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Some examples of complete Kähler metrics for which the Neumann operator is compact

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1 Introduction

In this paper we consider compactness of the Neumann operator for complete Kähler metrics.

For the case of the Neumann operator for the Euclidean metric, S. Fu and E. J. Straube [2] obtain a complete characterization of compactness of the Neumann operator as follows.

Theorem 1.1 ([2]). *Let D be a bounded convex domain in $\mathbf{C}^n$. Let q be an integer satisfying $1 \leq q \leq n$. Then the following are equivalent.*

(i) the Neumann operator $N : L^2_{0,q}(D) \longrightarrow L^2_{0,q}(D)$ is compact for the Euclidean metric on $\mathbf{C}^n$.

(ii) the boundary of D does not contain any analytic variety of dimension greater than or equal to q.

However we do not know any fact about compactness of the Neumann operator for complete Kähler metrics. Here we investigate the relation between compactness of the Neumann operator and complete Kähler metrics, and exhibit some examples for which the Neumann operator is compact or not compact.

2 Basic definitions and results

Let (D, ω) be an n-dimensional Kähler manifold. We denote by $L^2_{p,q}(D)$ the space of L^2-integrable (p,q)-forms on D with respect to ω. When we need to be more precise, we denote $L^2_{p,q}(D)$ by $L^2_{p,q}(D,\omega)$. Let $\overline{\partial} : L^2_{p,q}(D) \longrightarrow L^2_{p,q+1}(D)$ be the $\overline{\partial}$-operator in the sense of distribution. Let $\overline{\partial}^* : L^2_{p,q+1}(D) \longrightarrow L^2_{p,q}(D)$ be the L^2-adjoint operator of the $\overline{\partial}$-operator. Let $\Box := \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}|_{L^2_{p,q}(D)}$ be the $\overline{\partial}$-Laplacian. Then we have $L^2_{p,q}(D) = \mathrm{Ker}\Box \oplus \overline{\mathrm{Im}\Box}$.

To simplify arguments in this paper, we define the Neumann operator as

Definition. Suppose that $\mathrm{Ker}\Box = \{0\}$ and $\overline{\mathrm{Im}\Box} = \mathrm{Im}\Box$ hold on $L^2_{p,q}(D)$. Then there exists the inverse operator of the $\overline{\partial}$-Laplacian $N : L^2_{p,q}(D) \longrightarrow L^2_{p,q}(D)$ with $N\Box = id$. We call this operator N the Neumann operator of D for ω.

We know that there is the following result with respect to the existence of the Neumann operator for the Euclidean metric and complete Kähler metrics.

Theorem 2.1 ([7], cf:[6]). *Let D be a bounded domain in $\mathbf{C}^n$ such that there exists a complete Kähler metric on D. Then there exists the Neumann operator $N : L^2_{n,q}(D, \omega_E) \longrightarrow L^2_{n,q}(D, \omega_E)$ for $1 \leq q \leq n$, where we denote by ω_E the Euclidean metric on $\mathbf{C}^n$.*

To show the existence of the Neumann operator for complete Kähler metrics, we introduce the following condition. For example, the Bergman metric of strongly pseudoconvex bounded domains with smooth boundaries satisfies this one.

Definition ([3]). A Kähler metric ω is d–bounded if there exists a positive constant C and $(1,0)$–form η on D such that (i) $\omega = d\eta$ (ii) $\sup_{x \in D} |\eta_x| < C$, where $|\cdot|$ denotes the pointwise norm with respect to ω.

Then we have the following.

Theorem 2.2 ([3]). *Let ω be a complete Kähler and d-bounded metric on D. Then we have $\mathrm{Ker}\Box = \{0\}$ and $\overline{\mathrm{Im}\Box} = \mathrm{Im}\Box$ on $L^2_{p,q}(D)$ if $p + q \neq n$*

3 Examples

From now on, let D be a domain in $\mathbf{C}^n$ and ω be a complete Kähler metric on D.

Krantz [4] showed that the Neumann operator of the bidisc is not compact in $\mathbf{C}^2$ for the Euclidean metric. First of all, we consider the Bergman metric of the bidisc in $\mathbf{C}^2$. We see that this metric is d-bounded by the calculation. Then, by using the same argument in [4], we can show that the Neumann operator of the bidisc is not compact for the Bergman metric as follows.

Proposition 3.1. *Let $\Delta^2 := \{(z_1, z_2) \in \mathbf{C}^2 |\ |z_1| < 1,\ |z_2| < 2\}$ be the bidisc in $\mathbf{C}^2$. Let ω be the Bergman metric on Δ^2. Then the Neumann operator $N : L^2_{2,1}(\Delta^2) \longrightarrow L^2_{2,1}(\Delta^2)$ is not compact.*

Proof. We put $u_j := \overline{\partial}^*(|z_2|^2 z_1^j dz_1 \wedge dz_2 \wedge d\overline{z}_2)$ and $v_j := \sqrt{j+1}\overline{\partial} u_j$. Then $\|v_j\|$ is constant and there exists a positive constant c satisfying $(Nv_j, Nv_k) = c\delta_{jk}$. Hence $\{Nv_j\}$ does not have any subsequence which converges strongly. $\Box$

For proving compactness of the Neumann operator, we need the following. We say that a compactness estimate holds on (p,q)–forms, if for any $\varepsilon > 0$, there exists a positive constant C_ε satisfying that

$$\|u\|^2 \le \varepsilon(\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2) + C_\varepsilon \|u\|_{-1}^2$$

holds for any $u \in L^2_{p,q}(D) \cap \mathrm{Dom}\ \overline{\partial} \cap \mathrm{Dom}\ \overline{\partial}^*$. Here $\|\cdot\|_{-1}$ denotes Sobolev norm of order -1 with respect to ω. Then we have the following lemma.

Lemma 3.2 **([5], [1]).** *Suppose that a compactness estimate holds for (p,q)-forms. Then the Neumann operator $N : L^2_{p,q}(D) \longrightarrow L^2_{p,q}(D)$ is compact.*

Here we need some of the apparatus of Hermitian exterior algebra. if A and B are operators on forms, we put $[A, B] := AB - (-1)^{degA deg B} BA$. Let Λ be the adjoint of multiplication by the fundamental form of the metric ω.

To show that a compactness estimate holds, we use the following inequality.

Proposition 3.3 **(cf: [7], [8]).** *Let q be an integer satisfying $1 \le q \le n$. Let $b : D \longrightarrow (-\infty, 0)$ be a differentiable function. We put $g := 1 - e^b$. Then the following holds:*

$$\|\sqrt{g}\ \overline{\partial}u\|^2 + \|\sqrt{g}\ \overline{\partial}^* u\|^2 \ge (ie^b[\partial\overline{\partial}b, \Lambda]u, u) - \|e^{b/2}\overline{\partial}^* u\|^2$$

for any $u \in C_0^{n,q}(D)$, where $C_0^{n,q}(D)$ denotes the space of differentiable (n,q)-forms with compact support on D.

Then we can prove the following.

Proposition 3.4. *Let q be an integer satisfying $1 \le q \le n$. Let $\varphi : D \longrightarrow (-\infty, 0)$ be a bounded differentiable function such that $\omega = i\partial\overline{\partial}\varphi$ is a complete Kähler and d-bounded metric on D. Then $N : L^2_{n,q}(D) \longrightarrow L^2_{n,q}(D)$ is compact.*

Proof. From Theorem 2.2, the Neumann operator exists on $L^2_{p,q}(D)$ if $p+q \ne n$. For any $\varepsilon > 0$, we put $b := \frac{1}{\varepsilon}\varphi$. From Proposition 3.3 and the assumption, there exists a neighborhood $D_\varepsilon \subset D$ of the boundary of D such that $\frac{1}{2\varepsilon}\|u\|^2 \le 2(\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2)$ for any $u \in C_0^{n,q}(D_\varepsilon)$. Then we can show that a compactness estimate holds for (n,q)–forms in the similar way in pp. 45–46 in [1]. □

Example 1. Let $D := \{z \in \mathbf{C}^n \mid \|z\| < 1\}$ be the unit ball in $\mathbf{C}^n$. We put $\delta := 1 - \|z\|^2$. Let $\lambda : [-1, 0) \longrightarrow (-\infty, 0)$ be a strictly increasing convex bounded differentiable function such that

$$\lambda(t) = \frac{1}{\log(-t)} \quad \text{for} \quad -\frac{1}{2}e^{-2} < t < 0.$$

We put $\varphi = \lambda(-\delta)$ and $\omega := i\partial\overline{\partial}\varphi$. This metric is a complete Kähler and d-bounded on D. Hence the Neumann operator $N : L^2_{n,q}(D) \longrightarrow L^2_{n,q}(D)$ is compact for any $1 \le q \le n$ from Proposition 3.4.

Example 2. Let $D := \Delta^2$ be the bidisc in $\mathbf{C}^2$. In Proposition 3.1, we see that the Neumann operator is not compact for the Bergman metric. Here we show the existence of a complete Kähler metric such that the Neumann operator is compact. We put $\delta_i := 1 - |z_i|^2$ for $i = 1, 2$. Let $\lambda : [-1, 0) \longrightarrow (-\infty, 0)$ be the function which we have constructed in Example 1. We put $\varphi = \lambda(-\delta_1) + \lambda(-\delta_2)$ and $\omega := i\partial\overline{\partial}\varphi$. This metric is a complete Kähler and d-bounded on D. Then we see that $N : L^2_{n,q}(D) \longrightarrow L^2_{n,q}(D)$ is compact for any $1 \leq q \leq n$ from Proposition 3.4.

Example 3. We put $D := \{z \in \mathbf{C}^n | \|z\| < 1\} \setminus \{0\}$. We fix constants c_1 and c_2 satisfying $0 < c_1 < e^{-1}$ and $1 - e^{-2} < c_2 < 1$.

Let $\mu : (0, 1) \longrightarrow (-\infty, \infty)$ be a differentiable function such that (i) $\mu' > 0$ and $\mu'' \leq 0$ (ii) $\mu = \log t$ for $t \in (0, c_1]$ (iii) $\mu = t - 1$ for $t \in [c_2, 1)$. We fix constants c_3 and c_4 satisfying $c_3 < -1$ and $-1 < c_4 < 0$. Let $\lambda : (-\infty, 0) \longrightarrow (\infty, 0)$ be a strictly increasing convex bounded differentiable function such that

$$(i)\ \lambda(t) = \frac{1}{\log(-t)} - A \text{ for } -\infty < t < c_3,$$

$$(ii)\ \lambda(t) = \frac{1}{\log(-t)} \text{ for } c_4 \leq t < 0,$$

where A are a positive constant. We put $\varphi = (\lambda \circ \mu)(\|z\|^2)$ and $\omega := i\partial\overline{\partial}\varphi$. We see that ω is a complete Kähler and d-bounded metric on D by the calculation. Hence the Neumann operator $N : L^2_{n,q}(D) \longrightarrow L^2_{n,q}(D)$ is compact for $1 \leq q \leq n$ from Proposition 3.4.

Remark.
Unfortunately we can not know whether the Neumann operator is compact or not compact when D is the unit ball in $\mathbf{C}^n$ $(n \geq 2)$ and ω is the Bergman metric on D.

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