数理解析研究所講究録 1313 巻 2003 年 149-158

特異点近傍におけるパルス解の挙動について

Shin-Ichiro Ei (Yokohama City University)
栄 伸一郎 (横浜市立大学)

1 Introduction

In nature, many kinds of spatial and/or temporal patterns are observed, some of them are simple and the others are complicated. To understand theoretically the dynamics of such patterns, many model equations have been proposed and analyzed. Among them, some sort of reaction- diffusion systems are one of the most familar classes.

Recently, several reaction-diffusion model equations have been known as examples exhibiting various complicated behaviors of solutions; self-replicating behavior of pulses ([9] and its references), refleciton of pulses ([5]), the behavior of pulses like elastic objects (e.g. [?], [6], [11], [10]).

In this report, we consider reaction-diffusion systems

(1.1) $$\boldsymbol{u}_t = D\Delta \boldsymbol{u} + F(\boldsymbol{u}),\ t > 0,\ x \in \Omega \in \boldsymbol{R}^n,$$

where $D := diag(d_1, \cdots, d_N)$, $\boldsymbol{u} \in \boldsymbol{R}^N$ and $F : \boldsymbol{R}^N \to \boldsymbol{R}^N$ and suppose they possess pulse-like localized solutions, say $S(x)$ with $S(x) \to \mathbf{0}$ as $|x| \to \infty$ as in Fig1.

As the examples of such reaction-diffusion systems, we can show the following systems.

First, we can mention the Gierer-Meinhardt model (G-M).

(1.2) $$\left\{\begin{array}{rcl} u_t &=& d_1\Delta u - u + \dfrac{u^p}{v^q}, \\ \tau v_t &=& d_2\Delta v - v + \dfrac{u^r}{v^s}. \end{array}\right.$$

If d_1 is sufficiently small, (1.2) has a pulse-like solution for $\Omega \subset \boldsymbol{R}$ or $\boldsymbol{R}^2$ as in Fig2. We can also mention the Gray-Scott model as the similar examples to (1.2).

On the other hand, there is an example which shows a traveling pulse- like solution as follows. In [6], following reaction-diffuison systems which has a moving localized solution in two dimensional space was proposed:

(1.3) $$\left\{\begin{array}{rcl} \sigma u_t &=& \varepsilon\Delta u + \varepsilon^{-1} f(u, v, w), \\ v_t &=& \Delta v + u - v + h_1, \\ \tau w_t &=& d\Delta w + u - w + h_2, \end{array}\right.$$

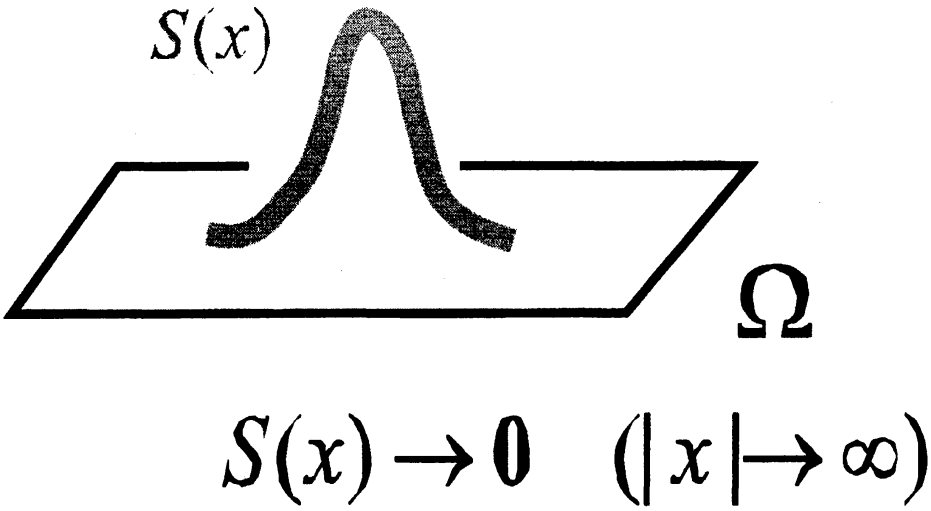

Figure 1: Pulse-like localized patterns

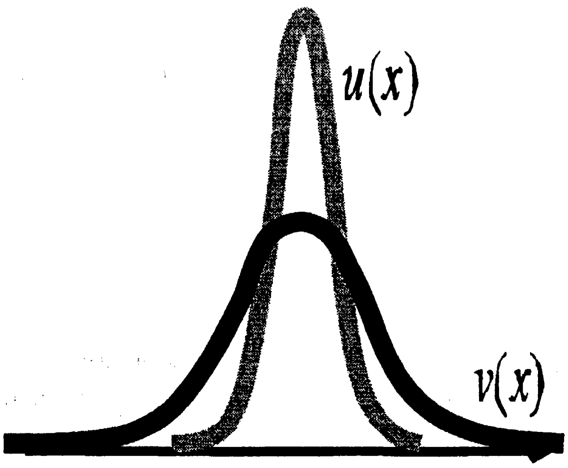

Figure 2: Pulse-like localized patterns for G-M in 1D.

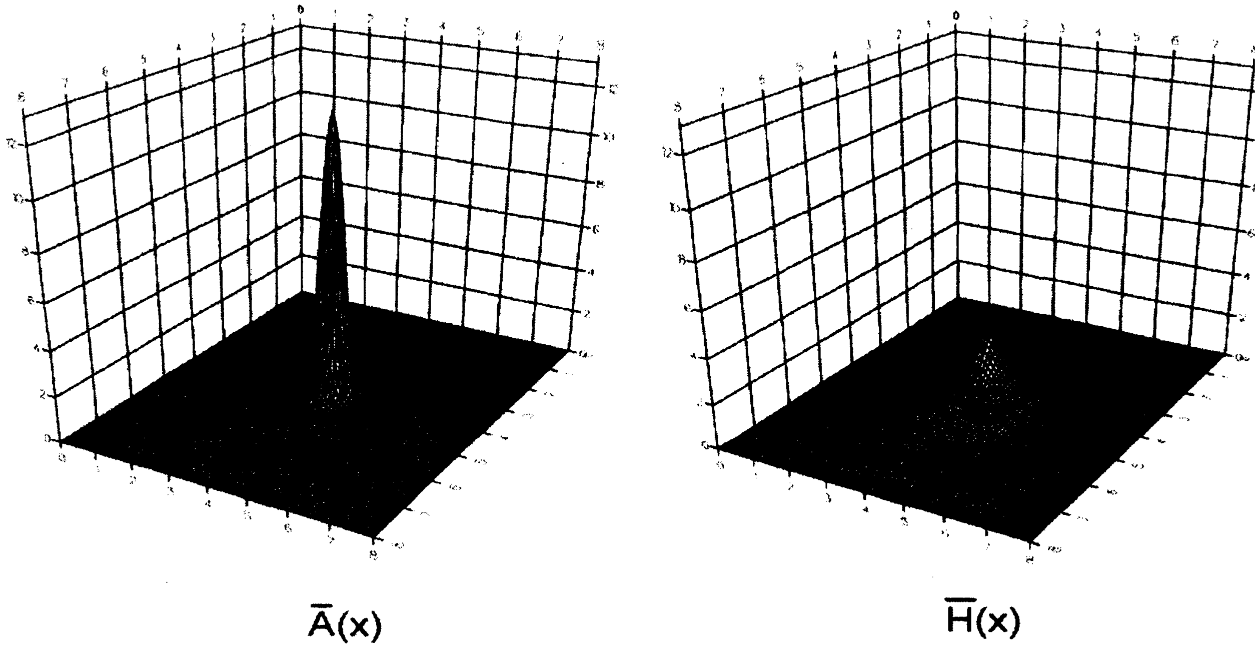

Figure 3: Pulse-like localized patterns for G-M in 2D ([4]).

where $f(u, v, w) = ru - u^3 - k_1 v - k_2 w$. They showed numerically the existence of a moving localized solution, say *travelling spot* , under suitable conditions (Fig.4).

Here, we consider how the dynamics of such pulse like localized solutions is if there exist many pulses.

For the dynamics of interacting pulses, the repulsiveness of spike solutions of (1.2) and the self-replicating dynamics of the Gray-Scott model have been analyzed by using weak interaction techniques ([4], [3]).

In this report, we specially consider the following particle-like dynamics of traveling spot solution ([2]).

In [6], the interaction like elastic objects between multi travelling spots was numerically shown (Fig.5).

Very recently, such particle like dynamics have been also observed in real experiments and several model equations which exhibit similar dynamics have been proposed ([8], [1]).

In order to understand such complicated phenomena, we first consider the existence of a travelling spot in two dimensional space near a critical point. That is, we assume the existence of stable (radially) symmetric stationary solutions and when it loses the stability, we construct a travelling spot as the bifurcating solutions from it.

Secondly, we analyze their interactions when there exist multiple travelling spots.

As a consequence, we can derive ODEs describing the particle like dynamics. The reduced ODEs show how pulses interact and reflection occur.

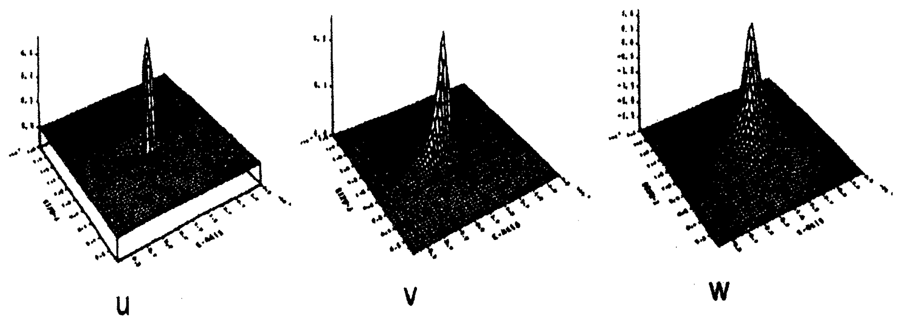

Figure 4: spatial profiles of a travelling spot. Parameter values are $\varepsilon = 0.1$, $\sigma = 0.04$, $r = 1.5$, $k_1 = 1.0$, $k_2 = 5.0$, $h_1 = 1.0$, $h_2 = 0.8$, $\tau = 0.01$, $d = 7.0$.

2 Construction of travelling spot

Let us consider general types of reaction-diffusion systems with bifurcation parameter k;

$$\boldsymbol{u}_t = \mathcal{L}(\boldsymbol{u}; k),\ x \in \boldsymbol{R}^2,\ t > 0, \tag{2.1}$$

where $\mathcal{L}(\boldsymbol{u}; k) = D\Delta\boldsymbol{u} + F(\boldsymbol{u}; k)$, $\boldsymbol{u} \in \boldsymbol{R}^N$ and D is a diagonal matrix with elements $\{d_j\}$ $(j = 1, 2, \cdots, N)$. We assume following assumptions.

H1) There exist a radially symmetric standing solution $S(x)$ such that $\mathcal{L}(S(x); k) = \boldsymbol{0}$ and $S(x) \to \boldsymbol{0}$ as $|x| \to \infty$, where $\boldsymbol{0} = (0, \cdots, 0) \in \boldsymbol{R}^N$.

Let $X = \{L^q(\boldsymbol{R}^2)\}^N$ $(q > 2)$ and let $L(k) = \mathcal{L}'(S(x); k)$ be the linearized operator of (2.1) with respect to $S(x)$ and $\Sigma(k)$ be the spectrum of $L(k)$. Note that $L(k)S_j = 0$ $(j = 1, 2)$ hold and 0 is necessarily eigenvalue of $L(k)$, where $S_j = \dfrac{\partial S}{\partial x_j}$ for $x = (x_1, x_2)$.

H2) There exists $k = k_c$ such that $\Sigma_c = \Sigma(k_c)$ consists of two sets $\Sigma_0 = \{0\}$ and $\Sigma_1 \subset \{z \in \boldsymbol{C};\ Re(z) < -\gamma_0\}$ for positive constant γ_0. The generalized eigenspace corresponding to Σ_0, say X_0, is given by $X_0 = span\{S_j, \Psi_j\}$ $(j = 1, 2)$, where Ψ_j are functions satisfying $L_c\Psi_j = -S_j$ $(j = 1, 2)$.

Let Q_c and R_c be projections at $k = k_c$ with respect to L_c corresponding to the spectral sets Σ_0 and Σ_1, respectively. Define a function $U(x; P, \boldsymbol{\zeta}) = S(x - P) + \sum_{j=1}^{2} \zeta_j \Psi_j$ for P, $\boldsymbol{\zeta} = (\zeta_1, \zeta_2) \in \boldsymbol{R}^2$ and a set $\mathcal{M} = \{S(x - P);\ P \in \boldsymbol{R}^2\}$.

We consider (2.1) in the neighborhood of the parameter $k = k_c$. To do so, we put $k = k_c + \eta$ and rewrite (2.1) as

$$\boldsymbol{u}_t = \mathcal{L}_c(\boldsymbol{u}) + \eta g(\boldsymbol{u}), \tag{2.2}$$

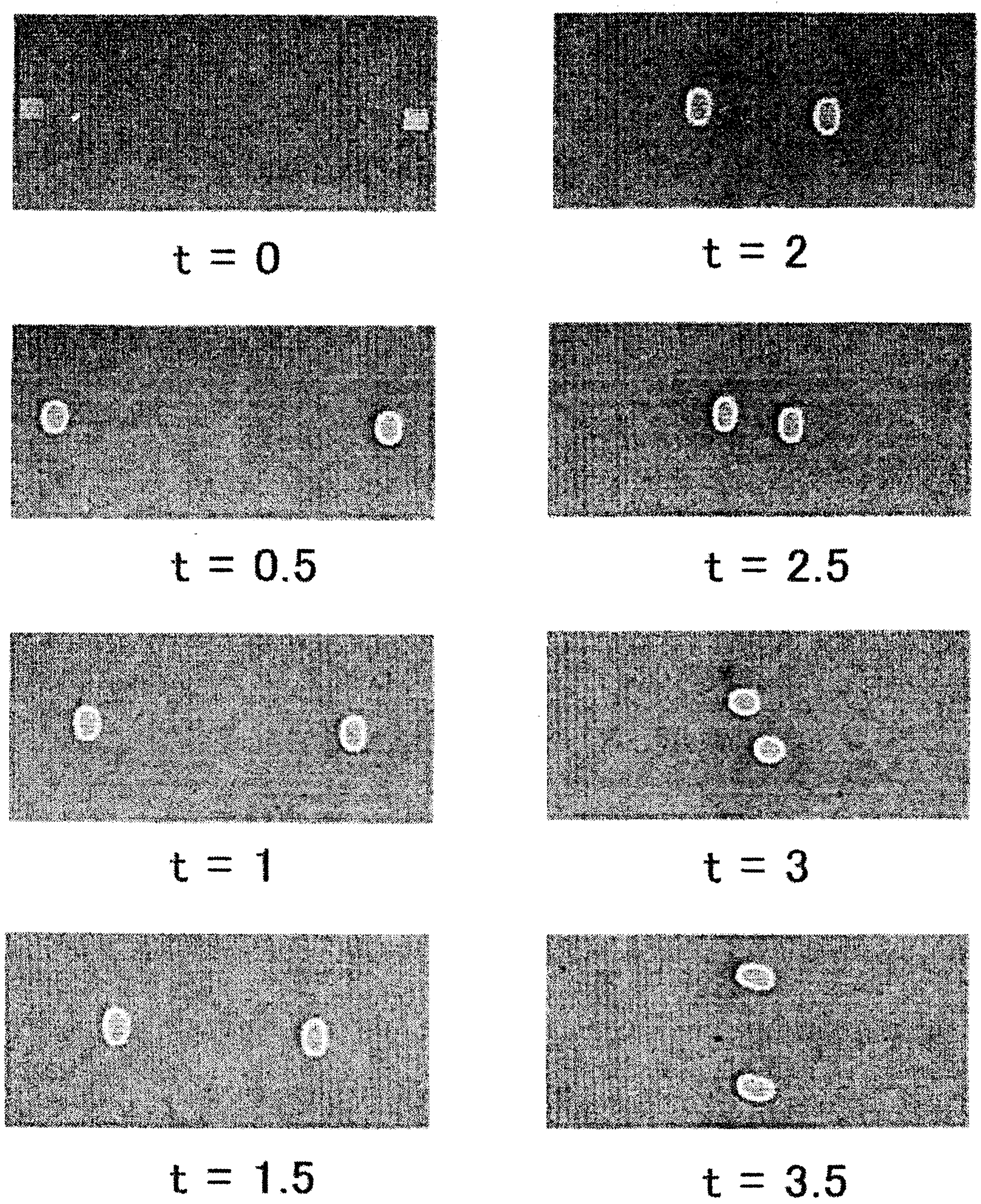

Figure 5: Particle like behavior of travelling spots. Each spot corresponds to the location of each travalling spot.

where $\mathcal{L}_c(\boldsymbol{u}) = \mathcal{L}(\boldsymbol{u}; k_c) = D\Delta\boldsymbol{u} + F(\boldsymbol{u}; k_c)$ and $\eta g(\boldsymbol{u}) = \eta g(\boldsymbol{u}; k) = \mathcal{L}(\boldsymbol{u}; k) - \mathcal{L}_c(\boldsymbol{u})$.

Then, we have the theorem:

Theorem 2.1 *If the initial data $\boldsymbol{u}(0)$ is in the neighborhood of $\mathcal{M}$ in $\{H^2(\boldsymbol{R}^2)\}^N$, then the solution $\boldsymbol{u}(t)$ of (2.2) satisfies*

$$\|\boldsymbol{u}(t) - U(\cdot, P(t), \boldsymbol{\zeta}(t))\|_\infty = O(|\boldsymbol{\zeta}(t)|^2 + |\eta|)$$

as long as $|\boldsymbol{\zeta}| < \zeta^*$ and $|\eta| < \eta^*$ for constants $\zeta^* > 0$ and $\eta^* > 0$. P and $\boldsymbol{\zeta}$ are estimated by

$$\dot{P} = O(|\boldsymbol{\zeta}(t)| + |\eta|^2),\ \dot{\boldsymbol{\zeta}} = O(|\boldsymbol{\zeta}(t)|^2 + |\eta|^2).$$

To obtain more accurate dynamics of P and $\boldsymbol{\zeta}$, we have to know the explicit form of the projection Q_c. In fact, the equation governing P and $\boldsymbol{\zeta}$ is formally derived in the similar manner to [4] as

$$Q_c \frac{d}{dt} U = Q_c \mathcal{L}(U; k_c + \eta) + h.o.t., \tag{2.3}$$

which is in general very difficult to calculate in explicit way.

In the following, we obtain the explicit form of Q_c under suitable assumptions and show the dynamics of P and $\boldsymbol{\zeta}$.

Since the standing solution $S(x)$ is radially symmetric, we write it as $S(x) = S(r)$, where $r = |x|$. Define the functional space consisting of radially symmetric functions by $X_R = \{L^2(0, \infty)\}^N$ with the inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle_R = \int_0^\infty r \langle \boldsymbol{u}(r), \boldsymbol{v}(r) \rangle dr$ for $\boldsymbol{u}$ and $\boldsymbol{v} \in X_R$.

Let $L_R(k)$ be the restriction of the linearized operator $L(k)$ on X_R, that is,

$$L_R(k)\boldsymbol{u} = D\{\boldsymbol{u}_{rr} + \frac{1}{r}\boldsymbol{u}_r\} + F'(S(r); k)\boldsymbol{u}$$

for $\boldsymbol{u} \in \mathcal{D}_R = \{\boldsymbol{u} \in H^2(0, \infty) \cap X_R;\ \boldsymbol{u}_r(0) = 0\}$.

H3) The spectrum of $L_R(k)$ in X_R is uniformly apart from the imaginary axis in the left hand side for the parameter k in the neighborhood of k_c.

Define an operator $\widetilde{L}(k)$ on X_R by

$$\widetilde{L}(k)\boldsymbol{u} = D\{\boldsymbol{u}_r + \frac{1}{r}\boldsymbol{u}\}_r + F'(S(r); k)\boldsymbol{u}$$

for $\boldsymbol{u} \in \widetilde{\mathcal{D}} = \{\boldsymbol{u} \in H^2(0, \infty) \cap X_R;\ \boldsymbol{u}(0) = \boldsymbol{0}\}$. Here, we note that $\widetilde{L}(k)S_r = \boldsymbol{0}$ holds while $L_R(k)S_r \neq \boldsymbol{0}$. This means 0 is necessarily an eigenvalue of $\widetilde{L}(k)$. Let $\widetilde{L}_c = \widetilde{L}(k_c)$ and $\widetilde{\Sigma}_c$ be the spectrum of $\widetilde{L}_c$.

H4) $\widetilde{\Sigma}_c$ consists of two sets $\widetilde{\Sigma}_0 = \{0\}$ and $\widetilde{\Sigma}_1 \subset \{z \in \boldsymbol{C};\ Re(z) < -\gamma_1\}$ for a positive constant γ_1. The generalized eigenspace corresponding to $\widetilde{\Sigma}_0$, say $\widetilde{X}_0$, is given by

$\widetilde{X}_0 = span\{S_r, \psi\}$, where ψ is a function satisfying $\widetilde{L}_c\psi = -S_r$.

Let $\widetilde{L}_c^*$ be the adjoint operator of $\widetilde{L}_c$ in X_R. Note that it is given by

$$\widetilde{L}_c^*\boldsymbol{u} = D\{\boldsymbol{u}_r + \frac{1}{r}\boldsymbol{u}\}_r + {}^tF'(S(r);k_c)\boldsymbol{u}.$$

$\widetilde{L}_c^*$ has also similar properties to $\widetilde{L}_c$, that is, there exist eigenfunctions ϕ^* and ψ^* in X_R satisfying $\widetilde{L}_c^*\phi^* = \mathbf{0}$ and $\widetilde{L}_c^*\psi^* = -\phi^*$.

Proposition 2.1 *Eigenfunctions ψ, ϕ^* and ψ^* are uniquely determined by the normalization*

$$\langle \psi, S_r \rangle_R = \langle \psi, \psi^* \rangle_R = 0, \ \langle S_r, \psi^* \rangle_R = 1.$$

We assume eigenfunctions are normalized according to the proposition. Put

$$\Psi(r) = \int_0^r \psi(r)dr - \int_0^\infty \psi(r)dr, \ \Phi^*(r) = \int_0^r \phi^*(r)dr - \int_0^\infty \phi^*(r)dr,$$

$$\Psi^*(r) = \int_0^r \psi^*(r)dr - \int_0^\infty \psi^*(r)dr.$$

Then, it is easily checked that

$$L_c\Psi_j = -S_j, \ L_c^*\Phi_j^* = \mathbf{0}, \ L_c^*\Psi_j^* = -\Phi_j^*$$

hold for $j = 1, 2$, where $\Psi_j = \dfrac{\partial\Psi}{\partial x_j}$ and so on. By this, we have

Proposition 2.2 *The projection Q_c is given by*

$$\begin{aligned}\pi Q_c\boldsymbol{u} \ = \ & \int_0^{2\pi} \langle \boldsymbol{u}, \phi^* \rangle_R \cos\theta d\theta \cdot \Psi_1 + \int_0^{2\pi} \langle \boldsymbol{u}, \psi^* \rangle_R \cos\theta d\theta \cdot S_1 \\ & + \int_0^{2\pi} \langle \boldsymbol{u}, \phi^* \rangle_R \sin\theta d\theta \cdot \Psi_2 + \int_0^{2\pi} \langle \boldsymbol{u}, \psi^* \rangle_R \sin\theta d\theta \cdot S_2\end{aligned}$$

for $\boldsymbol{u} = \boldsymbol{u}(r,\theta) \in X$.

By using this expression of Q_c, we can obtain the explicit dynamics of P and $\boldsymbol{\zeta}$.

Theorem 2.2 *Under assumptions H1) - H4), $P(t)$ and $\boldsymbol{\zeta}(t)$ in theorem 2.1 satisfy*

$$\begin{cases} \dot{P} &= \ \boldsymbol{\zeta} + O(|\boldsymbol{\zeta}(t)|^3 + |\eta|^{\frac{3}{2}}), \\ \dot{\boldsymbol{\zeta}} &= \ -\nabla W + O(|\boldsymbol{\zeta}(t)|^4 + |\eta|^2) \end{cases}$$

as long as $|\boldsymbol{\zeta}(t)| < \zeta^*$ and $|\eta| < \eta^*$, where $W = W(\boldsymbol{\zeta}) = \frac{1}{4}M_1|\boldsymbol{\zeta}|^4 + \frac{1}{2}M_2\eta|\boldsymbol{\zeta}|^2$ for constants M_1 and M_2.

Remark 2.1 *The values of constants M_j in Theorem 2.2 are obtained in explicit forms while we will not show them in this report, which will be written in [?]. For (1.3), it is numerically checked that both M_1 and M_2 are positive.*

Remark 2.2 *Theorem 2.2 suggests that $\boldsymbol{\zeta}$ denotes the velocity of the spot S because P denotes the location of the spot. $\boldsymbol{\zeta}$ also stands for the deformation from radial symmetry of spot since the solution $\boldsymbol{u}(t,x)$ is close to the function $U(x;P(t),\boldsymbol{\zeta}(t))$ as in Theorem 2.1.*

Corollary 2.1 *Suppose M_1 and M_2 are positive. If $\eta>0$, there exists a stable standing spot with profile $S(x)+O(|\eta|)$ while if $\eta<0$, there exists a travelling spot with velocity $(|\boldsymbol{\zeta}(t)|=)\sqrt{\dfrac{-2M_2\eta}{M_1}}(1+o(1))$.*

3 Interaction of two spots

Let us consider how two travelling spots interact.

H5) The standing spot $S(x)$ has an aysmptotic form $S(r)\to\dfrac{1}{\sqrt{r}}e^{-\alpha r}\boldsymbol{a}$ $(r\to\infty)$ for a constant $\alpha>0$ and a nonzero vector $\boldsymbol{a}\in\boldsymbol{R}^N$.

Remark 3.1 *The asymptotic form in H5) is true for many model equations in $\boldsymbol{R}^2$ such as the Gierer-Meinhardt model ([4]) and the Gray-Scott model ([12]).*

Define a function

$$U(x;P_1,P_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2)=\sum_{j=1}^{2}\{S(x-P_j)+\langle\,\boldsymbol{\zeta}_j,\nabla_x\Psi(x-P_j)\,\rangle\}$$

for P_j, $\boldsymbol{\zeta}_j\in\boldsymbol{R}^2$ and define a set

$$\mathcal{M}(h^*)=\{S(x-P_1)+S(x-P_2);\ |P_1-P_2|=h>h^*\}.$$

Theorem 3.1 *There exists a sufficiently large $h^*>0$ such that if the initial data $\boldsymbol{u}(0)$ is sufficiently close to the set $\mathcal{M}(h^*)$, then the solution $\boldsymbol{u}(t)$ of (2.2) keeps close to $U(x;P_1,P_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2)$ with*

$$\boldsymbol{u}(t)=U(x;P_1,P_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2)+O(e^{-\alpha h}+|\boldsymbol{\zeta}_1|^2+|\boldsymbol{\zeta}_2|^2+|\eta|)$$

and for $j=1,2$

$$(3.1)\qquad\left\{\begin{array}{rcl}\dot{P}_j&=&\boldsymbol{\zeta}_j\mp M_0\dfrac{1}{\sqrt{h}}e^{-\alpha h}\boldsymbol{e}+O(e^{-2\alpha h}+|\boldsymbol{\zeta}_1|^3+|\boldsymbol{\zeta}_2|^3+|\eta|^{\frac{3}{2}}),\\ \dot{\boldsymbol{\zeta}}_j&=&-\nabla W(\boldsymbol{\zeta}_j)\mp\overline{M}_0\dfrac{1}{\sqrt{h}}e^{-\alpha h}\boldsymbol{e}+O(e^{-2\alpha h}+|\boldsymbol{\zeta}_1|^4+|\boldsymbol{\zeta}_2|^4+|\eta|^2)\end{array}\right.$$

hold as long as $h>h^*$, $|\boldsymbol{\zeta}_j(t)|<\zeta^*$ and $|\eta|<\eta^*$ for constants M_0 and $\overline{M}_0$, where $h=|P_2-P_1|$ and $\boldsymbol{e}=\dfrac{1}{h}(P_2-P_1)$.

Remark 3.2 *Constants M_0 and $\overline{M}_0$ are obtained in explicit way as constants M_1 and M_2 stated in Remark 2.1 while we will not show them in this report, which will be written in [?]. For (1.3), it is numerically checked that both M_0 and $\overline{M}_0$ are positive.*

In the rest of this report, we will intuitively consider the dynamics of P_j and ζ_j in the case of $\eta < 0$ (the case of the existence of a travelling spot). Suppose both M_0 and $\overline{M}_0$ are positive. To understand the dynamics of ζ_j intuitively, we consider a simplified ODE

$$\dot{\zeta}_1 = -\nabla W(\zeta_1) - Ke \tag{3.2}$$

for a positive constant K. Since the right hand side of (3.2) is written by $-\nabla W_1(\zeta_1)$, where $W_1(\zeta) = W(\zeta) + K \langle \zeta, e \rangle$, (3.2) has one stable equilibrium with a form $-\beta e$ for $\beta > 0$. Thus, ζ_1 is pushed toward the direction of $-e$.

Similarly in (3.1), ζ_1 is pushed toward the direction of $-e$ and ζ_2 is done toward the direction of e. As a consequence, approaching two spots push each other toward opposite directions and they eventually part from each other (Fig.6).

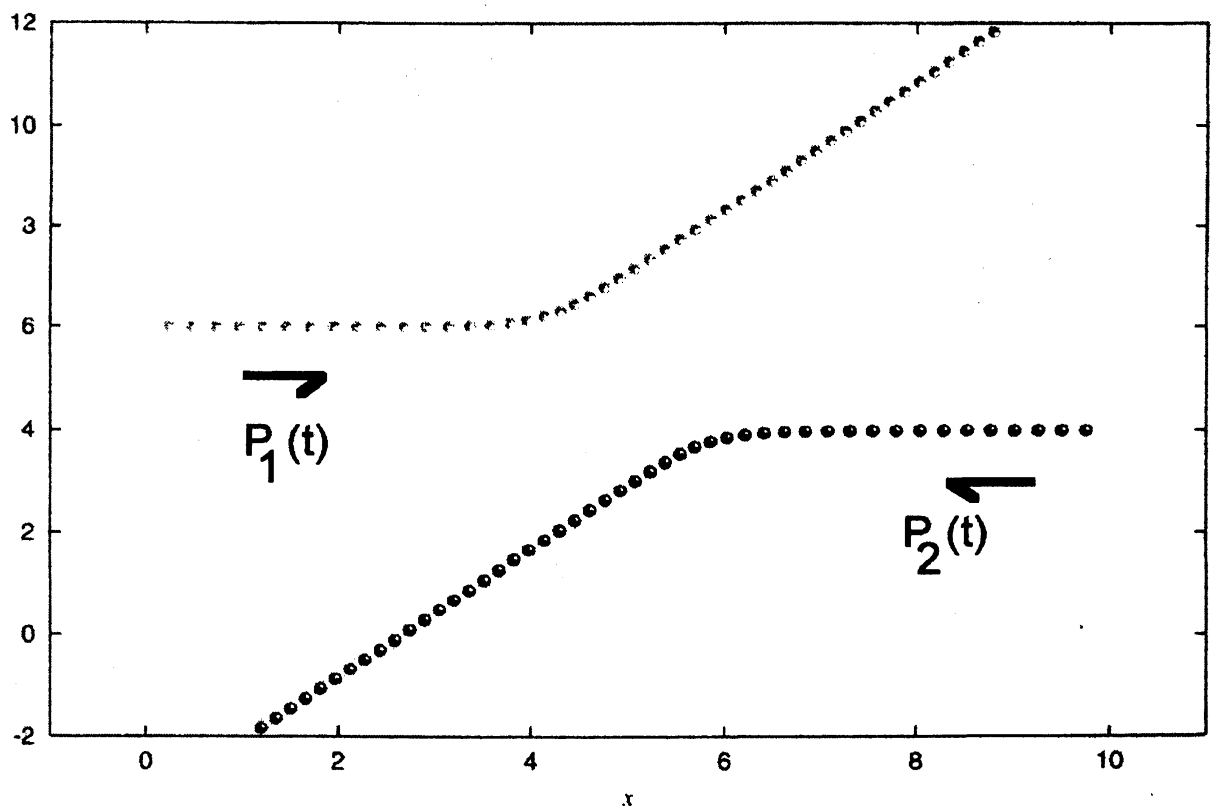

Figure 6: Movements of $P_1(t)$ and $P_2(t)$ which is the solution of ODE consisting of the principal parts of (3.1). Each dot stands for $P_j(t)$ in every time unit.

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