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On a crystalline approximation for an area-preserving motion*

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Abstract

An area-preserving motion of immersed, closed and convex curves by curvature is approximated by so-called crystalline algorithm. We establish a convergence result between a crystalline motion and the area-preserving motion. We also construct an implicit scheme which enjoys several nice properties such as the area-preserving and curve-shortening.

Key Words: area-preserving, motion by curvature, curve-shortening, singularity, crystalline curvature, implicit crystalline algorithm, adaptive time step control.

1 Introduction

We concern a motion of closed plane curves $\Gamma(t)$ which are evolved by the evolution law:

$$V=\kappa-\frac{2\eta\pi}{L}. \tag{1}$$

Here V is the inward normal velocity, κ the curvature, L the length and η the rotation number of the curve Γ. It can be verified that through the evolution,

$$A(t)=-\frac{1}{2}\int_{\Gamma(t)}(\boldsymbol{x},\boldsymbol{n})ds$$

is conserved, where $\boldsymbol{x}$ denotes point on $\Gamma(t)$, $\boldsymbol{n}$ the inward unit normal vector of $\Gamma(t)$ and s the arc-length parameter, respectively. In the case where $\eta=1$, $A(t)$ is the area enclosed by $\Gamma(t)$. Hence, we call this problem area-preserving motion by curvature. We

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can also see that the length $L(t)$ is monotone decreasing in time t. The purpose of this article is to propose a numerical scheme for this problem.

If the curvature of initial curve, say Γ_0, is positive everywhere, then it holds that the curvature of $\Gamma(t)$ is positive everywhere for all $t > 0$ and the evolution is described by an initial and periodic boundary value problem for the following nonlocal partial differential equation (see Gage [G]):

$$\kappa_t = \kappa^2 \kappa_{\theta\theta} + \kappa^3 - \frac{2\eta\pi}{L}\kappa^2, \quad L = \int_0^{2\eta\pi} \kappa^{-1}\, d\theta. \tag{2}$$

This problem possesses a classical time local solution. In the case where $\eta = 1$, the solution of the problem (2) exists globally in time and the solution converges to a constant function exponentially, namely $\Gamma(t)$ converges to a circle (Gage [G]). In the case where $\eta = 1$ with initial curve Γ_0 whose curvature changes sign, the solution is possible to develop singularities, but to our knowledge such a result might not be available. In the case where $\eta > 1$, the behavior of solution might not be also clarified yet.

Hereafter we only consider the case where

(A1) the curvature of Γ_0 is positive everywhere.

Let us introduce an approximate problem for (1) which is derived by so-called crystalline approximation. The notions of crystalline approximation, crystalline motion and crystalline curvature below are originally introduced by Taylor ([T, TCH]) and independently by Angenent and Gurtin ([AGu]) for studying crystal growth mathematically. As for the recent development in this direction, we refer [GY, KG] and references there in.

Firstly, we approximate the curve $\Gamma(t)$ by so-called admissible piecewise linear closed curve $\Gamma_\Delta(t)$. Here the admissible piecewise linear curve is the piecewise linear curve that satisfies the following properties:

1. there exists a finite set Θ that consists of unit vectors such that the all normal angles of the curve belong to the set Θ, which is called the angle set of the curve,
2. each normal angles of adjacent sides is adjacent in the angle set Θ.

Let n be the number of elements in the set Θ. We note that in the case where $\eta = 1$, admissible piecewise linear closed curves are n-polygon. Moreover, because of the assumption (A1), we only need to consider convex n-polygon, if $\eta = 1$.

Secondly, we define so-called crystalline curvature κ_j of the j-th side of the admissible curve as follows:

$$\kappa_j = \frac{\gamma_j}{d_j}.$$

Here d_j denotes the length of the j-th side, $\Delta\theta_j$ the angle between the j-th side and the $(j-1)$-th side, respectively, and $\gamma_j = \tan(\Delta\theta_{j+1}/2) + \tan(\Delta\theta_j/2)$. Setting $\Delta\theta_j \equiv \Delta\theta = 2\pi/n$, we can see that the crystalline curvature is the inverse of the radius of the largest inscribed regular polygon, while usual curvature is the inverse of the curvature radius. In this sense crystalline curvature is a generalization of usual curvature (see appendix B in [Y1]).

Finally, we approximate the original motion by so-called crystalline motion. Here the crystalline motion is a motion of the admissible curve which moves keeping admissibility. Namely, each sides of the admissible curve moves in normal direction only. Such motion can be described by a system of ordinary differential equations (see [AGu, Gu]):

$$\frac{d}{dt}d_j = -\gamma_j(\Delta_\theta v + v)_j, \quad (j = 0, 1, \ldots, \eta n - 1).$$

Here v_j denotes the normal velocity of the j-th side of the piecewise linear curve Γ_Δ and we set

$$(\Delta_\theta v)_j = \frac{(D_+ v)_j - (D_+ v)_{j-1}}{\gamma_j}, \quad (D_+ u)_j = \frac{u_{j+1} - u_j}{\sin \Delta\theta_{j+1}}.$$

We note that v_j, κ_j, d_j are periodic in the index j with period ηn since the curve Γ_Δ is closed. We approximate evolution law (1) by

$$v_j(t) = \kappa_j(t) - \frac{\eta \sum_i \gamma_i}{L_\Delta(t)}.$$

We note that solutions to this approximate evolution law also satisfy area-preserving and curve-shortening properties. Substituting the approximate evolution law into the ordinary differential equations above, we obtain the following approximate problem:

$$\text{(3)} \qquad \dot{\kappa}_j = \kappa_j^2(\Delta_\theta \kappa)_j + \kappa_j^3 - \frac{\eta \sum_i \gamma_i}{L_\Delta}\kappa_j^2, \quad L_\Delta = \sum_{i=0}^{\eta n - 1} \gamma_i \kappa_i^{-1}, \quad (j = 0, 1, \ldots, \eta n - 1).$$

Note that $\sum_i \gamma_i = \sum_{0 \le i < n} \gamma_i \sim 2\pi$ holds for large n. In the case where $\eta = 1$, the solution of (3) exists globally in time and converges to a so-called Wulf shape (Yazaki [Y2]). To our knowledge for $\eta > 1$ there might be almost no result.

Let us make a comment on the relation between our problem and the classical curvature flow:

(4)
$$V = \kappa.$$

The classical curvature flow (4) is the gradient flow of the length L of the curve, while our problem (1) is the gradient flow of L under a constraint that the area A is constant.

In the case of the classical curvature flow (4), the crystalline approximation works very well. The convergence results between the solution of the classical curvature flow (4) and the solution of corresponding crystalline motion are established by several authors([FG, GMHG2, Gir, GirK1, IS, UY1]). Moreover, the efficiency of the numerical scheme which based on crystalline approximation, we call it crystalline algorithm, is demonstrated in [GirK2, UY1, UY2]. One of our motivation is to extend the class of problems to which the crystalline algorithm is applicable.

We establish a convergence result between (2) and (3) in §2 (Theorem 2.1). We propose a numerical scheme based on this approximation in §3. We will see that this scheme enjoys several nice properties. We exhibit numerical examples in §4.

2 A main result

Hereafter we only consider the case where

(A2)
$$\Delta\theta_j \equiv \Delta\theta = \frac{2\pi}{n}.$$

In this case $\gamma_j \equiv \gamma = \tan(\Delta\theta/2)$, and the equations (3) reduce to

(5)
$$\dot{d}_j = -\gamma(\Delta_\theta\kappa + \kappa)_j + \frac{\eta n \gamma^2}{L_\Delta}.$$

Our main result is the following convergence theorem between the solution of the are-preserving motion (2) and the solution of the corresponding crystalline motion (5).

Theorem 2.1 *Assume $\eta = 1$, (A1) and (A2). We also assume that for small $\Delta\theta$*

(6)
$$\|\kappa(\theta,0) - \kappa_{\Delta\theta}(\theta,0)\|_\infty = O(\Delta\theta^2) \text{ and } \max_{0\le j<n}\left|\kappa_\theta(j\Delta\theta,0) - \frac{\kappa_{j+1}(0) - \kappa_{j-1}(0)}{2\sin(\Delta\theta/2)}\right| = O(\Delta\theta^2).$$

For all $T \in (0,\infty)$ there exists a constant C such that

$$\max_{0\le t\le T}\|\kappa(\theta,t) - \kappa_{\Delta\theta}(\theta,t)\|_\infty \le C(\Delta\theta)^2.$$

Here $\kappa(\theta,t)$ is solution of (2), $\kappa_{\Delta\theta}(\theta,t)$ is the continuous function which is derived by linear interpolation from solution $\kappa_j(t)$ of (5).

We will prove the theorem in appendix. We note that in the case of the classical curvature flow, convergence results are obtained by comparison theorem. However, in our case such comparison theorem is not available, because of the presence of the nonlocal term.

Remark 1 *In crystalline algorithm, the initial curve is usually approximated by circumscribed piecewise linear closed curve. By this way of approximation, the assumption on the initial data (6) is achieved (see [UY1, UY2, Y1]).*

3 Numerical scheme

In this section we propose a numerical scheme based on crystalline approximation. We assume (A1) and (A2).

In order to maintain the area-preserving property, we discretize the equation (5) as follows:

$$(D_t d)_j^m = -\gamma(\Delta_\theta \hat{k} + \hat{k})_j^{m+1/2} + \frac{\eta n \gamma^2}{L_\Delta^{m+1/2}}. \tag{7}$$

Here we set

$$(D_t d)_j^m = \frac{d_j^{m+1} - d_j^m}{\tau_m}, \quad \hat{k}_j^{m+1/2} = \frac{\gamma}{d_j^{m+1/2}},$$

$$d_j^{m+1/2} = \frac{d_j^{m+1} + d_j^m}{2}, \quad L_\Delta^{m+1/2} = \sum_{0\le j<\eta n} d_j^{m+1/2},$$

and τ_m denotes the m-th time increment. By this way of discretization, the area-preserving property holds in the following sense:

$$\begin{aligned}(D_t A_\Delta)^m &= \sum_{j=0}^{\eta n-1} \left(\frac{d_j^m + d_j^{m+1}}{2}\right) v_j^{m+1/2} \\ &= \sum_{j=0}^{\eta n-1} d_j^{m+1/2} \left(\hat{k}_j^{m+1/2} - \frac{\eta n \gamma}{L_\Delta^{m+1/2}}\right) = 0.\end{aligned}$$

Here A_Δ^m denotes the area enclosed by the solution curve, say Γ_Δ^m. Moreover, we can also verify the curve-shortening property, namely,

$$\begin{aligned}(D_t L_\Delta)^m &= -\gamma \sum_{j=0}^{\eta n-1} \hat{k}_j^{m+1/2} + \frac{\eta^2 n^2 \gamma^2}{L_\Delta^{m+1/2}}, \\ &= \frac{\gamma^2}{L_\Delta^{m+1/2}} \left(\eta^2 n^2 - \sum_{j=0}^{\eta n-1} \frac{1}{d_j^{m+1/2}} \sum_{j=0}^{n\eta-1} d_j^{m+1/2} \right) \leq 0.\end{aligned}$$

Here we have used summation by parts and the Schwarz inequality. Thus we have the following proposition.

Proposition 1 *The solution of equations (7) satisfies area-preserving and curve-shortening properties.*

Since the discretized problem (7) is fully implicit, to ensure the unique solvability, we use the following adaptive time step control:

$$\tau_m = \frac{\rho \Delta\theta_2}{4 + \Delta\theta_2} \left(\frac{\min_{0 \leq j < \eta n} d_j^m}{\gamma} \right)^2. \tag{8}$$

Here we set

$$\Delta\theta_2 = 2(1 - \cos\Delta\theta) = \Delta\theta^2 + O(\Delta\theta^4), \quad t_m = \sum_{0 \leq i < m} \tau_i, \quad \rho = \frac{(1-\lambda)\min\{\lambda, 1-\lambda\}}{1+\lambda},$$

and λ is an appropriate constant in $(0,1)$. Under this time step control we can prove the following unique solvability result.

Proposition 2 *The full discretized problem (7) possesses a unique solution $\{d_j^m\}_{0 \leq j < \eta n}$ and there holds*

$$(1-\lambda) \min_{0 \leq j < \eta n} d_j^m \leq d_j^{m+1} \leq (1+\lambda) \max_{0 \leq j < \eta n} d_j^m.$$

Proof. Hereafter we use the notations $(\cdot)_{\max}$, $(\cdot)_{\min}$ and $\sum_j (\cdot)_j$ for $\max_{0 \leq j < \eta n} (\cdot)_j$, $\min_{0 \leq j < \eta n} (\cdot)_j$ and $\sum_{0 \leq j < \eta n} (\cdot)_j$, respectively. To show the solvability, we use the next iteration: for $k = 0, 1, 2, \ldots$

$$z_j^{k+1} = z_j^0 - \gamma^2 \left((\Delta_\theta + 1) \left(\frac{2}{z_j^k + z_j^0} \right) - \frac{2n}{\sum_i z_i^k + \sum_i z_i^0} \right) \tau_m$$

with the initial data $z_j^0 = d_j^m$.

By using the variable time step (8), one can get the boundedness:

$$(1-\lambda)z_{\min}^{0} < z_j^k < (1+\lambda)z_{\max}^{0}$$

and the contraction map:

$$\|z^{k+1} - z^k\| \le \ell \|z^k - z^{k-1}\|, \ \ \ell = \frac{1}{2(1+\lambda)} < 1.$$

Here $\|(\cdot)\|$ denotes the maximum norm $\max_{0\le j<\eta n} |(\cdot)_j|$. Hence there exists the limit $\lim_{k\to\infty} z_j^k = d_j^{m+1}$ and also the boundedness above leads the second assertion. □

In the case where $\eta = 1$, we can prove a global existence result. It is a consequence of area-preserving and curve-shortening properties. This result can be proved in almost same manner as the proof of Lemma 3.1 of [Y2].

Proposition 3 *Assume $\eta = 1$, (A1), and (A2). Let us fix the number of sides n. The solution of the full discretized problem (7) exists globally in the following sense:*

$$\sum_{m=0}^{\infty} \tau_m = \infty.$$

Remark 2 *Generally speaking, we can expect that the discretization which maintains conservation laws such as area-preservation leads to numerical stability. So the numerical solution κ_j^m might be bounded uniformly in n and m.*

For fixed n, the convergence between the solution of (5) and (7) can be easily verified. However, this convergence depends on n and we could not obtain satisfying convergence result between the solution of (2) and (7), yet.

4 Numerical examples

Now let us exhibit several numerical examples which are obtained by the scheme explained in the previous section.

Firstly, we examine the convergence of our scheme numerically, in the case where $\eta = 1$. In Figure 1, we show the convergence of a numerical solutions whose initial data is an ellipse with the major axis 1.5 and the minor axis 1. Here the horizontal axis is the number of sides n and the vertical axis error between the numerical solutions for n and $n/2$, $\|\kappa_{\Delta\theta}(\cdot,t) - \kappa_{2\Delta\theta}(\cdot,t)\|_2$, both in logarithmic scale. We plot the graphs at time $t = 0.0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5,$ and 5.0, the upper graph is corresponding to the smaller t. From the figure we can see the rate of convergence is about $O(\Delta\theta^2)$. In Figure 2, we show a time evolution of the curve which starts from an ellipse ($n = 160$; from left to right). We can observe that the curve converges to a circle (see also Table 1).

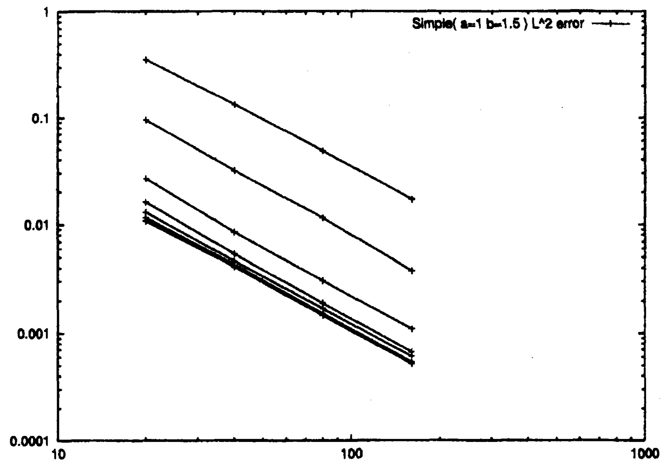

Figure 1: convergence of crystalline algorithm

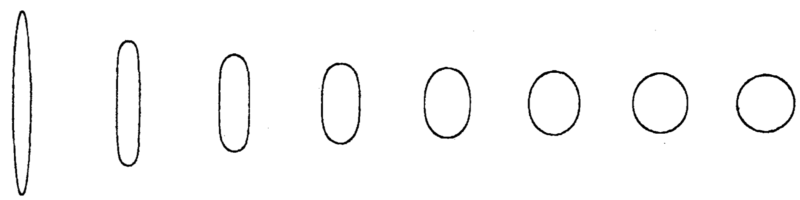

Figure 2: time evolution of a curve

Secondly, we examine the area-preserving and the curve shortening properties of our scheme. In Table 1, we show the time evolutions of the area and the length of the numerical solution that is same as in Figure 2. From the table we can see that the area-preserving property holds very precisely, and the isoperimetric ratio for polygon (see [Y2]) $I_\Delta^m = \frac{(L_\Delta^m)^2}{2n\gamma A_\Delta^m}$ converges to 1.

Thirdly, we investigate the case where $\eta > 1$, numerically. In Figure 3, we show a time evolution which starts from a self-intersected initial data ($\eta = 2$; from left to right). In Figure 4, for the same initial data, we observe the behavior of $M(t) = \max_j \kappa_j(t)$, $\frac{M_t}{M}$, $\frac{M_t}{M^2}$, $\frac{M_t}{M^3}$, and $\frac{M_t}{M^4}$, numerically. Here horizontal axis is the number of

Table 1: area-preserving and curve-shortening properties

Time(t_m)	Area(A_Δ^m)	Length(L_Δ^m)	Isoperimetric ratio(I_Δ^m)
0.00	9.4900475984	24.2846618519	4.944587
1.00	9.4900476781	14.1617906947	1.681521
2.00	9.4900477791	11.5715060040	1.122653
3.00	9.4900477951	11.0190270481	1.018011
4.00	9.4900477950	10.9347550372	1.002499
5.00	9.4900477941	10.9229931726	1.000344
13.80	9.4900477933	10.9211168364	1.000000

time step m. We can observe M, $\frac{M_t}{M}$, $\frac{M_t}{M^2}$ seems to diverge, $\frac{M_t}{M^3}$ seems to be bounded, and $\frac{M_t}{M^4}$ seems to decay. From these figures it seems to develop singularity in finite time, say T, with the order $O((T-t)^{-1/2})$.

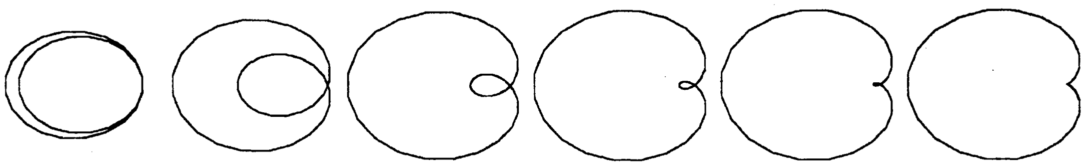

Figure 3: time evolution of a curve (case $\eta = 2$)

A Proof of the results

In this appendix we will prove Theorem 2.1. This result could be obtained from an a priori estimate (Theorem A.1) and a rather general convergence theorem (Theorem A.2) below. Firstly, we mention the a priori estimate. Secondly, we explain the convergence theorem.

A.1 A priori estimate

In this subsection we explain the a priori estimate for the solution of our problem (5). This is a discrete version of Proposition 3.6 in Gage [G]. We note that in our case

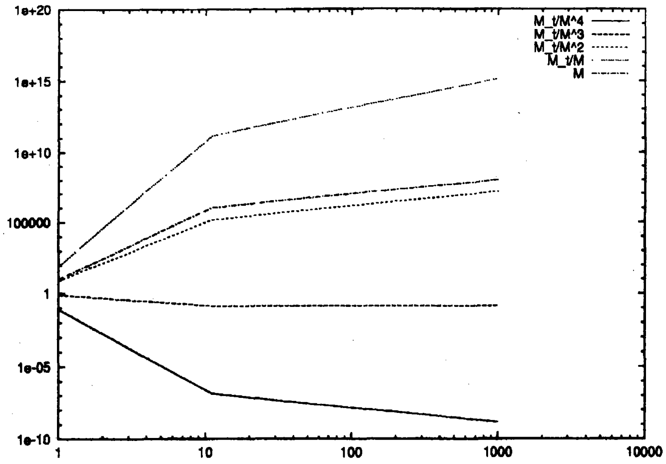

Figure 4: time evolution of M, $\frac{M_t}{M}$, $\frac{M_t}{M^2}$, $\frac{M_t}{M^3}$, $\frac{M_t}{M^4}$

admissible piecewise linear closed curves are convex polygons, say Γ_Δ, since we assume the rotation number $\eta = 1$ and (A1). Using isoperimetric inequality for polygons (see [Y2]) instead of usual one, Gage's proof also works well in our case. So we skip the proof.

Theorem A.1 *Let $\eta = 1$. Assume that $\kappa_j(t)$ is a solution of (5). Let M be $\sup\limits_{0\le t<T} \max\limits_{0\le j<n} \kappa_j(t)$, then there exist positive constants c_1, c_2 and c_3, which depend only on initial polygon Γ_0 such that*

$$c_1 \log M \le c_2 + c_3 T.$$

Remark 3 *Using the local existence theorem for the system of ordinary differential equations (5) and the theorem above, we can see that the solution of (5) exists globally in time and remains bounded. Moreover, this bound does not depend on n, since the constants c_1, c_2, and c_3 in the theorem depend only on the initial polygon Γ_0.*

A.2 Convergence theorem

In this subsection we establish a convergence theorem between a nonlocal partial differential equation (9) and a system of ordinary differential equations (10) below. Our goal is to obtain the following theorem.

Theorem A.2 (convergence theorem) *Assume (H1) to (H5). Let u be a solution of continuous problem (9) and v_h a solution of semi-discrete problem (10). Then there* *exists a positive constant* C_R *such that*

$$||u(\cdot,t)-v_h(\cdot,t)||_{C(0,R)} \le C_R h^2, \quad ||u_x(\cdot,t)-(\mathrm{D}_\varphi v_h)(\cdot,t)||_{C(0,R)} \le C_R h^2, \quad 0 \le t < T_\star.$$

Here, we set $T_\star = \min\{T,\ T_h\}$.

Let us make clear the problems (9) and (10) and the hypotheses (H1) to (H5).

Continuous problem

We call the following one dimensional periodic boundary value problem for a nonlocal partial differential equation the continuous problem.

$$\text{(9)} \qquad \left\{ \begin{array}{rcl} u_t(x,t) &=& a u_{xx}(x,t) + f, \quad 0 < x < R, \quad 0 < t < T, \\ u(x,0) &=& u^0(x), \quad 0 < x < R, \\ u(0,t) &=& u(R,t), \quad u_x(0,t) = u_x(R,t), \quad 0 < t < T. \end{array} \right.$$

Here, we set

$$a = a(x,t,u(x,t)), \quad f = f(x,t,u(x,t),G[u]), \quad G[u] = G[u](t) = \int_0^R g(u(\xi,t))\,d\xi.$$

We make the following assumptions on this problem.

(H1) $a = a(x,t,z),\ \ f(x,t,z,w),\ \ g = g(z)$ are smooth, bounded and

$$\inf_{0<x<R,\ 0<t<T,\ z\in\mathbb{R}} a(x,t,z) > 0.$$

(H2) There exists a unique time local classical solution $u(x,t)$ for the continuous problem (9). T denotes the maximal existence time of the solution $u(x,t)$.

Semi-discrete problem

We call the following initial value problem for a system of ordinary differential equations that is obtained via spatial discretization of the continuous problem (9) the semi-discrete problem.

$$\text{(10)} \qquad \left\{ \begin{array}{rcl} \dot{v}_j(t) &=& a_j(\Delta_c v)_j(t) + f_j, \quad 0 \le j < n, \quad 0 < t < T_h, \\ v_j(0) &=& v_j^0, \quad 0 \le j < n, \\ v_{-1}(t) &=& v_{n-1}(t), \quad v_n(t) = v_0(t), \quad 0 < t < T_h. \end{array} \right.$$

Here, we set

$$x_j = jh, \quad h = R/n, \quad a_j = a(x_j,t,v_j(t)),$$
$$f_j = f(x_j,t,v_j(t),G_h[v]), \quad G_h[v] = G_h[v](t) = \sum_{0\le k<n} g(v_k(t))h.$$

The difference operator Δ_c for $\{\zeta_j\}$, which is periodic in the index j with period n, is defined as follows:

$$(\Delta_c\zeta)_j = ({\rm D}_\varphi{}^2\zeta)_j = \frac{\zeta_{j+1} - 2\zeta_j + \zeta_{j-1}}{\varphi^2}, \quad \varphi = \varphi(h).$$

We will also use the following difference operators.

$$(\Delta_h\zeta)_j = (D_h^2\zeta)_j,$$
$$({\rm D}_\varphi\zeta)_j = \frac{\zeta_{j+\frac{1}{2}} - \zeta_{j-\frac{1}{2}}}{\varphi} = \frac{\zeta_{j+1} - \zeta_{j-1}}{2\varphi}, \quad (D_h\zeta)_j = \frac{\zeta_{j+\frac{1}{2}} - \zeta_{j-\frac{1}{2}}}{h},$$
$$(M_h\zeta)_j = \frac{\zeta_{j+\frac{1}{2}} + \zeta_{j-\frac{1}{2}}}{2} = \frac{\zeta_{j+1} + 2\zeta_j + \zeta_{j-1}}{4}.$$

We note that $({\rm D}_\varphi M_h\zeta)_j = (M_h{\rm D}_\varphi\zeta)_j = ({\rm D}_\varphi\zeta)_j$ and $(M_h^2\zeta)_j = (M_h\zeta)_j$.

We make the following assumptions on the semi-discrete problem (10).

(H3) There exists a unique time local solution $\{v_j(t)\}$ for the semi-discrete problem (10). T_h denotes the maximal existence time of the solution $\{v_j(t)\}$. There exists a constant C such that $\sup_{0\le j<n,\, 0<t<T_h} |v_j(t)| \le C < \infty$ and C does not depend on h.

(H4) (Assumption on initial condition): There exists a positive constant C_0 such that

$$\max_{0\le j<n} |u^0(x_j) - v_j^0| \le C_0h^2, \quad \max_{0\le j<n} \left| \frac{u^0(x_{j+1}) - u^0(x_{j-1})}{2\varphi} - \frac{v_{j+1}^0 - v_{j-1}^0}{2\varphi} \right| \le C_0h^2.$$

(H5) (Assumption on discretization parameter $\varphi = \varphi(h) > 0$): There exists a positive constant C_φ such that $|\varphi(h) - h| \le C_\varphi h^3$. We also assume $h \le 1$.

The solution of the ordinary differential equations (10) is a vector valued function $\{v_j\}$. From this function we can obtain a continuous function $v_h(x,t)$, which is defined on $[0,R] \times (0,T_h)$, by linear interpolation in spatial direction. We also call this function v_h the solution of the semi-discrete problem (10).

A.2.1 Proof of Theorem 2.1

Let us prove the Theorem 2.1, namely, convergence between the following two problems.

- $\kappa_t = \kappa^2\kappa_{\theta\theta} + \kappa^3 - \dfrac{2\eta\pi}{L}\kappa^2, \quad L = \displaystyle\int_0^{2\pi} \kappa^{-1}\, d\theta,$
- $\dot\kappa_j = \kappa_j^2(\Delta_\theta\kappa)_j + \kappa_j^3 - \dfrac{\eta n\gamma}{L_\Delta}\kappa_j^2, \quad L_\Delta = \gamma\displaystyle\sum_i \kappa_i^{-1}, \quad \gamma = 2\tan\dfrac{\Delta\theta}{2}.$

$$a(x,t,z) = z^2, \quad f(x,t,z,w) = z^3 - \frac{R}{w}z^2, \quad g(z) = \frac{1}{z}, \quad R = 2\pi, \quad \varphi(h) = 2\sin\frac{h}{2},$$

in Theorem A.2, we can see that hypotheses (H1) and (H5) are satisfied. And the hypothesis on the initial data (H4) is also clear (see Remark 1). As we noted in §1, the former problem possesses a global solution and it remains bounded. The later problem also possess a global solution and it remains bounded uniformly in n (see Remark 3). Hence the hypotheses (H2) and (H3) are also satisfied. Therefore we obtain the Theorem 2.1. □

A.2.2 Proof of Theorem A.2

As we noted in §2, the presence of nonlocal term prevents us from the use of comparison theorem. In stead of this powerful tool, we estimate discrete $W^{1,p}$ norms of the error $e_j(t) = u(x_j,t) - v_j(t)$ for all $p \geq 1$. Let us explain the outline of the proof. Firstly, for $p \geq 2$ we obtain the differential inequalities for discrete L^p norms of e_j and $(\mathrm{D}_\varphi e)_j$ ((15) and (16)). Secondly, we derive time global boundedness of $(\mathrm{D}_\varphi e)_j$ and the convergence in discrete H^1 norm, using (15), (16), and time global boundedness of e_j. Thirdly, we obtain the estimate for all $W^{1,p}$ norms. Passing p to infinity, we obtain the discrete $W^{1,\infty}$ estimate (20). Finally, by linear interpolation in spatial direction, we obtain the result.

Notations and formulae

Before to proceed, let us introduce several notations.

For vector valued function $\zeta(t) = \{\zeta_j(t)\} \in \mathbb{R}^n$, $||\zeta||_\infty(t)$ and $||\zeta||_p(t)$ denote discrete L^∞ norm $\max\limits_{0\leq j<n} |\zeta_j(t)|$ and discrete L^p norm $\left(\sum\limits_{0\leq j<n} |\zeta_j(t)|^p h\right)^{1/p}$, $(1 \leq p < \infty)$, respectively. For continuous function $\zeta(x,t)$, $||\zeta(\cdot,t)||_{C(0,R)}$ denotes $\sup\limits_{0<x<R} |\zeta(x,t)|$.

For the solution $u(x,t)$ of (9) we set $u_j(t) = u(x_j,t)$ and let $v_j(t)$ be the solution of (10). From vector valued function $\zeta_j(t) = u_j(t)$(or $v_j(t)$), we obtain continuous function $\zeta_h(x,t)$ on $0 \leq x \leq R$ by interpolation as follows:

$$\zeta_h(x,t) = \frac{x - x_j}{h}\zeta_{j+1}(t) + \frac{x_{j+1} - x}{h}\zeta_j(t), \quad x_j \leq x < x_{j+1}, \quad 0 \leq j < n.$$

We also interpolate the subscripts of $\zeta_j(t) = u_j(t)$(or $v_j(t)$) as follows:

$$\zeta_{j+s}(t) = s\zeta_{j+1}(t) + (1-s)\zeta_j(t), \quad 0 \leq s \leq 1, \quad 0 \leq j < n.$$

Then we have

$$u_h(x_{j+s},t) = \begin{cases} u_{j+s}(t) \neq u(x_{j+s},t), & 0 < s < 1, \\ u_{j+s}(t) = u(x_{j+s},t), & s = 0,\ 1, \end{cases} \qquad 0 \le j < n,$$
$$v_h(x_{j+s},t) = v_{j+s}(t), \quad 0 \le s \le 1, \quad 0 \le j < n.$$

Here, we set

$$x_{j+s} = x_j + sh = (j+s)h, \quad -1 \le s \le 1, \quad 0 \le j < n.$$

We interpolate the first order difference of $\zeta_h = u_h$ (or v_h) as follows:

$$(\mathrm{D}_\varphi \zeta_h)(x,t) = \frac{x - x_j}{h}(\mathrm{D}_\varphi \zeta)_{j+1}(t) + \frac{x_{j+1} - x}{h}(\mathrm{D}_\varphi \zeta)_j(t), \quad x_j \le x < x_{j+1}, \quad 0 \le j < n.$$

We will use the following formulae many times.

(F1) $$(\mathrm{D}_\varphi \xi\zeta)_j = (\mathrm{D}_\varphi \xi)_j (M_h\zeta)_j + (M_h\xi)_j(\mathrm{D}_\varphi\zeta)_j = (\mathrm{D}_\varphi \zeta\xi)_j,$$

(F2) $$\sum_j \xi_j(\mathrm{D}_\varphi\zeta)_j = -\sum_j (\mathrm{D}_\varphi\xi)_j\zeta_j,$$

(F3) $$\sum_j \xi_j(\Delta_c\zeta)_j = -\sum_j (\mathrm{D}_\varphi\xi)_j(\mathrm{D}_\varphi\zeta)_j,$$

(F4) $$\sum_j (M_h\xi)_j\eta_j(\mathrm{D}_\varphi\zeta)_j = \sum_j \xi_j\eta_j(\mathrm{D}_\varphi\zeta)_j.$$

Proof. We set $e_j(t) = u(x_j,t) - v_j(t) = u_j(t) - v_j(t)$. We note that the hypotheses (H2), (H3), and (H4) lead to

(11) $$\sup_{0<t<T_*} \|e\|_\infty(t) \text{ is bounded,}$$

and

(12) $$\|e\|_\infty(0) \le C_0h^2, \quad \|\mathrm{D}_\varphi e\|_\infty(0) \le C_0h^2.$$

1^o. *Differential equation for* e_j

Differentiating e_j with respect to t, after long calculation, we obtain

(13) $$\dot e_j(t) = a_j(\Delta_c e)_j + C_{1,j}e_j + \sum_k C_{3,k,j}e_kh + (a_jC_{2,j} + C_4,j)h^2.$$

Here $C_i (i = 1, \ldots, 4)$ are given by

$$C_{1,j} = \int_0^1 F(x_j,t,z_{s,j})\,ds,$$
$$F(x,t,z) = a_z(x,t,z)u_{xx}(x,t) + f_z(x,t,z,G[u]),$$

$$C_{2,j} = C'_{2,j}\frac{h^2}{\varphi^2} + \frac{\varphi^2 - h^2}{h^2\varphi^2}u_{xx}(x_j, t),$$

$$C'_{2,j} = -\frac{1}{6}\int_0^1 (1-s)^3(u_{xxxx}(x_{j+s}, t) + u_{xxxx}(x_{j-s}, t))\,ds,$$

$$C_{3,k,j} = C_{3,j}C'_{3,k},$$

$$C_{3,j} = \int_0^1 f_w(x_j, t, v_j, G_{s,h})\,ds, \quad C'_{3,k} = \int_0^1 g_z(z_{s,k})\,ds,$$

$$C_{4,j} = \int_0^1 f_w(x_j, t, v_j, G_{s,h})\,ds\, C'_4,$$

$$C'_4 = \frac{h}{8}\sum_k \int_{-1}^1\int_0^1 \sigma^2(1-s)\frac{\partial^2}{\partial x^2}g\left(u\left(x_{k+\frac{\sigma s}{2}}, t\right)\right)\,ds\,d\sigma.$$

Here we set $z_{s,j} = su_j + (1-s)v_j$. Those are all bounded:

$$\sup_{1\le i\le 4,\ 0\le j<n,\ 0<t<T_*} |C_{i,j}(t)| < \infty, \qquad \sup_{0\le k<n, 0<t<T_*} |C'_{3,k}(t)| < \infty.$$

The differences of $a_j, C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}$ are in the following forms

$$\text{(14)} \qquad \begin{cases} (\mathrm{D}_\varphi a)_j = K_{1,j} + K_{2,j}(\mathrm{D}_\varphi e)_j, & (\mathrm{D}_\varphi C_1)_j = K_{3,j} + K_{4,j}(\mathrm{D}_\varphi e)_j, \\ (\mathrm{D}_\varphi a C_2)_j = K_{5,j} + K_{6,j}(\mathrm{D}_\varphi e)_j, & (\mathrm{D}_\varphi C_3)_j = K_{7,j} + K_{8,j}(\mathrm{D}_\varphi e)_j, \\ (\mathrm{D}_\varphi C_4)_j = K_{9,j} + K_{10,j}(\mathrm{D}_\varphi e)_j. & \end{cases}$$

Here all constants K_i are bounded, namely,

$$\sup_{0<t<T_*} ||K_i||_\infty(t) < \infty, \quad i = 1, 2, \ldots, 10.$$

2^o. *Differential inequalities for $||e||_p$ and $||\mathrm{D}_\varphi e||_p$ ($p \ge 2$)*

Differentiating $||e||_p$ with respect to t and using (13), (14), the formulae above and $p \ge 2$, after long calculation we can obtain

(15)

$$\frac{1}{p}\frac{d}{dt}||e||_p^p \le C_{\mathrm{I}}\left(||e||_p^{p-1}||\mathrm{D}_\varphi e||_p + \sum_j |e_j|^{p-1}(\mathrm{D}_\varphi e)_j^2 h + ||e||_p^p + ||e||_{p-1}^{p-1}(||e||_1 + h^2)\right).$$

Here we set $C_{\mathrm{I}} = \sup_{0<t<T_*}\{||K_1||_\infty, ||K_2||_\infty, ||C_1||_\infty, ||C_3||_\infty, ||aC_2||_\infty + ||C_4||\}$.

In the same manner as above, we obtain

(16)

$$\frac{1}{p}\frac{d}{dt}||\mathrm{D}_\varphi e||_p^p$$
$$\le C_{\mathrm{II}}\left(\sum_j |e_j||(\mathrm{D}_\varphi e)_j|^{p-1}h + \sum_j |e_j||(\mathrm{D}_\varphi e)_j|^p h + ||\mathrm{D}_\varphi e||_{p-1}^{p-1}h^2 + ||\mathrm{D}_\varphi e||_p^p(1+h^2)\right).$$

Here the constant $C_{\rm II}$ depends on $\sup\limits_{0\le t<T}\{||C_1||_\infty, ||K_3||_\infty, ||K_4||_\infty, ||K_5||_\infty, ||K_6||_\infty, ||K_7||_\infty,$ $||K_8||_\infty, ||K_9||_\infty, ||K_{10}||_\infty\}$.

3^o. *Boundedness of* $||\mathrm{D}_\varphi e||_\infty$

Using the Hölder inequality, from the inequality (16) we have

$$\frac{1}{p}\frac{d}{dt}||\mathrm{D}_\varphi e||_p^p \le C_{\rm II}\left(||e||_p||\mathrm{D}_\varphi e||_p^{p-1} + ||e||_\infty||\mathrm{D}_\varphi e||_p^p + ||\mathrm{D}_\varphi e||_p^p(1+h^2) + c_{\rm R}||\mathrm{D}_\varphi e||_p^{p-1}h^2\right).$$

Here we set $c_{\rm R} = \max\{1, R\}$. Hence, using (11) we have

$$\begin{aligned}\frac{d}{dt}||\mathrm{D}_\varphi e||_p &\le C_{\rm II}\left(||e||_p + ||e||_\infty||\mathrm{D}_\varphi e||_p + ||\mathrm{D}_\varphi e||_p(1+h^2) + c_{\rm R}h^2\right)\\ &\le C_{\rm II}\left(c_{\rm R}||e||_\infty + (||e||_\infty + 1 + h^2)||\mathrm{D}_\varphi e||_p + c_{\rm R}h^2\right)\\ &\le C_{\rm III}\left(||\mathrm{D}_\varphi e||_p + 1\right).\end{aligned}$$

Here we set $C_{\rm III} = C_{\rm II}\sup_{0<t<T_\star}\{||e||_\infty + 2, c_{\rm R}(1+||e||_\infty)\}$. From the inequality above, we have

$$||\mathrm{D}_\varphi e||_p(t) \le (||\mathrm{D}_\varphi e||_p(0) + 1)e^{C_{\rm III}t}.$$

Using the Hölder inequality and (12), there holds

$$||\mathrm{D}_\varphi e||_p(t) \le C_{\rm IV}e^{C_{\rm III}T_\star}, \quad 0 \le t < T_\star \quad (C_{\rm IV} = c_{\rm R}C_0 + 1),$$

uniformly in p. Therefore, as $p \to \infty$, we obtain

$$||\mathrm{D}_\varphi e||_\infty(t) \le C_{\rm IV}e^{C_{\rm III}T_\star} \tag{17}$$

uniformly in $0 \le t < T_\star$ and $h \le 1$.

4^o. *Convergence of* $||e||_2$ *and* $||\mathrm{D}_\varphi e||_2$

Setting $p = 2$ in (15) and (16) and using the Schwarz inequality and (11), we have

$$\begin{aligned}\frac{d}{dt}||e||_2^2 &\le 2C_{\rm I}\left(||e||_2||\mathrm{D}_\varphi e||_2 + \sum_j |e_j|(\mathrm{D}_\varphi e)_j^2 h + ||e||_2^2 + ||e||_1^2 + ||e||_1 h^2\right),\\ &\le 2C_{\rm I}\left(\frac{||e||_2^2 + ||\mathrm{D}_\varphi e||_2^2}{2} + ||e||_\infty\sum_j(\mathrm{D}_\varphi e)_j^2 h + ||e||_2^2 + R||e||_2^2 + \frac{||e||_1^2 + h^4}{2}\right),\\ &\le C_{\rm I}'\left(||e||_2^2 + ||\mathrm{D}_\varphi e||_2^2 + h^4\right),\end{aligned}$$

$$
\begin{aligned}
&\frac{d}{dt}||\mathrm{D}_\varphi e||_2^2 \\
&\leq 2C_{\mathrm{II}}\left(\sum_j |e_j||(\mathrm{D}_\varphi e)_j|h + ||e||_\infty \sum_j (\mathrm{D}_\varphi e)_j^2 h + ||\mathrm{D}_\varphi e||_1 h^2 + ||\mathrm{D}_\varphi e||_2^2(1+h^2)\right), \\
&\leq 2C_{\mathrm{II}}\left(\frac{||e||_2^2 + ||\mathrm{D}_\varphi e||_2^2}{2} + ||e||_\infty ||\mathrm{D}_\varphi e||_2^2 + \frac{R||\mathrm{D}_\varphi e||_2^2 + h^4}{2} + ||\mathrm{D}_\varphi e||_2^2(1+h^2)\right), \\
&\leq C'_{\mathrm{II}}(||e||_2^2 + ||\mathrm{D}_\varphi e||_2^2 + h^4).
\end{aligned}
$$

Here, we set $C'_{\mathrm{I}} = C_{\mathrm{I}} \sup_{0<t<T_\star} \{3(1+R), 1+2||e||_\infty\}$, $C'_{\mathrm{II}} = C_{\mathrm{II}} \sup_{0<t<T_\star} \{5+R+2||e||_\infty\}$. Hence setting $J_2(t) = ||e||_2^2(t) + ||\mathrm{D}_\varphi e||_2^2(t)$, we have

$$
\frac{d}{dt}J_2(t) \leq C_V(J_2(t) + h^4) \qquad (C_V = C'_{\mathrm{I}} + C'_{\mathrm{II}}).
$$

Therefore we obtain $J_2(t) \leq (J_2(0) + h^4)e^{C_V T_\star}$, $0 \leq t < T_\star$, namely,

$$
||e||_2^2(t) + ||\mathrm{D}_\varphi e||_2^2(t) \leq (||e||_2^2(0) + ||\mathrm{D}_\varphi e||_2^2(0) + h^4)e^{C_V T_\star}, \quad 0 \leq t < T_\star.
$$

Taking (12) into account, we can derive

$$
||e||_2(t) \leq C_{\mathrm{VI}}h^2, \quad ||\mathrm{D}_\varphi e||_2(t) \leq C_{\mathrm{VI}}h^2. \tag{18}
$$

Here we set $C_{\mathrm{VI}} = \sqrt{2(c_{\mathrm{R}}C_0)^2 + 1}\, e^{C_V T_\star/2}$. By the Schwarz inequality, we can also obtain

$$
||e||_1(t) \leq \sqrt{R}C_{\mathrm{VI}}h^2, \quad ||\mathrm{D}_\varphi e||_1(t) \leq \sqrt{R}C_{\mathrm{VI}}h^2. \tag{19}
$$

5^o. *Convergence of* $||e||_\infty$ *and* $||\mathrm{D}_\varphi e||_\infty$

Using (11), (17), (18), and the Hölder inequality, we have

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}||e||_p^p \leq C_{\mathrm{I}}\left(||e||_p^{p-1}||\mathrm{D}_\varphi e||_p + \sum_j |e_j|^{p-1}(\mathrm{D}_\varphi e)_j^2 h + ||e||_p^p + ||e||_{p-1}^{p-1}(||e||_1 + h^2)\right), \\
&\leq C_{\mathrm{I}} \\
&\left(||e||_p^{p-1}||\mathrm{D}_\varphi e||_p + ||\mathrm{D}_\varphi e||_\infty \sum_j |e_j|^{p-1}|(\mathrm{D}_\varphi e)_j|h + ||e||_p^p + c_{\mathrm{R}}||e||_p^{p-1}(||e||_1 + h^2)\right), \\
&\leq C_{\mathrm{I}}\left(||e||_p^{p-1}||\mathrm{D}_\varphi e||_p + ||\mathrm{D}_\varphi e||_\infty ||e||_p^{p-1}||\mathrm{D}_\varphi e||_p + ||e||_p^p + c_{\mathrm{R}}||e||_p^{p-1}(\sqrt{R}||e||_2 + h^2)\right), \\
&\leq C_{\mathrm{VII}}\left(||e||_p^{p-1}||\mathrm{D}_\varphi e||_p + ||e||_p^p + ||e||_p^{p-1}h^2\right).
\end{aligned}
$$

Here, we set $C_{\mathrm{VII}} = C_{\mathrm{I}} \sup_{0<t<T_\star}\{1 + ||\mathrm{D}_\varphi e||_\infty, c_{\mathrm{R}}(C_{\mathrm{VI}}\sqrt{R} + 1)\}$. Hence, we obtain

$$
\frac{d}{dt}||e||_p \leq C_{\mathrm{VII}}(||e||_p + ||\mathrm{D}_\varphi e||_p + h^2).
$$

We can also estimate as follows,

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}||\mathrm{D}_\varphi e||_p^p \\
&\le C_{\mathrm{II}}\left(\sum_j |e_j||(\mathrm{D}_\varphi e)_j|^{p-1}h + \sum_j |e_j||(\mathrm{D}_\varphi e)_j|^p h + ||\mathrm{D}_\varphi e||_{p-1}^{p-1}h^2 + ||\mathrm{D}_\varphi e||_p^p(1+h^2)\right) \\
&\le C_{\mathrm{II}}\left(||e||_p||\mathrm{D}_\varphi e||_p^{p-1} + ||e||_\infty||\mathrm{D}_\varphi e||_p^p + c_{\mathrm{R}}||\mathrm{D}_\varphi e||_p^{p-1}h^2 + ||\mathrm{D}_\varphi e||_p^p(1+h^2)\right).
\end{aligned}$$

Hence we obtain

$$\frac{d}{dt}||\mathrm{D}_\varphi e||_p \le C_{\mathrm{VIII}}(||e||_p + ||\mathrm{D}_\varphi e||_p + h^2),$$

here we set $C_{\mathrm{VIII}} = C_{\mathrm{II}} \sup_{0<t<T_\star}\{2 + ||e||_\infty, c_{\mathrm{R}}\}$. Now we can see that $I(t) = ||e||_p(t) + ||\mathrm{D}_\varphi e||_p(t)$ satisfies

$$\frac{d}{dt}I(t) \le C_{\mathrm{IX}}(I(t) + h^2) \quad (C_{\mathrm{IX}} = \max\{C_{\mathrm{VII}}, C_{\mathrm{VIII}}\}).$$

Therefore $I(t) \le (I(0) + h^2)e^{C_{\mathrm{IX}}T_\star}$, $0 \le t < T_\star$. From this inequality and (12), we see that

$$||e||_p(t) \le C_X h^2, \quad ||\mathrm{D}_\varphi e||_p(t) \le C_X h^2, \quad 0 \le t < T_\star,$$

hold uniformly in p. Here we set $C_X = (2c_{\mathrm{R}}C_0 + 1)e^{C_{\mathrm{IX}}T_\star}$. Tending p to infinity, we obtain

$$||e||_\infty(t) \le C_X h^2, \quad ||\mathrm{D}_\varphi e||_\infty(t) \le C_X h^2, \quad 0 \le t < T_\star, \tag{20}$$

for all $h \le 1$.

6^o. *Convergence of* $||e_h||_{C(0,R)}$

Finally, we prove the convergence result. For $x_j \le x < x_{j+1}$, we set $\lambda = \lambda(x) = \dfrac{x - x_j}{h}$. Then at $x = x_j + \lambda h$ we have

$$\begin{aligned}
&|u(x,t) - u_h(x,t)| \le C_R' h^2, \\
&\quad C_R' = \sup_{0<t<T_\star}\max_{0\le j<n}\sup_{x_j<x<x_{j+1}}|B_{0,j}(x,t)|, \\
&\quad B_{0,j}(x,t) = \lambda\int_0^1 (1-s)\{\lambda u_{xx}(x_{j+\lambda s}, t) - u_{xx}(x_{j+s}, t)\}\,ds,
\end{aligned}$$

and

$$\begin{aligned}
|u_h(x,t) - v_h(x,t)| &\le \lambda|u(x_{j+1},t) - v_{j+1}(t)| + (1-\lambda)|u(x_j,t) - v_j(t)| \\
&\le \lambda||u - v||_\infty(t) + (1-\lambda)||u - v||_\infty(t) \\
&= ||u - v||_\infty(t) \le C_X h^2, \quad 0 \le t < T_\star.
\end{aligned}$$

Hence, we obtain

$$
\begin{aligned}
& |u(x,t) - v_h(x,t)| \le |u(x,t) - u_h(x,t)| + |u_h(x,t) - v_h(x,t)| \\
\le \ & (C'_R + C_X)h^2, \quad 0 \le x < R, \quad 0 \le t < T_\star.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
& |u_x(x,t) - (\mathrm{D}_\varphi u_h)(x,t)| \\
& = |u_x(x_{j+\lambda},t) - \lambda(\mathrm{D}_\varphi u_h)_{j+1} - (1-\lambda)(\mathrm{D}_\varphi u_h)_j| \\
& \le \frac{|\varphi - h|}{\varphi}|u_x(x_j,t)| + \lambda h \frac{|\varphi - h|}{\varphi}|u_{xx}(x_j,t)| + |B_{1,j}|h^2 + \frac{h}{\varphi}|B_{2,j+\lambda}|h^2 \\
& \le C''_R h^2, \\
& C''_R = \sup_{0<t<T_\star} \max_{0\le j<n} \sup_{x_j<x<x_{j+1}} |B_{3,j}(x,t)|, \\
& B_{1,j} = \lambda \int_0^1 (1-s)\{\lambda u_{xxx}(x_{j+s\lambda},t) - \frac{h}{\varphi}u_{xxx}(x_{j+s},t)\}\, ds, \\
& B_{2,j} = -\frac{1}{4}\int_0^1 (1-s)^2\{u_{xxx}(x_{j+s},t) + u_{xxx}(x_{j-s},t)\}\, ds, \\
& B_{3,j} = \frac{h}{\varphi}C_\varphi |u_x(x_j,t)| + \lambda h \frac{h}{\varphi}C_\varphi |u_{xx}(x_j,t)| + |B_{1,j}| + \frac{h}{\varphi}|B_{2,j+\lambda}|,
\end{aligned}
$$

and

$$
\begin{aligned}
& |(\mathrm{D}_\varphi u_h)(x,t) - (\mathrm{D}_\varphi v_h)(x,t)| \\
\le \ & \lambda|(\mathrm{D}_\varphi u)_{j+1}(t) - (\mathrm{D}_\varphi v)_{j+1}(t)| + (1-\lambda)|(\mathrm{D}_\varphi u)_j(t) - (\mathrm{D}_\varphi v)_j(t)| \\
= \ & ||(\mathrm{D}_\varphi u) - (\mathrm{D}_\varphi v)||_\infty(t) \le C_X h^2, \quad 0 \le t < T_\star.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
& |u_x(x,t) - (\mathrm{D}_\varphi v_h)(x,t)| \\
\le \ & |u_x(x,t) - (\mathrm{D}_\varphi u_h)(x,t)| + |(\mathrm{D}_\varphi u_h)(x,t) - (\mathrm{D}_\varphi v_h)(x,t)| \\
\le \ & (C''_R + C_X)h^2, \quad 0 \le x < R, \quad 0 \le t < T_\star.
\end{aligned}
$$

Therefore setting $C_R = \max\{C'_R, C''_R\} + C_X$, we finally obtain

(21)

$$
||u(\cdot,t) - v_h(\cdot,t)||_{C(0,R)} \le C_R h^2, \quad ||u_x(\cdot,t) - (\mathrm{D}_\varphi v_h)(\cdot,t)||_{C(0,R)} \le C_R h^2, \quad 0 \le t
$$

References

[AGu] Angenent, S. and Gurtin, M. E., *Multiphase thermomechanics with interfacial structure, 2. Evolution of an isothermal interface,* Arch. Rational Mech. Anal. **108** (1989) 323–391.

[FG] Fukui, T. and Giga, Y., *Motion of a graph by nonsmooth weighted curvature,* World Congress of Nonlinear Analysis '92 (ed. Lakshmikantham, V.), Walter de Gruyter, Berlin (1996) 47–56.

[G] Gage, M., *On an area-preserving evolution equation for plane curves,* Nonlinear Promlems in Geometry (D. M. DeTurck, ed.) Contemp. Math. **51** (1986) 51–62.

[GH] Gage, M. and Hamilton, R. S., *The heat equation shrinking convex plane curves,* J. Diff. Geom. **23** (1986) 69–96.

[GY] Giga, Y., *Moving boundary equations with anisotropic curvature,* (Japanese), Sūgaku **52** (2000) 113–127.

[GMHG1] Giga, M.-H. and Giga, Y., *Consistency in evolutions by crystalline curvature,* Free boundary problems, theory and applications (Zakopane, 1995), Pitman Res. Notes Math. Ser. **363** (1996) 186–202, Longman, Harlow.

[GMHG2] Giga, M.-H. and Giga, Y., *Remarks on convergence of evolving graphs by nonlocal curvature,* Progress in partial differential equations, Vol. 1 (Pont-à-Mousson, 1997), Pitman Res. Notes Math. Ser. **383** (1998) 99–116, Longman, Harlow.

[Gir] Girão, P. M., *Convergence of a crystalline algorithm for the motion of a simple closed convex curve by weighted curvature,* SIAM J. Numer. Anal. **32** (1995) 886–899.

[GirK1] Girão, P. M. and Kohn, R. V., *Convergence of a crystalline algorithm for the heat equation in one dimension and for the motion of a graph by weighted curvature,* Numer. Math. **67** (1994) 41–70.

[GirK2] Girão, P. M. and Kohn, R. V., *The crystalline algorithm for computing motion by curvature,* Variational methods for discontinuous structures (eds.

Serapioni, R and Tomarelli, F.), Birkhäuser, Progress in Nonlinear Differential Equations and Their Applications **25** (1996) 7–18.

[Gu] Gurtin, M. E., *Thermomechanics of evolving phase boundaries in the plane,* Oxford, Clarendon Press (1993).

[IS] Ishii, K. and Soner, H. M., *Regularity and convergence of crystalline motion,* SIAM J. Math. Anal. **30** (1999) 19–37 (electronic).

[KG] Kobayashi, R. and Giga, Y., *On anisotropy and curvature effects for growing crystals*, Recent topics in mathematics moving toward science and engineering, Japan J. Indust. Appl. Math. **18** (2001) 207–230.

[T] Taylor, J. E., *Constructions and conjectures in crystalline nondifferential geometry,* Proceedings of the Conference on Differential Geometry, Rio de Janeiro, Pitman Monographs Surveys Pure Appl. Math. **52** (1991) 321–336, Pitman London.

[TCH] Taylor, J. E., Cahn, J. W. and Handwerker, C. A., *Geometric models of crystal growth,* Acta Metall. Mater. **40** (1992) 1443–1474.

[UY1] Ushijima, T. K. and Yazaki, S., *Convergence of a crystalline algorithm for the motion of a closed convex curve by a power of curvature* $V = K^{\alpha}$, SIAM J. Numer. Anal. **37** (2000) 500–522.

[UY2] Ushijima, T. K. and Yazaki, S., *Convergence of a crystalline algorithm for the motion of a closed convex curve by a power of curvature,* Proceeding of the 4-th. Japan-China Joint Seminar on Numerical Mathematics (1999) 261–270, Gakkou-tosyo.

[Y1] Yazaki, S., *Point-extinction and geometric expansion of solutions to a crystalline motion,* Hokkaido Math. J. **30** (2001), 327–357.

[Y2] Yazaki, S., *On an area-preserving crystalline motion,* Calc. Var. **14** (2002), 85–105.