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# Applications of symbolic construction of the fundamental solution

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**§0. Introduction.** Let $M$ be a n dimensional compact complex manifold, and let $L_q=\bar{\partial}_{q-1}\bar{\partial}_{q-1}^{*}+\bar{\partial}_q^{*}\bar{\partial}_q$ be the Laplacian actiong on differential $(0,q)$-forms $A^{(0,q)}(M)=\Gamma(\wedge^{(0,q)}T^{*}(M))$.

Then Riemann-Roch theorem states as follows:

$$\sum_{q=0}^{n}(-1)^q\dim H_q=\int_M(2\pi i)^{-n}[Td(TM)]_{2n}, \tag{0.1}$$

where $H_q$ is the set of all harmoniq $(0,q)$-forms for $L_q$ and $Td(TM)$ is the Todd class defined as

$$Td(TM)=\det(\frac{\Omega}{e^{\Omega}-1}) \tag{0.2}$$

with the curvature form $\Omega$.

Analytical prooves of the above thorem are based on

$$\sum_{q=0}^{n}(-1)^q\dim H_q=\int_M\sum_{q=0}^{n}(-1)^q\operatorname{tr}e_q(t,x,x)dv_x, \tag{0.3}$$

where $\operatorname{tr}e_q(t,x,x)$ is the trace of the kernel $e_q(t,x,x)$ of the fundamental solution $E_q(t)$ of the following heat equation;

$$\begin{cases}(\dfrac{\partial}{\partial t}+L_q)E_q(t)=0 \ \text{ in } (0,T)\times M,\\ E_q(0)=I \ \text{ in } M.\end{cases}$$

So, by (0.3) an analytical proof for Riemann-Roch theorem is complete, if the following equation holds;

$$\int_M \operatorname{str} e(t,x,x)dv = \int_M (2\pi i)^{-n}[Td(TM)]_{2n},$$

where $\operatorname{str} e(t,x,x) = \sum_{q=0}^{n}(-1)^q \operatorname{tr} e_q(t,x,x)$ and we call it the supertrace of the fundamental solution.

We call the following equation "a local version of Riemann-Roch theorem "

$$\operatorname{str} e(t,x,x)dv_x = (2\pi i)^{-n}[Td(TM)]_{2n} + O(t^{\frac{1}{2}}). \tag{0.4}$$

Our aim of this paper is to give a rough sketch of a proof of the above formula (0.4), constructing the fundamental solution according to the method of symbolic caluculus for a degenerate parabolic operator instead of that of a parabolic operator. Our point is that if $M$ is a Kaehler manifold, we can prove the above formula by only calculating the main term of the fundamental solution, introducing a new weight of symbols of pseudodifferential operators (See C.Iwasaki[10]).

In this paper, we study also "a local version of Riemann-Roch theorem " under the condition that $M$ is a compact complex manifold. In this situation, does "a local version of Riemann-Roch theorem " hold? It is known that the answer of this question is negative(See P.B.Gilkey[4].) So the problem is to characterize complex manifolds where "a local version of Riemann-Roch theorem " holds. Our results is that the characterization is given by $\partial\bar{\partial}\Phi = 0$, where $\Phi$ is the Kaehler form of $M$.

If $M$ is a Kaehler manifold, the above equation (0.4) is obtained by many authors. T.Kotake[6] proved this formula for manifolds of dimension one. Then V.K.Patodi[13] has proved for Kaehler manifolds of any dimension. P.B.Gilkey[5] also has shown, using invariant theory. E.Getzler[3] treated this problem by different approach.

In section 1 we give an rough skech of the proof of Gauss-Bonnet-Chern theorem for a Reimann manifold by constructing a fundamental solution of a parabolic equation according to Ċ.Iwasaki[9]. We give the representation of the operator $L_q$ on a Kaehler manifold, which is called Bochner-Kodaira formula in section 2. In section 3, we give an algebraic lemma which works essentially in calculating the supertrace of the fundamental solution. In section 5, we give a rough sketch of a proof of a "a local version of Riemann-Roch theorem" by the method of symbolic construction of the fundamental solution for degenerate parabolic equation discussed in section 4. Last section is devoted to discuss on a complex manifold. The method of a proof is almost similar with that of on a Kaehler manifold, if $\partial\bar{\partial}\Phi = 0$. On the other hand, unde the conditon $\partial\bar{\partial}\Phi \neq 0$, the method is similar to that of local version of Gauss-Bonnet-Chern theorem.

**§1. A proof of Gauss-Bonnet-Chern theorem.** Let $M$ be a smooth Riemannian manifold of dimension $n$ without boundary . We give a rough sketch of a local version of Gauss-Bonnet-Chern theorem according to [9].

**local version of Gauss-Bonnet-Chern theorem**

$$\sum_{p=0}^{n}(-1)^p \operatorname{tr} e_p(t,x,x)dv_x = \begin{cases} \text{the Euler form} + O(\sqrt{t}) & \text{as } t \to 0, \quad \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

where $e_p(t,x,x)$ is the kernel of the fundamental solution $E_p(t)$ for the Cauchy problem of the heat equaiton on $A^p(M) = \Gamma(\Lambda^p T^*(M))$;

$$(1.1) \qquad \begin{cases} (\dfrac{\partial}{\partial t} + \Delta_p)E_p(t) = 0 & \text{in } (0,T)\times M, \\ E_p(0) = I & \text{in } M. \end{cases}$$

Let $g$ be the Riemannian metric and $\nabla$ denotes the Levi-Civita connection. Let $U$ be a local patch of $M$. Choose $X_1, X_2, \cdots, X_n$ a local orthonormal frame of of $T(M)$ in $U$ and let $\omega^1, \omega^2, \cdots, \omega^n$ be its dual.

In this section we use the following notations:

**Notations.**

$$e(\omega^j)\omega = \omega^j \wedge \omega = a_j^* \omega, \ \imath(X_j)\omega(Y_1, \cdots, Y_{p-1}) = \omega(X_j, Y_1, \cdots, Y_{p-1}) = a_j \omega.$$

We have the following representation for $\Delta = d\vartheta + \vartheta d$, which is known as Weitzenböck's formula.

**Lemma 1.** *The Laplacian $\Delta$ on $A^*(M) = \sum_{p=0}^n A^p(M)$ has the formula*

$$\Delta = -\{\sum_{j=1}^n \nabla_{X_j} \nabla_{X_j} - \nabla_D + \sum_{i,j=1}^n e(\omega^i)\imath(X_j)R(X_i, X_j)\},$$

*where*

$$D = \sum_{j=1}^n \nabla_{X_j} X_j$$

*and $R(X, Y)$ is the curvature transformation.*

By the above lemma, using $c_{i,j}^\ell$ such that

$$\nabla_{X_i} X_j = \sum_{\ell=1}^n c_{i,j}^\ell X_\ell$$

and the coefficients $R^m_{\ \ell ij}$ of the curvature tranformation, we can obtain

$$\Delta = -\{\sum_{j=1}^n (X_j I - G_j)^2 - \sum_{i,j=1}^n c_{i,i}^j (X_j I - G_j) - \sum_{i,j,\ell,m=1}^n R^m_{\ \ell ij} a_i^* a_j a_\ell^* a_m\}$$

on $A^*(M)$. Here

$$G_j = \sum_{\ell,m=1}^n c_{j,\ell}^m a_\ell^* a_m$$

and $I$ is the identity operator on $\wedge^*(T^*(M))$.

Now let us consider the Cauchy problem on $\mathbf{R}^n$:

$$(1.2)\qquad \begin{cases} (\dfrac{\partial}{\partial t} + R(x,D))U(t) = 0 & \text{in } (0,T)\times\mathbf{R}^n, \\ U(0) = I & \text{in } \mathbf{R}^n, \end{cases}$$

where $R(x,D)$ is a differential operator of which symbol $r(x,\xi) = p_2(x,\xi)I + p_1(x,\xi)$. Here $I$ is the identity matrix, $p_j(x,\xi)$ belongs to $S^j_{1,0}$ $(j = 1,2)$, $p_2(x,\xi)$ is a scalar function satisfying $p_2 \geq \delta|\xi|^2$ $(\delta > 0)$ .

Let us give a matrix $\mathcal{A}$ by $\mathcal{A} = (\mathcal{A}_{ij})$, $\quad \mathcal{A}_{ij} = a_i^* a_j \quad 1 \leq i,j \leq n$.

**Definition 1.** A subset $K^m$ of $S^m_{1,0}$ is given by $K^m = \{p(x,\xi : \mathcal{A})$; polynomials with respect to $\xi$ and $\mathcal{A}_{i,j}, (i,j = 1,2,\cdots,n)$ of order $m$ with coefficients in $\mathcal{B}(\mathbf{R}^n)\}$. We define a pseudo-differential operator $P = p(x,D : \mathcal{A})$ acting on $A^*(M)$ of a symbol $\sigma(P) = p(x,\xi : \mathcal{A}) = \sum_{I,J} p_{I,J}(x,\xi)a_I^* a_J \in K^m$ as follows;

$$p(x,D : \mathcal{A})(\varphi_K \omega^K) = \sum_{I,J} p_{I,J}(x,D)\varphi_K a_I^* a_J(\omega^K).$$

In our case the symbol $r$ is of the form $r = r_2 + r_1$, where

$$r_2 = -\sum_{j=1}^{n}(\alpha_j I - G_j)^2 + R.$$

Here

$$R = \sum_{i,j,\ell,m=1}^{n} R^m_{\ell ij} a_i^* a_j a_\ell^* a_m.$$

We note that $p_2 = -\sum_{j=1}^{n}(\alpha_j)^2 I$. It is proved that the fundamental solution $U(t)$ of (1.2) is constructed as a pseudo-differential operator with parameter $t$. Moreover the main part of its symbol is given by

$$u_0(t,x,\xi) = e^{-tr_2(x,\xi:A)}.$$

So we can calculate the trace of the kernel of pseudo-differential operator $U_0(t)$

$$\tilde{u}_0(t,x,x) = (2\pi)^{-n}\int_{\mathbf{R}^n} u_0(t,x,\xi)d\xi = (\frac{1}{2\sqrt{\pi t}})^n\sqrt{\det g}\{1+O(\sqrt{t})\}e^{-tR}.$$

Applying Berezin-Patodi theorem which is stated in section 3, we have

$$\operatorname{str}\tilde{u}_0(t,x,x) = \begin{cases} (\frac{1}{2\sqrt{\pi t}})^n\sqrt{\det g}\operatorname{str}\{\frac{(-1)^m}{m!}R^m t^m\} + O(t), & \text{if } n=2m\ ; \\ O(\sqrt{t}), & \text{if } n \text{ is odd.} \end{cases}$$

Also we can show

$$\operatorname{str} e(t,x,x)dv_x - \operatorname{str}\tilde{u}_0(t,x,x)dx = \begin{cases} O(t), & \text{if } n \text{ is even ;} \\ O(\sqrt{t}), & \text{if } n \text{ is odd} \end{cases}$$

as $t \to 0$. By the following fact we achieve the local version of Gauss-Bonnet-Chern theorem. If $n = 2m$, it holds that

$$(\frac{1}{2\sqrt{\pi}})^n\operatorname{str}\{\frac{(-1)^m}{m!}R^m\} = (\frac{1}{2\sqrt{\pi}})^n\frac{1}{m!}\sum_{\pi,\sigma\in S_n}(\frac{1}{2})^m sign(\pi)sign(\sigma)$$
$$\times R_{\pi(1)\pi(2)\sigma(1)\sigma(2)}\cdots R_{\pi(n-1)\pi(n)\sigma(n-1)\sigma(n)}.$$

**§2.The representation of $L$.** Let $M$ be a smooth Kaehler manifold with a hermitian metric $g$. Set $Z_1, Z_2, \cdots, Z_n$ be a local orthonormal frame of $T^{1,0}(M)$ in a local patch of chart $U$, that is $g(Z_i,Z_j)=0, g(\bar{Z}_i,\bar{Z}_j)=0, g(Z_i,\bar{Z}_j)=\delta_{i,j}$. And let $\omega^1,\omega^2,\cdots,\omega^n$ be its dual. The differential $\bar{\partial}$ and its dual $\bar{\partial}^*$ acting on $A^{0,q}(M)$ are given as follows, using the Levi-Civita connection $\nabla$;

$$\bar{\partial} = \sum_{j=1}^n e(\bar{\omega}^j)\nabla_{\bar{Z}_j}, \qquad \bar{\partial}^* = -\sum_{j=1}^n \imath(\bar{Z}_j)\nabla_{Z_j},$$

where we use the following notations.

**Notations.**

$$e(\eta)\omega = \eta \wedge \omega, \ \imath(Z)\omega(Y_1, \cdots, Y_{p-1}) = \omega(Z, Y_1, \cdots, Y_{p-1}).$$

$$Z_{\bar{j}}' = \bar{Z}_j, \qquad \omega^{\bar{j}} = \bar{\omega}^j \qquad (j = 1, \cdots, n).$$

$$\Lambda = \{1, \cdots, n, \bar{1}, \cdots, \bar{n}\}$$

$$e(\omega^\alpha) = a_\alpha^*, \qquad \iota(Z_\beta) = a_\beta, \qquad (\alpha, \beta \in \Lambda)$$

$$\bar{a}_I = a_{\bar{i}_1} a_{\bar{i}_2} \cdots a_{\bar{i}_p}, \qquad \bar{a}_I^* = a_{\bar{i}_p}^* \cdots a_{\bar{i}_1}^* \qquad \text{for } I = \{i_1 < i_2 < \cdots < i_p\},$$

$$\bar{\omega}^I = \bar{\omega}^{i_1} \wedge \bar{\omega}^{i_2} \wedge \cdots \wedge \bar{\omega}^{i_p} \qquad \text{for } I = \{i_1 < i_2 < \cdots < i_p\}.$$

Let $R(Z_\alpha, Z_\beta)$ be the curvature transformation, that is

$$R(Z_\alpha, Z_\beta) = [\nabla_{Z_\alpha}, \nabla_{Z_\beta}] - \nabla_{[Z_\alpha, Z_\beta]}.$$

*Remark* 1. Owing to that $\{Z_j, \bar{Z}_j\}_{j=1,\cdots,n}$ is an orthnormal frame, we have $R^{\bar{i}}_{\bar{j}k\bar{\ell}} = R_{i\bar{j}k\bar{\ell}} = g(R(Z_k, \bar{Z}_\ell)\bar{Z}_j, Z_i) = R(Z_i, \bar{Z}_j, Z_k, \bar{Z}_\ell)$.

From the fact that our connection is the Riemannian connection, we have

$$R(W_1, W_2, W_3, W_4) + R(W_1, W_3, W_4, W_2) + R(W_1, W_4, W_2, W_3) = 0,$$

$$R(W_1, W_2, W_3, W_4) = -R(W_1, W_2, W_4, W_3),$$

$$R(W_1, W_2, W_3, W_4) = -R(W_2, W_1, W_3, W_4).$$

The assumption $M$ is a Kaehler manifold leads that the curvature transformation satisfies

$$R(Z_i, Z_j) = 0, \quad R(\bar{Z}_i, \bar{Z}_j) = 0,$$

because $\nabla$ preserves the type of vector fields.

Let $c^{\gamma}_{\alpha,\beta}, (\alpha, \beta, \gamma \in \Lambda)$ and $R^{\gamma}{}_{\beta i \bar{j}} (\alpha, \beta, \gamma, \delta \in \Lambda)$ be the following functions:

$$\nabla_{Z_\alpha} Z_\beta = \sum_{\gamma \in \Lambda} c^{\gamma}_{\alpha,\beta} Z_\gamma,$$

$$R(Z_i, Z_{\bar{j}}) Z_\beta = \sum_{\gamma \in \Lambda} R^{\gamma}{}_{\beta i \bar{j}} Z_\gamma.$$

**Proposition 1.** *The coefficients $c^{\gamma}_{\alpha,\beta}$ of connection form satisfy*

$$c^{\bar{i}}_{\alpha j} = c^{i}_{\alpha,\bar{j}} = 0, \qquad c^{i}_{\alpha,j} = -c^{\bar{j}}_{\alpha,\bar{i}}, \qquad (\alpha \in \Lambda, i, j \in \{1, \cdots, n,\})$$

$$[Z_\alpha, Z_\beta] = \sum_{\gamma \in \Lambda} (c^{\gamma}_{\alpha,\beta} - c^{\gamma}_{\beta,\alpha}) Z_\gamma \qquad (\alpha, \beta \in \Lambda).$$

We have the following representation for $L = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ which is known as Bochner-Kodaira formula.

**Theorem 1.** *On $A^{0,*}(M) = \sum_{q=0}^{n} A^{0,q}(M)$ we have*

$$L = -\frac{1}{2}\{\sum_{j=1}^{n} (\nabla_{Z_j} \nabla_{\bar{Z}_j} + \nabla_{\bar{Z}_j} \nabla_{Z_j}) - \nabla_D - \sum_{j=1}^{n} R(Z_j, \bar{Z}_j)\},$$

*where*

$$D = \sum_{j=1}^{n} (\nabla_{\bar{Z}_j} Z_j + \nabla_{Z_j} \bar{Z}_j).$$

By Theorem 1 and Proposition 1 we have

$$L = -\frac{1}{2}\sum_{j=1}^{n}\{(Z_jI - G_j)(\bar{Z}_jI - G_{\bar{j}}) + (\bar{Z}_jI - G_{\bar{j}})(Z_jI - G_j)\}$$

$$+\frac{1}{2}\sum_{i,j=1}^{n}\{c^{\bar{j}}_{\bar{i},\bar{i}}(\bar{Z}_jI - G_{\bar{j}}) + c^{j}_{\bar{i},i}(Z_jI - G_j)\} - \frac{1}{2}\sum_{j,k,\ell=1}^{n} R_{\ell\bar{k}j\bar{j}}a^*_{\bar{k}}a_{\bar{\ell}}$$

on $A^{0,*}(M)$. Here

$$G_\alpha = \sum_{\ell,m=1}^{n} c^m_{\alpha,\ell}a^*_{\bar{\ell}}a_{\bar{m}} \qquad (\alpha \in \Lambda)$$

and $I$ is an identity operator on $\wedge^*(T^{0,*}(M))$.

The following proposition is fundamental for $a_\alpha, a^*_\beta$.

**Proposition 2.** *For any $\alpha, \beta \in \Lambda$, we have*

$$a_\alpha a_\beta + a_\beta a_\alpha = 0,$$

$$a^*_\alpha a^*_\beta + a^*_\beta a^*_\alpha = 0,$$

$$a_\alpha a^*_\beta + a^*_\beta a_\alpha = \delta_{\alpha\beta},$$

**§3. Berezin-Patodi's formula.** Let $V$ be a vector space of dimention $n$ with inner product and let $\wedge^p(V)$ be its anti-symmetric $p$ tensors. Set $\wedge^*(V) = \sum_{p=0}^{n}\wedge^p(V)$. Let $\{v_1, \cdots, v_n\}$ be an orthnormal basis for $V$. Set $a^*_i$ be a linear transformation on $\wedge^*(V)$ defined by $a^*_i v = v_i \wedge v$ and set $a_i$ be an adjoint operator of $a^*_i$ on $\wedge^*(V)$. Then $\{a^*_i, a_j\}$ satisfy Proposition 2. The following Theorem 2 and Corollary are shown in [2] under the above assumptions.

**Theorem 2(Berezin-Patodi[2]).** *For any linear operator $A$ on $\wedge^*(V)$, we can write uniquely in the form $A = \sum_{I,J} \alpha_{I,J} a_I^* a_J$ and*

$$\sum_{p=0}^{n} \mathrm{tr}[(-1)^p A_p] = (-1)^n \alpha_{\{1,2,\cdots,n\}\{1,2,\cdots,n\}},$$

*where $A_p = A|_{\wedge^p(V)}$.*

**Corollary .** *(1)If multi index $I$ and $J$ satisfy $\sharp(I) < n$ or $\sharp(J) < n$, we have*

$$\sum_{p=0}^{n} \mathrm{tr}[(-1)^p a_I^* a_J] = 0.$$

*(2)Let $\pi$ and $\sigma$ be elements of permutation of $n$. Then*

$$\sum_{p=0}^{n} \mathrm{tr}[(-1)^p a_{\pi(1)}^* a_{\sigma(1)} a_{\pi(2)}^* a_{\sigma(2)} \cdots a_{\pi(n)}^* a_{\sigma(n)}] = (-1)^n sign(\pi) sign(\sigma).$$

**§4. Fundamental solution for a degenerate operator.** In this paper we use the pseudo-differential operators of Weyl symbols, that is, a symbol $p(x,\xi) \in S^m_{\rho,\delta}(\mathbf{R}^n)$ defines an operator as

$$Pu(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi} p(\frac{x+y}{2},\xi) u(y) dy d\xi, \qquad u \in \mathcal{S}(\mathbf{R}^n).$$

**Definition 2.** (1) $\nabla p$ means a vector

$$\nabla p = \binom{\nabla_x p}{\nabla_\xi p} = {}^t(\frac{\partial}{\partial x_1}p, \cdots, \frac{\partial}{\partial x_n}p, \frac{\partial}{\partial \xi_1}p, \cdots, \frac{\partial}{\partial \xi_n}p)$$

for a linear transformation $p(x,\xi)$ with parameter $(x,\xi)$.

(2) $J$ is a transformation on $\mathbf{C}^n \times \mathbf{C}^n$ defined by

$$J\binom{u}{v} = \binom{v}{-u}$$

for $u, v \in \mathbf{C}^n$. We also use the same notation $J$ in case $u = {}^t(u_1, \cdots, u_n), u_j$ is a linear tranformation on some vector space.

(3)$< t, s > = \sum_{j=1}^{k} t_j s_j$ for a pair of vectors $t = {}^t(t_1, \cdots, t_k), s = {}^t(s_1, \cdots, s_k)$.

We consider the construction of the fundamental solution $U(t)$ for a degenerate parabolic system

$$\begin{cases} (\frac{d}{dt} + P)U(t) = 0 & \text{in } (0, \mathrm{T}) \times \mathbf{R}^{\mathrm{m}}, \\ U(0) = I & \text{on } \mathbf{R}^{\mathrm{m}}, \end{cases}$$

for the Cauchy problem on $\mathbf{R}^m$ (See I-Iwasaki [7], C.Iwasaki[8]). Here $P$ is a differential operator of a symbol $p(x,\xi) = p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi)$, where $p_j(x,\xi)$ are homogeneous of order j with respect to $\xi$. In [7] construction of the fundamental solution is studied for degenerate parabolic equations of more general case, which had been charactarized by A.Melin[12]. We restrict ourselves to an application to the Laplacian.

**Condition (A).**

$(A) - (1)$
$$p_2(x,\xi) = \sum_{j=1}^{d} b_j(x,\xi) c_j(x,\xi) \quad (c_j = \bar{b}_j),$$

where $b_j (\in S^1_{1,0})$ are scalar symbols.

$(A) - (2)$
$$p_1 + \mathrm{tr}_+ (\frac{\mathcal{M}}{2}) \geq c|\xi|$$

for some positive constant $c$ on the characteristic set $\Sigma = \{(x,\xi) \in \mathbf{R}^m \times \mathbf{R}^m;$ $b_j(x,\xi) = 0$ for any j $\}$, where $\mathrm{tr}_+ \mathcal{M}$ is the sum of all positive eigenvales of $\mathcal{M}$:

$$\mathcal{M} = i\Xi^* J \Xi.$$

Here $\Xi$ is $2n \times 2n$ matrix such that $\Xi = (\nabla c_1, \cdots, \nabla c_d, \nabla b_1, \cdots, \nabla b_d)$.

Set $\mathbf{b} = {}^t(b_1, \cdots, b_d)$, $\mathbf{c} = {}^t(c_1, \cdots, c_d)$. Then we have

**Theorem 3.** *Let $p(x, \xi)$ satisfy Condition (A). Then the fundamental solution $U(t)$ is constructed as a pseudo-differential operator of a symbol $u(t)$ belonging to $S^0_{\frac{1}{2},\frac{1}{2}}$ with parameter $t$. Moreover $u(t)$ has the following expansion for any $N$:*

$$u(t) - \sum_{j=0}^{N-1} u_j(t) \text{ belongs to } S^{-\frac{N}{2}}_{\frac{1}{2},\frac{1}{2}},$$

$$u_0(t) = \exp\varphi, \quad u_j(t) = f_j(t)u_0(t) \in S^{-\frac{j}{2}}_{\frac{1}{2},\frac{1}{2}},$$

*where*

$$\varphi = -\frac{t}{2}\left\langle \begin{pmatrix} \mathbf{c} \\ \mathbf{b} \end{pmatrix}, F\left(\frac{\mathcal{M}t}{2}\right)\begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix} \right\rangle - \frac{1}{2}\mathrm{tr}[\log\{\cosh(\frac{\mathcal{M}t}{2})\}] - \mathrm{p}_1 t,$$

$$F(s) = s^{-1}\tanh s.$$

## §5. Sketch of the proof for Kaehler manifolds.

We can not obtain (0.4) by the same method in section one, because the length of the complex are half of the dimension of manifolds. So we must modify our operator.

Fix a point $\hat{z} \in M$ and choose a local chart $U$ of a neighborhood $\hat{z}$. Choose a local coordinate $z_1, z_2, \cdots, z_n$ of $U$ such that

$$\hat{z} = 0, \qquad Z_j = \left(\frac{\partial}{\partial z_j}\right) + \sum_{k=1}^{n} \kappa_{jk}(z, \bar{z})\left(\frac{\partial}{\partial z_k}\right), \qquad \kappa_{jk}|_{z=\hat{z}} = 0.$$

We use the following notation.

$$\bar{z}_j = z_{\bar{j}}, \qquad \hat{G}_j = G_j|_{z=0}.$$

Set

$$W(z, \bar{z}) = \sum_{j=1}^{n}(z_j\hat{G}_j + \bar{z}_j\hat{G}_{\bar{j}}) = \sum_{\alpha\in\Lambda} z_\alpha \hat{G}_\alpha, \tag{5.1}$$

$$F_\alpha = G_\alpha - \hat{G}_\alpha - \sum_{\beta\in\Lambda} \kappa_{\alpha\beta}\hat{G}_\beta + \frac{1}{2}\sum_{\beta\in\Lambda}[\hat{G}_\alpha, \hat{G}_\beta]z_\beta + \tilde{F}_\alpha$$

with

$$\tilde{F}_\alpha = \sum_{\beta\in\Lambda} z_\beta[G_\alpha - \hat{G}_\alpha, \hat{G}_\beta] - \frac{1}{2}\sum_{\beta\in\Lambda}[\kappa_{\alpha_\beta}\hat{G}_\beta, W]$$

$$-I_3(1, C_2(Z_\alpha W : W) : W) + I_2(1, C_2(G_\alpha : W) : W),$$

where

$$I_j(t, B : A) = \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} e^{-sA} B e^{sA} ds$$

and

$$C_2(B : A) = [[B, A], A].$$

**Lemma 2.** *We have for $W(z, \bar{z})$ defined by (5.1)*

$$Le^W = e^W \tilde{L},$$

*where*

$$\tilde{L} = -\frac{1}{2}\sum_{j=1}^{n}\{(Z_jI - F_j)(\bar{Z}_jI - F_{\bar{j}}) + (\bar{Z}_jI - F_{\bar{j}})(Z_jI - F_j)\} \tag{5.2}$$

$$+\frac{1}{2}\sum_{i,j=1}^{n}\{c^{\bar{j}}_{i,\bar{i}}(\bar{Z}_jI - F_{\bar{j}}) + c^{j}_{\bar{i},i}(Z_jI - F_j)\} - \frac{1}{2}R$$

*with*

$$R = e^{-W}(\sum_{j,k,\ell=1}^{n} R_{\ell\bar{k}j\bar{j}} a^*_{\bar{k}} a_{\bar{\ell}})e^W.$$

In order to construct the fundamental soluiton for $(\frac{\partial}{\partial t} + \tilde{L})$, we introduce a new class of symbols instead of Definiton 1.

**Definition 3.**

$$K^m = \{\sum_{I,J} p_{I,J}(x,\xi)\bar{a}_I^*\bar{a}_J : p_{I,J}(x,\xi) \in S_{1,0}^k,\ k + \frac{|I|+|J|}{2} \leq m\}$$

If we apply Theorem 3 in symbol class $K^m$ to construction of the fundamental solution for $(\frac{\partial}{\partial t} + \tilde{L})$, we obtain, in this case, the fundamental solution whose main part is given by $u_0(t) = e^{\varphi}$, where $\varphi$ is of the form (4.1) with

$$\mathcal{M} = -\begin{pmatrix} \mathcal{M}_0 & 0 \\ 0 & \bar{\mathcal{M}}_0 \end{pmatrix},$$

where $(\mathcal{M}_0)_{ij} = R(Z_i, \bar{Z}_j)$. So the kernel $\tilde{u}_0(t,x,x)$ of pseudodifferential operator with symbol $u_0(t,x,\xi)$ is obtained as

$$\tilde{u}_0(t,x,x) = (2\pi t)^{-n}\det(\frac{t\mathcal{M}_0}{\exp(t\mathcal{M}_0)-1})\sqrt{\det g}.$$

Then we have

$$\mathrm{str}\,\tilde{u}_0(t,x,x) = (2\pi t)^{-n}\mathrm{str}\ [\det(\frac{t\mathcal{M}_0}{\exp(t\mathcal{M}_0)-1})]\sqrt{\det g}.$$

Applying Theorem 2 , we have

$$\mathrm{str}\,\tilde{u}_0(t,x,x)dx = (2\pi i)^{-n}[Td(TM)]_{2n},$$

where

$$Td(T(M)) = \det(\frac{\Omega}{e^{\Omega}-1})$$

with curvature form $\Omega$, that is,

$$\Omega = (\Omega_\ell^k),\quad \Omega_\ell^k = \sum_{i,j=1}^{n} R_{\bar{k},\ell,i,\bar{j}}\omega^i \wedge \bar{\omega}^j.$$

Noting

$$\mathrm{str}\,e(t,x,x)dv = \mathrm{str}\,\tilde{u}_0(t,x,x)dx + 0(t^{\frac{1}{2}}),\quad t \to 0,$$

we obtain (0.4).

**§6. The representation of $L$ on a complex manifold and calculation of the supertrace.** Let $M$ be a complex manifold with a Hermitian metric $g$. Choose a local frame as in secton 2.

The differential $d$ and its dual $\vartheta$ acting on $A^{p,q}(M)$ are given as follow, using the Levi-Civita connection $\nabla$ .

$$d = \sum_{j=1}^{n} e(\omega^j)\nabla_{Z_j} + \sum_{j=1}^{n} e(\bar{\omega}^j)\nabla_{\bar{Z}_j}, \quad \vartheta = -\sum_{j=1}^{n} \imath(Z_j)\nabla_{\bar{Z}_j} - \sum_{j=1}^{n} \imath(\bar{Z}_j)\nabla_{Z_j}.$$

From the fact that our connection is the Levi-Civita connection, that is, $\nabla g = 0$ and the torsion is free. But, $\nabla$ does not preserve the type of vector fields, that is, $\nabla I \neq 0$, where $I$ is the complex structure. So we have

**Proposition 3.** *The coefficients $c^{\gamma}_{\alpha,\beta}$ of connection satisfy*

$$c^{\gamma}_{\alpha,\beta} = -c^{\bar{\beta}}_{\alpha,\bar{\gamma}}, \qquad \alpha, \beta, \gamma \in \Lambda,$$

$$c^{\bar{j}}_{\ell,k} = 0, \qquad c^{j}_{\bar{\ell},\bar{k}} = 0 \qquad j, k, \ell \in \{1, \cdots, n\},$$

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta} \qquad \alpha, \beta, \gamma, \delta \in \Lambda,$$

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0 \qquad \alpha, \beta, \gamma, \delta \in \Lambda.$$

*Remark* 2. We can not assume that in our case

$$c^{\bar{i}}_{\alpha j} = c^{i}_{\alpha,\bar{j}} = 0, \qquad R(Z_i, Z_j) = 0, \;\; R(\bar{Z}_i, \bar{Z}_j) = 0,$$

which hold if $M$ is a Kaehler manifold.

We introduce new connections $\nabla^S$ and $\nabla^{\tilde{S}}$ and give their characterization.

**Definition 4.**

(1) Let $\nabla^{\tilde{S}}$ be the Hermitian connection on $M$, which is the unique connectioı satisfing with its torsion $T^{\tilde{S}}$

$$\nabla^{\tilde{S}} g = 0, \quad \nabla^{\tilde{S}} I = 0, \quad T^{\tilde{S}}(V, \bar{W}) = 0, \; V \in T^{(1,0)}(M), \bar{W} \in T^{(0,1)}(M).$$

Let $\tilde{S}^{\gamma}_{\alpha\beta}$ be the coefficients of the connection $\nabla^{\tilde{S}}$, that is

$$\nabla^{\tilde{S}}_{Z_\alpha} Z_j = \sum_{k=1}^{n} \tilde{S}^k_{\alpha j} Z_k, \quad \nabla^{\tilde{S}}_{Z_\alpha} Z_{\bar{j}} = \sum_{k=1}^{n} \tilde{S}^{\bar{k}}_{\alpha \bar{j}} \bar{Z}_k.$$

(2)Let $\nabla^S$ be the unique connection on $M$, satisfing with its torsion $T^S$

$$\nabla^S g = 0, \quad \nabla^S I = 0,$$

and

$$g(\bar{W}, T^S(U, V)) + g(U, T^S(\bar{W}, V)) = 0 \quad \text{for } U, V \in T^{(1,0)}(M), \bar{W} \in T^{(0,1)}(M).$$

**Proposition 4.** *Using the above connections, we have*

$$\bar{\partial} = \sum_{r=1}^{n} a^*_{\bar{r}} D_{\bar{r}}, \quad \bar{\partial}^* = -\sum_{r=1}^{n} a_{\bar{r}} (D_r + \sum_{j=1}^{n} c^{\bar{j}}_{\bar{j}r})$$

*with for any* $\alpha \in \Lambda$

$$D_\alpha = Z_\alpha - \sum_{j,k=1}^{n} c^{\bar{k}}_{\alpha \bar{j}} a^*_{\bar{j}} a_{\bar{k}} - \sum_{j,k=1}^{n} \tilde{S}^k_{\alpha j} a^*_j a_k.$$

The Kaehler form is given $\Phi(u, v) = g(Iu, v)$ and also we get the following formula for $\Phi = i \sum_{j=1}^{n} \omega^j \wedge \bar{\omega}^j$.

**Proposition 5.**

$$i\partial\bar{\partial}\Phi = \sum_{j,k,\ell,m=1}^{n} \omega_{\bar{\ell}\bar{m}jk}\bar{\omega}^{\ell} \wedge \bar{\omega}^{m} \wedge \omega^{j} \wedge \bar{\omega}^{k},$$

*where*

$$\omega_{\bar{\ell}\bar{m}jk} = -\frac{1}{2}R_{\bar{\ell}\bar{m}jk} + \frac{1}{2}\sum_{r=1}^{n}\{c^{\bar{r}}_{\bar{m}j}c^{r}_{k\bar{\ell}} + c^{\bar{r}}_{\bar{\ell}k}c^{r}_{j\bar{m}} - c^{\bar{r}}_{\bar{m}k}c^{r}_{j\bar{\ell}} - c^{\bar{r}}_{\bar{\ell}j}c^{r}_{k\bar{m}}\}. \tag{6.1}$$

Instead of the Bochner-Kodaira formula, we have a representaion of $L$ on complex manifolds, using (6.1).

**Theorem 4.** *On $A^{0,*}(M) = \sum_{q=0}^{n} A^{0,q}(M)$ we have*

$$L = \bar{\partial}\bar{\partial}^{*} + \bar{\partial}^{*}\bar{\partial} = -\frac{1}{2}\{\sum_{j=1}^{n}(\nabla^{S}_{\bar{Z}_j}\nabla^{S}_{\bar{Z}_j} + \nabla^{S}_{\bar{Z}_j}\nabla^{S}_{Z_j}) - \nabla^{S}_{D}$$

$$+ \sum_{j,k,r=1}^{n} e(\bar{\omega}^{j})\imath(\bar{Z}_k)g(R^{\bar{S}}(\bar{Z}_j, Z_k)Z_r, \bar{Z}_r)\}$$

$$- \sum_{\ell,m,j,k=1}^{n} \omega_{\bar{\ell}\bar{m}jk}\bar{a}^{*}_{\ell}\bar{a}^{*}_{m}\bar{a}_j\bar{a}_k - 2\sum_{r,\ell,k=1}^{n} \omega_{\bar{r}\bar{\ell}rk}\bar{a}^{*}_{\ell}\bar{a}_k,$$

*where*

$$D = \sum_{r=1}^{n}\{\nabla^{S}_{\bar{Z}_r}Z_r + \nabla^{S}_{Z_r}\bar{Z}_r + T^{S}(Z_r, \bar{Z}_r)\}.$$

*Remark* 3. If $M$ is a Kaehler manifold, the we have the following equalities

$$\partial\bar{\partial}\Phi = 0,\ \nabla^{S} = \nabla^{\bar{S}} = \nabla,\ T^{S} = T^{\bar{S}} = 0.$$

In case $\partial\bar{\partial}\Phi \neq 0$ using the above Theorem 4 instead of Lemma 1, a fundamental soluiton of parabolic equation $(\frac{\partial}{\partial t} + L)$ on a complex manifold can be constructed by symbolic calculus in the similar way. But in this case, calculating the super trace of the main part of symbol of the fundamental soluiton, we have a singularity with respect to $t$. In case $\partial\bar{\partial}\Phi = 0$, by (6.1) and Theorem 4 we obtain the assertion of Theorem 5, following the method similar to that of section 5.

**Theorem 5.**

*(1) If $\partial\bar{\partial}\Phi \neq 0$ and $n$ is even, then we have*

$$\operatorname{str} e(t,x,x)dv_x = (2\pi)^{-n}(-1)^{\frac{n}{2}} \frac{(i\partial\bar{\partial}\Phi)^{\frac{n}{2}}}{(\frac{n}{2})!} t^{-\frac{n}{2}} + O(t^{-\frac{n}{2}+\frac{1}{2}}).$$

*(2)If $\partial\bar{\partial}\Phi = 0$, then we have*

$$\operatorname{str} e(t,x,x)dv_x = O(1).$$

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