + \frac12\Big\langle f(AX)^*f(AX)x, x\Big\rangle\right.\\
&\qquad\qquad \left. + \frac{k}{2}\Big\langle\Re\{e^{i\theta}f(XA)\}x, x\Big\rangle + \frac{k}{2}\Big\langle\Re\{e^{i\theta}f(AX)\}x, x\Big\rangle\right]\\
&\le \frac12\max_{x\in\mathbf{C}^n,\|x\|=1}\left[\Big\langle f(XA)^*f(XA)x, x\Big\rangle + k\Big\langle\Re\{e^{i\theta}f(XA)\}x, x\Big\rangle\right]\\
&\qquad\qquad + \frac12\max_{x\in\mathbf{C}^n,\|x\|=1}\left[\Big\langle f(AX)^*f(AX)x, x\Big\rangle + k\Big\langle\Re\{e^{i\theta}f(AX)\}x, x\Big\rangle\right]\\
&= \frac12\lambda_1\Big(f(XA)^*f(XA) + k\Re\{e^{i\theta}f(XA)\}\Big)\\
&\qquad\qquad + \frac12\lambda_1\Big(f(AX)^*f(AX) + k\Re\{e^{i\theta}f(AX)\}\Big).
\end{aligned}
$$

shall use a polar decomposition $T = U|T|$ where U is a unitary matrix
;ute $A = |T|$, $X = U$ in (6.5). Since $(|T|U)^n = U^*T^nU$ for every integer $n \geq$

have the equation $f(|T|U) = U^* f(T)U$ for any polynomial f, so that

$$\begin{aligned}
&\lambda_1\Big(f(\widetilde{T})^* f(\widetilde{T}) + k\Re\{e^{i\theta} f(\widetilde{T})\}\Big) \\
&\leq \frac{1}{2}\lambda_1\Big(f(T)^* f(T) + k\Re\{e^{i\theta} f(T)\}\Big) \\
&\qquad + \frac{1}{2}\lambda_1\Big(f(|T|U)^* f(|T|U) + k\Re\{e^{i\theta} f(|T|U)\}\Big) \\
&= \frac{1}{2}\lambda_1\Big(f(T)^* f(T) + k\Re\{e^{i\theta} f(T)\}\Big) \\
&\qquad + \frac{1}{2}\lambda_1\Big(U^* f(T)^* f(T)U + kU^*\Re\{e^{i\theta} f(T)\}U\Big) \\
&= \lambda_1\Big(f(T)^* f(T) + k\Re\{e^{i\theta} f(T)\}\Big).
\end{aligned}$$

Hence the proof of Theorem 6.1 is complete. □

In particular, by putting $q = 1$ in Theorem 6.1, we have the following relation.

Corollary 6.3. *If T is an $n \times n$ matrix. Then*

$$W(f(\widetilde{T})) \subset W(f(T)) \quad \textit{holds for all polynomial } f.$$

Moreover, we obtain the inequalities on the numerical radius and the spectral norm.

Corollary 6.4. *Let T be an $n \times n$ matrices. Then the following assertions hold:*

(i) $w(f(\widetilde{T})) \leq w(f(T))$ *for all polynomials f.*

(ii) $\|f(\widetilde{T})\| \leq \|f(T)\|$ *for all polynomials f,*

where $\|\cdot\|$ means the spectral norm.

Corollary 6.4 is easily obtained by the following Proposition 6.5.

Proposition 6.5. *Let A and B be $n \times n$ matrices. Then the following assertions are mutually equivalent:*

(i) $W(f(A)) \subset W(f(B))$ *for all polynomials f.*

(ii) $w(f(A)) \leq w(f(B))$ *for all polynomials f.*

(iii) $\|f(A)\| \leq \|f(B)\|$ *for all polynomials f,*

where $\|\cdot\|$ means the spectral norm.

Proof. Proofs of (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$(i) are obvious by

$$W(A) = \bigcap_{\mu \in \mathbf{C}} \{z \ : \ |z - \mu| \leq w(A - \mu)\}$$

and

$$W(A) = \bigcap_{\mu \in \mathbf{C}} \{z \ : \ |z - \mu| \leq \|A - \mu\|\}.$$

Proof of (i) $\Longrightarrow$ (ii) is also obvious. Hence we shall show (i) $\Longrightarrow$ (iii). In fact, we have only to show that

$$W(f(A)) \subset W(f(B)) \quad \text{for all polynomials } f \quad \Longrightarrow \quad \|A\| \leq \|B\|.$$

So we shall show

$$\|B\| < 1 \Longrightarrow \|A\| \leq 1.$$

Let $r(A)$ be the spectral radius of A. Since

$$r(A) \leq w(A) \leq w(B) \leq \|B\| < 1$$

hold, the inverses $(1+A)^{-1}$ and $(1+B)^{-1}$ exist, and we can consider the Cayley transform of A and B as follows:

$$\Phi(A) \equiv (1-A)(1+A)^{-1}, \quad \Phi(B) \equiv (1-B)(1+B)^{-1}.$$

On the other hand, setting

$$g_n(z) \equiv 1 + 2\sum_{k=1}^{n}(-1)^k z^k,$$

we have

$$\Phi(A) = \lim_{n\to\infty} g_n(A), \quad \Phi(B) = \lim_{n\to\infty} g_n(B)$$

since

$$\frac{1-z}{1+z} = 1 + 2\sum_{k=1}^{\infty}(-1)^k z^k$$

holds. By the assumption, we have

$$W(g_n(A)) \subset W(g_n(B)) \quad (n = 1, 2, \ldots),$$

then we obtain

$$W(\Phi(A)) \subset W(\Phi(B)).$$

On the other hand, since B is a contraction, we have $\Re(\Phi(B)) \geq 0$, that is, $W(\Phi(B))$ is included in the right-half plane. Then $W(\Phi(A))$ is also included in the right-half plane, that is, $\Re(\Phi(A)) \geq 0$ holds. Therefore, $1 + \Phi(A)$ is invertible, and $A = \Phi(\Phi(A))$ is a contraction, so that the proof is complete. □

Proof of Corollary 6.4. Put $A = \tilde{T}$ and $B = T$ in Proposition 6.5, and we have Corollary 6.4 by Corollary 6.3. □

Lastly, we summarize Theorems 5.1 and 6.1 as follows:

Theorem 6.6. *Suppose that T and C are $n \times n$ complex matrices and f is a complex polynomial. If C is a Hermitian matrix or a rank-one matrix, then the following inclusion relation holds:*

$$W_C(f(\tilde{T})) \subset W_C(f(T)).$$

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References

[1] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations Operator Theory, **13** (1990), 307–315.

[2] Y. H. Au-Yeung and N. K. Tsing, *An extension of the Hausdorff-Toeplitz theorem on the numerical range*, Proc. Amer. Math. Soc., **89** (1983), 215–218.

[3] C. A. Berger and J. G. Stampfli, *Mapping theorems for numerical range*, Amer. J. Math., **89** (1967), 1047–1055.

[4] W. S. Cheung and N. K. Tsing, *The C-numerical range of matrices is star-shaped*, Linear and Multilinear Algebra, **41** (1996), 245–250.

[5] K. Fan:, *On a theorem of Weyl concerning eigenvalues of linear transformations I*, Proc. Nat. Acad. Sci. U.S.A., **35** (1949), 652–655.

[6] K. Fan, *On a theorem of Weyl concerning eigenvalues of linear transformations II*, Proc. Nat. Acad. Sci. U.S.A., **36** (1950), 31–35.

[7] M. Goldberg and E. G. Straus, *Elementary inclusion relations for generalized numerical ranges*, Linear Algebra Appl., **18** (1977), 1–24.

[8] R. A. Horn and C. R. Johnson, *Topics in matrix analysis*, Cambridge University Press, Cambridge, New York, Melbourne, 1991.

[9] T. Huruya, *A note on p-hyponormal operators*, Proc. Amer. Math. Soc., **125** (1997), 3617–3624.

[10] I. B. Jung, E. Ko and C. Pearcy, *Aluthge transforms of operators*, Integral Equations Operator Theory, **37** (2000), 437–448.

[11] C. K. Li and H. Nakazato, *Some results on the q-numerical range*, Linear and Multilinear Algebra, **43** (1998), 385–409.

[12] C. K. Li, C. H. Sung and N. K. Tsing, *c-Convex matrices: Characterizations inclusions relations and normality*, Linear and Multilinear Algebra, **25** (1989), 275–287.

[13] A. W. Marshall and I. Olkin, *Inequalities: Theory of majorization and Its Applications*, Academic Press, San Diego, New York, 1979.

[14] Y. T. Poon, *Another proof of a result of Westwick*, Linear and Multilinear Algebra, **9** (1980), 35–37.

[15] A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York, San Francisco, London 1973.

[16] N. K. Tsing, *The constrained bilinear form and the C-numerical range*, Linear Algebra Appl., **56** (1984), 195–206.

[17] R. Westwick, *A theorem on numerical range*, Linear and Multilinear Algebra, **2** (1975), 311–315.

[18] P. Y. Wu, *Numerical range of Aluthge transform of operator*, Linear Algebra Appl., **357** (2002), 295–298.

[19] T. Yamazaki, *On numerical range of the Aluthge transformation*, Linear Algebra Appl., **341** (2002), 111–117.

[20] T. Yamazaki, *An expression of spectral radius via Aluthge transformation*, Proc. Amer. Math. Soc., **130** (2002), 1130–1137.